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Certain Types of Regular and T_3 *Spaces*

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Abstract: We introduced certaın types of separation axioms which are Z-regular space, ZT_3 space, semi-Z-regular space and semi-ZT_3 space, by using boundary points and semi boundary points. The relationships among them and relationships with each of T_3, T_1 and semi-T_1 axioms have been given.

Keywords T_1 ***space, semi-***T_1 ***space, regular space,*** T_3***, semi-regular space, semi-***T_3***, Z-regular space,*** ZT_3 ***space, semi-regular space, semi-***ZT_3 ***space.***

1 Introduction

There are many types of the separation axioms: T_0, T_1, T_2, regular, T_3, semi-T_1, semi-T_2, semi-regular, and semi-T_3. T_0 was introduced by A. Kolmogorov, T_1 was introduced by Frechet in 1923, T_2 was introduced by Hausdorff in 1914,[15]. Semi-T_0, semi-T_1, and semi-T_2 were introduced by S.N. Maheshwari and R. Prasad in 1975,[4]. Regular and T_3 space were introduced by V. Neumann in 1925,[15], and semi-regular and semi-T_3 space were introduced by C. Dorsett in 1982, [7]. In [12], Levine defined a set A in a topological space X to be semi-open if there exists an open set U such that $U \subseteq A \subseteq \overline{U}$. In [3], S. Gene Crossley and S. K. Hildebrand defined semi-closed set as the complement of semi-open set, the union of all semi-open sets of X contained in A and the intersection of all semi-closed sets containing A, respectively. In [13], P.Das defined semi-boundary of A as $\overline{A}^s - A^{\circ s}$. We shall use $\overline{A}^s$, A^{bs}, $\overline{A}$, A^b, and $\mathbb{Z}$ to the semi-closure of A, the semi-boundary of A, the closure of A, the boundary of A, and the integer numbers, respectively.We shall refer to the topological space as a space.

2 Preliminaries

Definition 2.1[[11] **A space (X, τ) be defined a T_1 space if and only if for any distinct points x and y in X,**

there exists open sets U, V such that $x \in U, y \notin U, x \notin V$ and $y \in V$.

Definition 2.2[5] A space (X, τ) is called a semi-T_1 space if and only if for any distinct points x, y in X, there are semi-open sets U and V such that $x \in U, y \notin U, x \notin V$ and $y \in V$.

Definition 2.3 [11]A space (X, τ) is called a regular space if and only if for every closed set F and every point $x \notin F$, there exists disjoint sets $U, V \in \tau$, $F \subseteq U$, and $x \in V$.
A regular T_1 space is called a T_3 space.

Definition 2.4 [7]A space (X, τ) be defined semi-regular space if and only if for every $x \in X$ and semi-closed set F, with $x \notin F$, there exists disjoint semi-open sets U and V such that $F \subseteq U, x \in V$.
A semi-regular semi-T_1 space is called a semi-T_3 space.

Theorem 2.5[2] If $f : X \to Y$ is a homeomorphism, then $f(A^b) = (f(A))^b$, for all $A \subset X$

Theorem 2.6[10] If $f : X \to Y$ be a semi-homeomorphism, then $f(A^{bs}) = (f(A))^{bs}$, for all $A \subset X$.

Theorem 2.7 [14] The product of semi-open(semi-closed) sets is a semi-open(semi-closed) set.

proposition 2.8 [11] In any space, $A^b = \phi$ iff A is a clopen set.

Theorem 2.9 [10] In a space X, A is a semi-clopen (semi-open and semi-closed) set iff $A^{bs} = \phi$.

Remark 2.10 [9] Every open (closed) set is semi-open (semi-closed) set.

Remark 2.11[3] Let X be a space and A be a subset of X, then A is semi-closed set iff $A = \overline{A}^s$.

Remark 2.12[14] The concept of semi-open set and open set in cofinite space, discrete space and indiscrete space are the same.

Theorem 2.13[8] Let (X, τ) be a space (Y, τ_y) a subspace of (X, τ) and let $A \subseteq Y$, then:

i. $\overline{A}^y = Y \cap \overline{A}$

ii. $A^{b_y} \subseteq Y \cap A^b$

Where $\overline{A}^y$ and A^{b_y} are the closure of A in Y and boundary of A in Y, respectively.

Theorem 2.14[11] Let (X, τ) be a space and let $A, B \subseteq X$, then $(A \cap B)^b \subseteq A^b \cup B^b$.

Remark 2.15 [10] In a space X, $A^{bs} \subset A^b$

Theorem 2.16[11],[6] The product space of T_1 (semi-T_1) spaces is T_1 (semi-T_1) space.

Theorem 2.17[3] A semi-topological property is a topological property.

3 Main Results

Definition 3.1 A space (X, τ) be defined a Z-regular space if and only if for every closed set F and every point $x \notin F$, there exists disjoint open sets U and V such that $F \subseteq U, x \in V$, and $U^b \cap V^b = \phi$.
A Z-regular T_1 space is called a ZT_3 space.
Note that if (X, τ) is Z-regular space then it is regular space and if X is ZT_3 space then it is a T_3 space and Z-regular space.

Example 3.2 Let $X = \{a,b,c\}, \tau = \{X, \phi, \{a\}, \{b,c\},\}$,$(X,\tau)$ is a Z-regular space:
Consider the closed set $\{b,c\}$ and $a \notin \{b,c\}$, $\{a\}$ and $\{b,c\}$ are disjoint open sets, $\{b,c\} \subseteq \{b,c\}, a \in \{a\}$ and $\{a\}^b \cap \{b,c\}^b = \phi$.
The closed sets $\{a\}$ with each of b and c, there exist disjoint open sets $\{a\}$ and $\{b,c\}$ such that $\{a\} \subseteq \{a\}, b \in \{b,c\}$ and $c \in \{b,c\}$, respectively, and $\{a\}^b \cap \{b,c\}^b = \phi$. Hence, it is Z-regular space.
Note that (X,τ) is not ZT_3 space for it is not T_1 space.

Example 3.3 The discrete space of more than one point, (X,τ_d), is Z-regular space.

Example 3.4 The discrete space of more than one point, (X,τ_d), is ZT_3 space.

Example 3.5 An infinite Cofinite space (X,τ_c) is not Z-regular space since there are no disjoint open sets in the cofinite space except X and ϕ.

Example 3.6 An infinite Cofinite space (X,τ_c) is not ZT_3 since it is not Z-regular.

Example 3.7 The indiscrete space of more than one point, (X,τ_{ind}) is Z-regular space, but not ZT_3 space.

Example 3.8 The Usual space, $(\mathbb{R},\tau_u)$, is ZT_3 space.

Theorem 3.9 The property of being Z-regular be a topological property.

Proof: Suppose that $X \cong Y$, where X is Z-regular space. Then there exists a homeomorphism $f : X \to Y$.
To show that Y be a Z-regular space, let $F \subset Y$ be a closed set and $a \in Y$ with $a \notin F$.
$f^{-1}(F)$ is closed set in X. Then there exists $b \in X$ such that $f(b) = a$.
Since $a \notin F$ then $b \notin f^{-1}(F)$. Since X is a Z-regular space then there exist disjoint open sets U and V such that $f^{-1}(F) \subset U, b \in V$, and $U^b \cap V^b = \phi$. So $f(f^{-1}(F)) = F \subseteq f(U)$,and $f(b) = a \in f(V)$.
$f(U)$ and $f(V)$ are open in Y, and $f(U \cap V) = f(U) \cap f(V) = f(\phi) = \phi$.
Now, $(f(U))^b \cap (f(V))^b = f(U^b) \cap f(V^b)$, [Theorem 2.5)]

$$= f(U^b \cap V^b)$$
$$= f(\phi) = \phi.$$

Hence (Y,τ^*) is Z-regular space.

□

Theorem 3.10 The property of being ZT_3 is a topological property.

Proof: Since Z-regular space and T_1 space is a topological properties then ZT_3 is a topological property.

□

Theorem 3.11 An open subspace of a Z-regular space be a Z-regular space.

Proof: Suppose that (Y,τ_y) be an open subspace of (X,τ), where X be a Z-regular space. To show that (Y,T_y) is Z-regular space, let F be a closed set in Y, and $a \in Y$ with $a \notin F$.
$\overline{F}^y = \overline{F} \cap Y$[Theorem 2.13 (i)] Also, $F = \overline{F}^y$ for F be a closed in Y, and so $F = \overline{F} \cap Y$. Since $a \notin F$, then $a \notin \overline{F} \cap Y$, and so $a \notin \overline{F}$.
Note that $\overline{F}$ is a closed set in X, and $a \notin \overline{F}$, There exist disjoint sets $U, V \in \tau$, $\overline{F} \subseteq U$, $a \in V$, and $U^b \cap V^b = \phi$, for X be a Z-regular space.
Since $a \in V$ and $a \in Y$ then $a \in V \cap Y$, and so $\overline{F} \subseteq U$ then $\overline{F} \cap Y = F \subseteq U \cap Y$.
Note that $(U \cap Y) \in \tau_y$ and $(V \cap Y) \in \tau_y$.
$(U \cap Y) \cap (V \cap Y) = (U \cap V) \cap Y = \phi \cap Y = \phi$
Now,

$(U\cap Y)^{b_y} \subseteq Y\cap(U\cap Y)^b$, [Theorem 2.13 (ii)]
$\subseteq Y \cap (U^b \cup Y^b)$, [Theorem 2.14]
$= (Y \cap U^b) \cup (Y \cap Y^b)$
$= (Y \cap U^b) \cup \phi$, [Proposition 2.8]
$= Y \cap U^b$
Similarity, $(V \cap Y)^{b_y} \subseteq Y \cap V^b$
So,$(U \cap Y)^{b_y} \cap (V \cap Y)^{b_y} \subseteq (Y\cap U^b)\cap(Y\cap V^b)$
$= (U^b\cap V^b)\cap Y = \phi \cap Y = \phi$.
Hence, (Y, τ_y) is a Z-regular space.

□

Theorem 3.12 An open subspace of a ZT_3 space is a ZT_3 space.

Proof: Since an open subspace of a Z-regular and T_1 spaces is a Z-regular T_1 space then an open subspace of a ZT_3 space is a ZT_3 space.

Remark 3.13 The continuous image of Z-regular space needs not be a Z-regular space :
$f : (\mathbb{Z}, \tau_d) \to (\mathbb{R}, \tau_c)$, $f(x) = x$ is continuous function and, $(\mathbb{Z}, \tau_d)$ is Z-regular space but $f(\mathbb{Z}) = \mathbb{Z}$ with the relative cofinite topology is not Z-regular space.

Remark 3.14 The continuous image of ZT_3 space needs not be a ZT_3 space.

Definition 3.15 A space (X, τ) be defined a semi-Z-regular space if and only if for every $x \in X$ and semi-closed set F, with $x \notin F$, there are disjoint semi-open sets U and V such that $F \subseteq U, x \in V$, and $U^{bs} \cap V^{bs} = \phi$.
A semi-Z-regular semi-T_1 space is called a semi-ZT_3 space.

Example 3.16 Let $X = \{a, b, c\}, \tau = \{X, \phi, \{a\}, \{b, c\}, \}$. (X, τ) is a semi-Z-regular space:
Consider semi-closed set $\{b, c\}$ and $a \notin \{b, c\}$, then there are disjoint semi-open sets $\{a\}$ and $\{b, c\}$ such that $\{b, c\} \subseteq \{b, c\}, a \in \{a\}$ and $\{a\}^{bs} \cap \{b, c\}^{bs} = \phi$.
The semi-closed sets $\{a\}$ with each of b and c, there exist disjoint semi-open sets $\{a\}$ and $\{b, c\}$ such that $\{a\} \subseteq \{a\}, b \in \{b, c\}$ and $c \in \{b, c\}$, respectively, and $\{a\}^{bs} \cap \{b, c\}^{bs} = \phi$.
Hence, (X, τ) is semi-Z-regular space.
Note that (X, τ) is not semi-ZT_3 space for it is not semi-T_1 space.

Note 3.17 :

i. Every semi-Z-regular space is semi-regular space
ii. Every semi-ZT_3 space is a semi-T_3 space
iii. Every semi-ZT_3 space is a semi-Z-regular space, but the converse is not true in general for the above example is semi-Z-regular space but not semi-ZT_3 space
iv. Every semi-T_3 space is a semi-regular space

Example 3.18 The discrete space of more than one point, (X, τ_d), is semi-Z-regular space:

Since the concepts of open set and semi-open set are the same in the discrete spaces, [Remark 2.12], then the concept of closed and semi closed are the same in discrete space. We have the result as in [Example 3.3].

Example 3.19 The discrete space of more than one point, (X, τ_d), is semi-ZT_3 space for it is semi-Z-regular and semi-T_1 space.

Example 3.20 Let X be an infinite

set. The Cofinite space (X, τ_c) is not semi-Z-regular space for there are no disjoint semi-open sets in the cofinite space except X and ϕ.

Example 3.21 Let X be an infinite set. The Cofinite space (X, τ_c) is not semi-ZT_3 space.

Example 3.22 The indiscrete space of more than one point (X, τ_{ind}) is semi-Z-regular space but not semi-ZT_3 space.

Theorem 3.23 The property of being semi-Z-regular is a semi-topological property.

Proof: Suppose that X is semi-homomorphic to Y, where X is semi-Z-regular space. Then $f : X \to Y$ be a semi-homeomorphism .
To show that Y is semi-Z-regular space, let $F \subset Y$ be a semi-closed set and $a \in Y$ with $a \notin F$.
Then there are $b \in X$, $f(b) = a$. Since f is irresolute function then $f^{-1}(F)$ is semi-closed set.
Since $a \notin F$, then $b \notin f^{-1}(F)$. Since X is a semi-Z-regular space, then there are disjoint semi-open sets U and V such that $f^{-1}(F) \subseteq U, b \in V$, and $U^{bs} \cap V^{bs} = \phi$. So $f(f^{-1}(F)) = F \subseteq f(U)$, and $f(b) = a \in f(V)$.
Since f is pre-semi-open function, then $f(U)$ and $f(V)$ are semi-open in Y. Since f is injective then $f(U \cap V) = f(U) \cap f(V) = f(\phi) = \phi$.
Now, $(f(U))^{bs} \cap (f(V))^{bs} = f(U^{bs}) \cap f(V^{bs})$, [Theorem 2.6],

$$= f(U^{bs} \cap V^{bs})$$
$$= f(\phi) = \phi.$$

Hence, (Y, τ^*) is a semi-Z-regular space.

□

Theorem 3.24 The property of being a semi-ZT_3 space is a semi-topological property.

Proof: Suppose that X is semi-homomorphic to Y, where X is semi-T_1 space. Then $f : X \to Y$ be a semi-homeomorphism. For any $y_1, y_2 \in Y, y_1 \neq y_2$, we have $f^{-1}(y_1), f^{-1}(y_2) \in X$ and $f^{-1}(y_1) \neq f^{-1}(y_2)$ for f is one to one. Since X is a semi-T_1 space, then there exists semi-open sets U and V such that $f^{-1}(y_1) \in U$,$f^{-1}(y_2) \notin U$, $f^{-1}(y_2) \in V$, $f^{-1}(y_1) \notin V$, we have $y_1 \in f(U)$, $y_2 \notin f(U)$, $y_2 \in f(V)$, and $y_1 \notin f(V)$. $f(U)$ and $f(V)$ are semi-open sets in Y for f be pre-semi-open. Then Y is a semi-T_1 space and Y semi-Z-regular space is a semi-topological property, then Y is a semi-ZT_3 space. Hence, the property of being a semi-ZT_3 space is a semi-topological property.

□

Theorem 3.25 The property of being a semi-Z-regular space is a topological property, [Remark 2.17].

Theorem 3.26 The property of being a semi-ZT_3 space is a topological property, [Remark 2.17].

Remark 3.27 The semi-continuous image of a semi-Z-regular space needs not be a semi-Z-regular space as shown in the following example :
$f : (\mathbb{Z}, \tau_d) \to (\mathbb{R}, \tau_c)$, $f(x) = x$ for all $x \in \mathbb{Z}$ is a semi-continuous function and, $(\mathbb{Z}, \tau_d)$ is semi-Z-regular space, but $f(\mathbb{Z}) = \mathbb{Z}$ with the relative cofinite topology is not semi-Z-regular space.

Remark 3.28 The semi-continuous image of a semi-ZT_3 space needs not be a semi-ZT_3 space.

4 Conclusion

After we have been introduced a Z-regular space, semi-Z-regular space, ZT_3 space and a semi-ZT_3 space, we have the following diagram.

$$\begin{array}{ccccc} T_1 \text{ space} & \Leftarrow & T_3 \text{ space} & \Rightarrow & \text{regular space} \\ & & \Uparrow & & \Uparrow \\ & & ZT_3 \text{ space} & \Rightarrow & \text{Z-regular space} \end{array}$$

$$\begin{array}{ccccc} & & \text{semi-}ZT_3 \text{ space} & \Rightarrow & \text{semi-Z-regular space} \\ & & \Downarrow & & \Downarrow \\ \text{semi-}T_1 & \Leftarrow & \text{semi-}T_3 \text{ space} & \Rightarrow & \text{semi-regular space} \end{array}$$

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