Iraqi Journal of Information and Technology Systems
DOI:
ISSN 3079-580X (Online), Vol.1, No.1, P.P. 17-23
Received February 11, 2025 , accepted February 23, 2025

α-Transform Method for Solving Fractional Wave Equation Using α-Fractional Derivative

Areej S. Salih[1], Mohammed S. Mechee [2],* Norazak Senu [3]

[1] Dept. of Math., College of Science, University of Derna, Derna, Libya.
[2, *] Information Technology Research and Development Center (ITRDC), University of Kufa, Najaf, Iraq.
[3] Dept. of Math., Institute for Mathematical Research, University Putra Malaysia (UPM), 43400 UPM Serdang, Selangor, Malaysia
Corresponding email: mohammeds.abed@uokufa.edu.iq

Received February 11, 2025, Revised February 22, 2025, Accepted February 23, 2025, Published March 4, 2025

Abstract: Differential equation (DE) plays a big role in applied science and engineering. One important topic and an extensive field of research involves determining the analytical or numerical solutions to the DEs. Utilizing the α-fractional derivative definition to transform the quasi-linear fractional partial differential equation (FPDE) to a quasi-linear partial differential equation (PDE) and then, solving a quasi-linear PDE utilizing the method of lines (MOL) is the objective of this article. The definition of the α-fractional derivative has appropriate, essential, and strong qualities. Furthermore, a quasi-linear PDE is generated from a quasi-linear FPDE using the characteristics of the definition of the α-fractional derivative. A system of ordinary differential equations (ODEs) can therefore be constructed from the PDE by employing the MOL. After applying the proposed approach to solve an implementation, the exact solution is compared to the numerical solution. Based on the test implementation, the recommended approach performs well with its solution. Consequently, the algorithm employed for this method has been shown to be accurate and efficient.

Keyword: α-Fractional Derivative; ODEs; PDEs; FPDEs, MOL.

1. INTRODUCTION

Fractional calculus (FC) is a great interest subject in the twenty-first century, especially over the past three decades. It is widely used in applied engineering and applied sciences, also in both pure and applied mathematics. However, some applications of fractional differential equations (FDEs) in these fields have been studied and introduced by numerous authors. A lot of researchers have examined some definitions of FDEs. For instance, they have presented some modern concepts in the definitions of fractional derivatives (FDs), as well as development of the methods of solving FDEs [1–9]. However, several authors introduced some new definitions for FDs. For instance, Mechee et al. [10] provided an important definition of the α-fractional integral and the α-FD of real functions, while Khalil et al. [11] introduced a novel definition of the FD. Also, Z. Zheng et al. proposed a new definition of Caputo-type for FCs then, they studied its properties [12]. Furthermore, in this paper, we will use some properties of these definitions of FDs in solving the FPDEs.
Accordingly, the literature review on FDs and solving FDEs can be introduced as follows: conformable fractional differential transform (CFDT) and its application introduced by Unal and Gan. On the other hand, the second-order conjugate boundary value problems (BVPs) have been reformulated by Anderson and Avery using the new definition of FD while the fundamental characteristics of Legendre conformable FDEs have been studied by Hammad and Khalil. Likewise, Abdel Hakim proves the existence of conformable fractional. The heat-conformable FDE's exact solution has been examined by Khalil and Abu-Hammad. Furthermore, Abdel Jawad established fundamental concepts in FDs, and then, he developed the definition of the conformable FD. Additionally, Ortega and Rosales were presented the conformable FD features, whereas Qasim and Holel study how to deal with particular composition DEs of fractional order types that have been linked with optimal control challenges. In a similar, Euler and RK methods for solving some classes of FDEs have been generalized by the Mechee et al. Lastly, numerous authors analyzed some types of the FDEs and then, they studied their solutions [13-23].

In this article, we solved the quasi-linear FPDEs by using some properties of the specific definition of α-fractional FD of real functions. Accordingly, FPDE is transformed into a PDE using α-fractional transformation. This α-transformation proved to be the most effective approach for transforming the FPDE into a PDE.

2. BACKGROUND

Several concepts and definitions have been provided in this section as follows.

2.1 *QUASI-LINEAR SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS*

The general class of second-order ODEs has the following definition

Definition 2.1: ***GENERAL CLASS OF SECOND-ORDER ODES***

The initial value problem of general formula of second-order ODE is written as follows:

$$\Psi\big(\zeta, \nu(\zeta), \nu'(\zeta), \nu''(\zeta)\big) = 0, \ \ \zeta_0 \le \zeta \le \zeta_1 \tag{1}$$

with the initial-conditions (ICs),

$$\nu^{(j)}(a) = \zeta^j, \ \ j = 0,1 \tag{2}$$

while the initial-value problem of quasi-linear second-order ODE can be written in the following formula:

$$\nu^{(2)}(\xi) = \phi\big(\zeta, \nu(\zeta), \nu'(\zeta)\big) = 0, \quad \zeta_0 \le \zeta \le \zeta_1 \tag{3}$$

with the ICs in Equation (2).

The general class of second-order FDEs has the following definition

Definition 2.2: GENERAL CLASS OF SECOND-ORDER FDES

The definition of the general class of second-order ODEs is as follows:

$$\phi\left(\zeta, \nu(\zeta), \nu'(\zeta), \nu''(\zeta), \nu^{(\alpha)}(\zeta), \nu^{(2\alpha)}(\zeta)\right) = 0, \ \ \zeta_0 \le \zeta \le \zeta_1, \qquad 0 \le \alpha \le 1 \tag{4}$$

with the ICs,

$$\nu^{(j\alpha)}(a) = \zeta^j, \qquad j = 0,1 \tag{5}$$

The special class of quasi-linear second-order ODEs has the following definition

Definition 2.2: SPECIAL CLASS OF QUASI-LINEAR FDES OF SECOND-ORDER

The class of quasi-linear second-order FDEs is written in the following formula:

$$\nu^{(2\alpha)}(\zeta) = \mathrm{f}\big(\zeta, \nu(\zeta), \nu'(\zeta), \nu''(\zeta)\big), \ \ \zeta_0 \le \zeta \le \zeta_1 \ \ 0 \le \alpha \le 1 \tag{6}$$

with the ICs in Equation (5).

2.2 *FRACTIONAL DERIVATIVES*

In the following definition, Mechee et al. [15] introduced a novel of α-fractional derivative definition

Definition 2.3: **α -FRACTIONAL DERIVATIVE [15]**

The definition of α-fractional derivative definition for the function $\phi(\zeta) : [\zeta_0, \infty) \to \mathcal{R}$ by following:

$$\mathrm{T}_\alpha\big(\phi(\zeta)\big) = \phi^{(\alpha)}(\zeta) = \lim_{\epsilon \to 0} \frac{\phi(\zeta + \epsilon\zeta^{1-\alpha}) - \phi(\zeta - \epsilon\zeta^{1-\alpha})}{2\epsilon}, \tag{7}$$

for $\alpha \in (0,1]$.

THEOREM 2.1 [6]

Consider the real function $\phi(\zeta) : [\zeta_0, \infty) \to \mathcal{R}$ defined in the domain I_α then, we obtain the following property of α-fractional derivative of the function $\phi(\zeta)$ using the Definition 2.3 of α-fractional derivative:

$$\mathrm{T}_\alpha\big(\phi(\zeta)\big) = \phi^{(\alpha)}(\zeta) = \zeta^{1-\alpha}\,\phi'(\zeta). \tag{8}$$

We obtain the following property of the second α-fractional derivative of the function $\phi(\zeta)$ Using the Definition 2.3 of α-fractional derivative for the function $\phi^{(\alpha)}(\zeta)$, in Equation (8),

$$\mathrm{T}_{2\alpha}\big(\phi(\zeta)\big) = \zeta^{1-2\alpha}\ \big(\zeta\phi''(\zeta) + (1-\alpha)\,\phi'(\zeta)\big), \tag{9}$$

where $\mathrm{T}_{2\alpha}\big(\phi(\zeta)\big)$ =second α-fractional derivative of the real function $\phi(\zeta)$ in the domain I_α.

3. MAIN RESULTS

In this section, we introduce the main results in this paper.

3.1. FRACTIONAL SECOND-ORDER PARTIAL DIFFERENTIAL EQUATION (FPDE)

In general, the quasi-linear second FPDE is written as follows:

$$\emptyset\left(\begin{array}{c} u(\tau,\zeta),\frac{\partial u(\tau,\zeta)}{\partial\zeta},\frac{\partial u(\tau,\zeta)}{\partial\tau},\frac{\partial^{\alpha}u(\tau,\zeta)}{\partial\tau^{\alpha}},\frac{\partial^{\alpha}u(\tau,\zeta)}{\partial\zeta^{\alpha}},\frac{\partial^{2}u(\tau,\zeta)}{\partial\zeta^{2}},\frac{\partial^{2}(\tau,\zeta)}{\partial\zeta\,\partial\tau},\frac{\partial^{2}u(\tau,\zeta)}{\partial\tau^{2}}, \\ \frac{\partial^{2\alpha}u(\tau,\zeta)}{\partial\tau^{2\alpha}},\frac{\partial^{2\alpha}u(\tau,\zeta)}{\partial\zeta^{2\alpha}} \end{array}\right)=0,\qquad 0<\zeta<l,\quad 0\le\alpha\le 1,$$
$$\tau>0, \tag{10}$$

with the initial-conditions (ICs)

$$\mathrm{u}\,(0,\xi)=\mathrm{f}(\xi),\ \frac{\partial^{\alpha}u(0,\xi)}{\partial\xi^{\alpha}}=\mathrm{g}(\xi)\quad \text{for } 0<\xi<l, \tag{11}$$

and the Dirichlet-boundary conditions (DBCs)

$$\mathrm{u}(\tau,\xi)=0 \text{ for } \xi=0{,}1 \text{ and } \tau>0. \tag{12}$$

3.2 FRACTIONAL DIFFUSION EQUATION

In general, the quasi-linear FPDEs of second-order have the following form

$$\frac{\partial^{\alpha}u(\tau,\xi)}{\partial\tau^{\alpha}}=\emptyset\left(u(\tau,\xi),\frac{\partial u(\tau,\xi)}{\partial\xi},\frac{\partial^{2}u(\tau,\xi)}{\partial\xi^{2}}\right),\qquad 0<\xi<l,\tau>0, \tag{13}$$

with the ICs in Equation (11) and the Dirichlet-boundary conditions (DBCs) in Equation (12).

In special case consider a fractional diffusion equation in one dimension:

$$\frac{\partial^{\alpha}u(\tau,\xi)}{\partial\tau^{\alpha}}=\beta^{2}\frac{\partial^{2}u(\tau,\xi)}{\partial\xi^{2}},\qquad 0<\xi<l,\qquad \tau>0, \tag{14}$$

with the ICs and the DBCs in equations (11)-(12) and

3.2.1 FRACTIONAL WAVE EQUATION

In general, the quasi-linear second FPDE has the following form

$$\frac{\partial^{2\alpha}u(\tau,\zeta)}{\partial\tau^{2\alpha}}-\frac{\partial^{2\alpha}u(\tau,\zeta)}{\partial\zeta^{2\alpha}}=\emptyset\left(u(\tau,\zeta),\frac{\partial u(\tau,\zeta)}{\partial\zeta},\frac{\partial u(\tau,\zeta)}{\partial\tau},\frac{\partial^{\alpha}u(\tau,\zeta)}{\partial\tau^{\alpha}},\frac{\partial^{\alpha}u(\tau,\zeta)}{\partial\zeta^{\alpha}},\frac{\partial^{2}u(\tau,\zeta)}{\partial\zeta^{2}},\frac{\partial^{2}(\tau,\zeta)}{\partial\zeta\,\partial\tau},\frac{\partial^{2}u(\tau,\zeta)}{\partial\tau^{2}},\right),$$
$$0<\zeta<l,\quad 0\le\alpha\le 1,\qquad \tau>0, \tag{15}$$

with the initial-condition (ICs) in Equation (11) and the Dirichlet-boundary conditions (DBCs) in Equation (12)

In special case consider a homogenous fractional wave equation in one dimension:

$$\frac{\partial^{2\alpha}u(\tau,\zeta)}{\partial\tau^{2\alpha}}=\beta^{2}\frac{\partial^{2\alpha}u(\tau,\zeta)}{\partial\zeta^{2\alpha}},\qquad 0<\zeta<l,\qquad \tau>0, \tag{16}$$

with the ICs and the DBCs in Equations (11)-(12) respectively.

3.2.2 PROPOSITION 1

Let $w(\tau,\zeta):[\tau_0,\infty)\to\mathcal{R}$ be a real function in the domain I_α then, using the Definition 2.3 of α-fractional derivative for the function $\phi(\zeta)$, and property of the second α-fractional derivative of this function. Then, we obtain the following property of α-fractional second derivative of the function $w(\tau,\xi)$ with respect to τ and ζ respectively as follows:

$$T_{2\alpha}\big(w(\tau,\zeta)\big)=\frac{\partial^{2\alpha}w(\tau,\zeta)}{\partial\zeta^{2\alpha}}=\zeta^{1-2\alpha}\left(\zeta\frac{\partial w(\tau,\zeta)}{\partial\zeta}+(1-\alpha)\frac{\partial^2 w(\tau,\zeta)}{\partial\zeta^2}\right), \tag{17}$$

$$T_{2\alpha}\big(w(\tau,\zeta)\big)=\frac{\partial^{2\alpha}w(\tau,\zeta)}{\partial\tau^{2\alpha}}=\tau^{1-2\alpha}\left(\tau\frac{\partial w(\tau,\zeta)}{\partial\tau}+(1-\alpha)\frac{\partial^2 w(\tau,\zeta)}{\partial\tau^2}\right), \tag{18}$$

*3.3 **PROPOSED METHOD***

For using the ICs and DBCs in Equations (11)-(12) to solve the quasi-linear FPDE in Equation (16). The proposed approach was developed by combining the numerical method of type RK, the finite difference method, and MOL to obtain system of ODEs. The following steps of the algorithm should be do

3.3.1 PROPOSED ALGORITHM

1. The FPDE in Equation (15) is converted to a PDE using the properties of α-fractional second derivative of the function $w(\tau,\xi)$ with respect to τ and ζ in Equations (17) and (18) with its ICs and BCs in Equations (11) and (12) respectively.
2. Divide the domain of the problem in the variable ξ in [0, l] by n subinterval with the norm of partition $h=\frac{l}{n}$ and in the variable τ in [0, T] by m subinterval with the norm of partition $k=\frac{T}{m}$.
3. Do steps 4-6 while $1\le j\le m$.
4. Put instead the point w (τ,ξ) by $w_{ij}=\big(\tau_i,\xi_j\big)$ of the PDE in step 1 for j=0,1, 2, …, n and i=0,1, 2, …, m.
5. Fix $\tau=\tau_j$ at the left side of the equation in step 1 and put the formulas of finite difference in the derivatives in the right side which converting this PDE in following system of ODEs:
   $$w_i''\big(\tau_j\big)=\tau_j^{\ \alpha-1}\,\emptyset(w_{i-3},\ w_{i-2},w_{i-1},w_i,w_{i+1},w_{i+2},w_{i+3}), \tag{19}$$
   with the I. Cs,
   $$w_{0j}=w\big(\tau_0,x_j\big)=g\big(x_j\big), \tag{20}$$
   and the Dirichlet-boundary conditions (DBCs)
   $$w_{j0}=w\big(\tau_j,\xi\big)=0,\quad \text{for } \xi=0,1 \tag{21}$$
   for $1\le j\le m$ and i=1, 2…, n.
6. Solve the system of ODEs of second-order in Equation (19) with the I.Cs and B.Cs in Equation (20)-(21) using the Runge-Kutta-Nystrom method.

In general, this algorithm can generalize for solving a FPDE of n^{th}-order with I.Cs. and B.Cs.

4. IMPLEMENTATION

To use the proposed approach, we provide an example in this section.

EXAMPLE1: Consider the following second-order FPDE

$$\tau^{2\alpha-1}\frac{\partial^{2\alpha}u(\tau,\xi)}{\partial\tau^{2\alpha}}-\beta^{2}\,\xi^{2\alpha-1}\frac{\partial^{2\alpha}u(\tau,\xi)}{\partial\xi^{2\alpha}}=\tau\frac{\partial u(\tau,\xi)}{\partial\tau}-\beta^{2}\xi\frac{\partial u(\tau,\xi)}{\partial\xi},0<\xi<l,\tau>0, \qquad (22)$$

with the ICs $\mathrm{u}(0,\xi)=\mathrm{e}^{-\xi}$, $\frac{\partial^{\alpha}u(0,\xi)}{\partial\xi^{\alpha}}=\xi^{1-\alpha}\mathrm{e}^{-\xi}$ for $0<\xi<l$, and the Dirichlet-boundary conditions (DBCs) $\mathrm{u}(\tau,\xi)=\mathrm{e}^{-\tau}$ and $\mathrm{u}(\tau,\xi)=\mathrm{e}^{1-\tau}$for $\xi=0{,}1$ respstively and $\tau>0$.

Using the relation in Equations (17)-(18), Equation (22) converted to the following fractional wave equation

$$\frac{\partial^{2}u(\tau,x)}{\partial\tau^{2}}=\beta^{2}\,\frac{\partial^{2}u(\tau,x)}{\partial\mathrm{x}^{2}},\qquad 0<x<l,\quad \tau>0, \qquad (23)$$

with the ICs $\mathrm{u}(0,\xi)=\mathrm{e}^{-\xi}$, $\frac{\partial\,u(0,\xi)}{\partial\,\xi}=\mathrm{e}^{-\xi}$ for $0<\xi<l$, and the Dirichlet-boundary conditions (DBCs) $\mathrm{u}(\tau,\xi)=\mathrm{e}^{-\tau}$ and $\mathrm{u}(\tau,\xi)=\mathrm{e}^{1-\tau}$ for $\xi=0{,}1$ respstively and $\tau>0$.

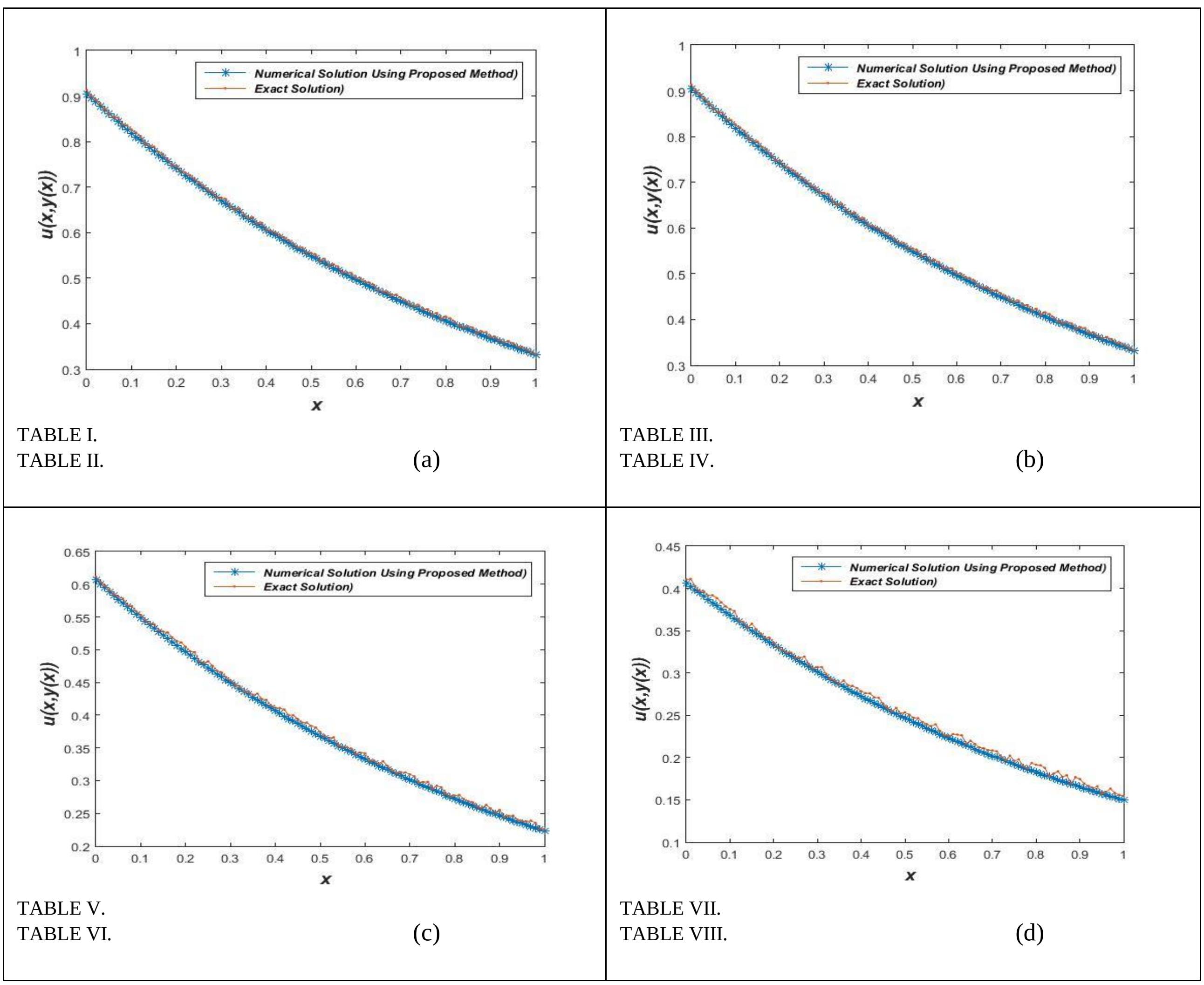

Figure 1: A Numerical Comparison of the Solutions of the Example with Exact Solution for (a) $\tau=0.1$, (b) $\tau=0.25$, (c) $\tau=0.5$ and (d) $\tau=1$ where β=1 Using MOL with RKN.

5. DISCUSSION AND CONCLUSION

The proposed method converts FPDE to PDE, and then it converts it to a system of ODEs that can be solved analytically or numerically. The numerical approach is used by combining MOL with classical numerical RKN method for solving the systems of second-order ODEs. This approach is used for solving test problems, showing that it agrees well with the exact solution. This implementation shows the accuracy and efficiency of this approach. Accordingly, for the first-case in in Example, the approximated solution is calculated by using the proposed method by compounding the MOL method with the numerical RKN. However, this numerical solution is identical with its exact solution. This numerical solution is compared and plotted in Figure 1 for different four cases. Meanwhile, we can conclude the power and efficiency of the proposed α-FD method.

DATA AVAILABILITY STATEMENT

The data used are included in the manuscript

CONFLICT OF INTEREST

The authors declare no conflict of interest in influencing the work in this paper

6. REFERENCES

[1] K. Oldham and J. Spanier, "The fractional calculus theory and applications of differentiation and integration to arbitrary order," Elsevier, 1974.
[2] K. S. Miller and B. Ross, "An introduction to the fractional calculus and fractional differential equations," Wiley, 1993.
[3] Anatoly A. Kilbas, Hari M Srivastava, and Juan J Trujillo, "Theory and applications of fractional differential equations," Elsevier, vol 204, 2006.
[4] I. Podlubny, "Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications," Elsevier, 1998.
[5] S. Das, "Concept of Fractional Divergence and Fractional Curl, " Funct. Fract. Calc., pp. 157–211, 2011, doi: 10.1007/978-3-642-20545-3_4
[6] S. Qasim Hasan and M. Abbas Holel, "Solution of some types for composition fractional order differential equations corresponding to optimal control problems," Journal of Control Science and Engineering, vol. 2018, 2018.
[7] Atangana, Abdon and Baleanu, Dumitru, New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model, arXiv preprint arXiv:1602.03408, 2016.
[8] Bayati, Basil S, Fractional diffusion-reaction stochastic simulations, The Journal of Chemical Physics, AIP Publishing, vol. 138, no. 10, 2013.
[9] Scalas, Enrico and Gorenflo, Rudolf and Mainardi, Francesco, Fractional calculus and continuous-time finance, Physica A: Statistical Mechanics and its Applications, Elsevier, vol.284, no.1-4, pp 376-384, 2000.
[10] M. S. Mechee, Z. I. Salman, and S. Y. Alkufi, "Novel Definitions of α -Fractional Integral and Derivative of the Functions," vol. 64, no. 11, pp. 5866–5877, 2023, doi: 10.24996/ijs.2023.64.11.32.
[11] R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh, "A new definition of fractional derivative," Journal of computational and applied mathematics, vol. 264, pp. 65-70, 2014.
[12] Z. Zheng, W. Zhao, and H. Dai, "A new definition of fractional derivative," International Journal of Non-Linear Mechanics, vol. 108, pp. 1-6, 2019.
[13] E. Unal ¨ and A. G¨ Okdo˘ Gan, "Solution of conformable fractional ordinary differential equations via differential transform method," Optik, vol. 128, pp. 264-273, 2017.
[14] D. R. Anderson and R. I. Avery, "Fractional-order boundary value problem with sturmliouville boundary conditions," arXiv preprint arXiv:1411.5622, 2014.
[15] M. A. Hammad and R. Khalil, "Legendre fractional differential equation and Legendre fractional polynomials," International Journal of Applied Mathematics Research, vol. 3, no. 3, p. 214, 2014.
[16] A. A. Abdelhakim, "Precise interpretation of the conformable fractional derivative," arXiv preprint arXiv:1805.02309, 2018.
[17] T. Abdeljawad, "On conformable fractional calculus," Journal of computational and Applied Mathematics, vol. 279, pp. 57-66, 2015.
[18] A. Ortega and J. J. Rosales, "Newtons law of cooling with fractional conformable derivative," Revista mexicana de f´ısica, vol. 64, no. 2, pp. 172-175, 2018.
[19] M. S. Mechee and N. Senu, "Numerical study of fractional differential equations of Laneem den type by method of collocation," 2012.

[20] Mohammed Mechee, and Sameeah Aidi,"Generalized Euler and Runge-Kutta methods for solving classes of fractional ordinary differential equations, International Journal of Nonlinear Analysis and Applications, publisher Semnan University, vol 13, no. 1, pp 1737—1745, 2022.
[21] Metzler, Ralf and Klafter, Joseph, The random walk's guide to anomalous diffusion: a fractional dynamics approach, Physics reports, Elsevier, vol. 339, no. 1, pp 1-77, 2000.
[22] Odibat, ZM and Momani, Shaher, Application of variational iteration method to nonlinear differential equations of fractional order, International Journal of Nonlinear Sciences and Numerical Simulation, De Gruyter, vol. 7, no. 1, pp 27-34, 2006.
[23] Y. Cenesiz and A. Kurt, "The solution of time fractional heat equation with new fractional derivative definition," in 8th International Conference on Applied Mathematics, Simulation, Modelling (ASM 2014), 2014, pp. 195-198.