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ZT_1 *spaces and semi-*ZT_1 *spaces*

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Abstract: In this paper, two separation axioms have been introduced: ZT_1 space and semi-ZT_1 space. Relationships between them and with each of T_1 and semi-T_1 spaces have been given.

Keywords: T_1 ***space, semi-***T_1 ***space,*** ZT_1 ***space, semi-***ZT_1 ***space.***

1 Introduction

There are many types of the separation axioms: T_0, T_1, T_2, regular, T_3, semi-T_1, semi-T_2. T_0 was introduced by A. Kolmogorov, T_1 was introduced by Frechet in 1923, T_2 was introduced by Hausdorff in 1914,[10]. Semi-T_0, semi-T_1, and semi-T_2 were introduced by S.N. Maheshwari and R. Prasad in 1975,[6]. Regular and T_3 space were introduced by V. Neumann in 1925,[10]. In [7], Levine defined a set A in a topological space X to be semi-open if there exists an open set U such that $U \subseteq A \subseteq \overline{U}$. In [8], S. Gene Crossley and S. K. Hildebrand defined semi-closed set as the complement of semi-open set, the union of all semi-open sets of X contained in A and the intersection of all semi-closed sets containing A, respectively. In [9], P.Das defined semi-boundary of A as $\overline{A}^s - A^{\circ s}$. We shall use A^c, $A^{\circ s}$, $\overline{A}^s$, A^{bs}, A°, $\overline{A}$, A^e, A^b, $(X \times Y, \tau_{pro})$, τ_y, A^{b_y}, $\mathbb{R}$, $\mathbb{Z}$ to the complement of A, the semi-interior, the semi-closure, the semi-boundary, the interior of A, the closure of A, the exterior of A, the boundary of A, the product space, the sub space topology, the boundary of A in a sub space, the real numbers, the integer numbers, respectively.

2 Preliminaries

Definition 2.1 [5] **A topological space (X, τ) is called T_1 space if and only if for any distinct points x, y in X, there are open sets U, V such that $x \in U, y \notin U, x \notin V$ and $y \in V$.**

Definition 2.2 [6] **A topological space (X, τ) is called semi-T_1 space if and only if for any distinct points x, y in X, there are semi-open sets U, V such that $x \in U, y \notin U, x \notin V$ and $y \in V$.**

Theorem 2.3 [4] **Let (X, τ) and (Y, τ^*) be spaces, non empty subsets A and B of X and Y,respectively, then $(A \times B)^b = (A^b \times \overline{B}) \cup (\overline{A} \times B^b)$.**

Theorem 2.4 [2] **If $f : X \to Y$ is a homeomorphism, then $f(A^b) = (f(A))^b$, for all $A \subset X$**

Theorem 2.5 [3] **Let (X, τ) and (Y, τ^*) be spaces, $A \subset X$ and $B \subset Y$, then $\overline{(A \times B)}^s \subset \overline{A}^s \times \overline{B}^s$.**

Remark 2.6 [1] **In a space X, $A^{bs} = \overline{A}^s \cap \overline{A^c}^s$.**

Remark 2.7 [9] **In a space X, $A^{\circ s} = A - A^{bs}$.**

Theorem 2.8 [7] **The product of semi-open sets is a semi-open set.**

Theorem 2.9 [8] **If $g : X \longrightarrow Y$ is a semi-homeomorphism then for any subsets $A \subset X$ and $B \subseteq Y$:**

- **i.** $\overline{g^{-1}(B)}^s = g^{-1}(\overline{B}^s)$
- **ii.** $(g^{-1}(B))^{\circ s} = g^{-1}(B^{\circ s})$
- **iii.** $\overline{g(A)}^s = g(\overline{A}^s)$
- **iv.** $(g(B))^{\circ s} = g(B^{\circ s})$

Remark 2.10 [5] **In any space, $A^b = \phi$ iff A is a clopen set.**

Theorem 2.11 [8] **In any space:**

- **i.** A **is semi-open set iff** $A = A^{\circ s}$
- **ii.** A **is semi-closed set iff** $A = \overline{A}^s$

Theorem 2.12 If $A \subset X$, $B \subset Y$ and $f : X \longrightarrow Y$ be a semi-homeomorphism, then:

- **i.** $f(A^{bs}) = [f(A)]^{bs}$
- **ii.** $f^{-1}(A^{bs}) = [f^{-1}(A)]^{bs}$

proof: i. Since f is a semi-homeomorphism, then $f(\overline{A}^s) = \overline{[f(A)]}^s$ and $f(A^{\circ s}) = [f(A)]^{\circ s}$, for all $A \subset X$, [Theorem 2.9 (iii, iv)].
$A^{bs} = \overline{A}^s - A^{\circ s}$ implies $f(A^{bs}) = f(\overline{A}^s - A^{\circ s}) = f(\overline{A}^s) - f(A^{\circ s}) = \overline{[f(A)]}^s - [f(A)]^{\circ s} = [f(A)]^{bs}$.

ii. Since f is a semi-homeomorphism, then $f^{-1}(\overline{B}^s) = \overline{[f^{-1}(B)]}^s$ and $f^{-1}(B^{\circ s}) = [f^{-1}(B)]^{\circ s}$, for all $B \subset Y$, [Theorem 2.9 (i, ii)].
$B^{bs} = \overline{B}^s - B^{\circ s}$ implies $f^{-1}(B^{bs}) = f^{-1}(\overline{B}^s - B^{\circ s}) = f^{-1}(\overline{B}^s) - f^{-1}(B^{\circ s}) = \overline{[f^{-1}(B)]}^s - [f^{-1}(B)]^{\circ s} = [f^{-1}(B)]^{bs}$.

Theorem 2.13 In a space X, A is a semi-clopen (semi-open and semi-closed) set iff $A^{bs} = \phi$.

proof: $\Rightarrow$) Since A is semi-clopen, then $\overline{A}^s = A$ and $A^{\circ s} = A$, [Theorem 2.11 (i, ii)], so $A^{bs} = \overline{A}^s - A^{\circ s} = A - A = \phi$
$\Leftarrow$) Since $A^{\circ s} = A - A^{bs}$, [Remark 2.7], and $A^{bs} = \phi$, then $A^{\circ s} = A$, so A is a semi-open set, [Theorem 2.11 (i)].
$A^{bs} = \overline{A}^s - A^{\circ s} = \phi$ implies $\overline{A}^s \subseteq A^{\circ s}$. Since $A^{\circ s} \subseteq A \subseteq \overline{A}^s$, then $A^{\circ s} = A = \overline{A}^s$ and A is semi-closed and semi-open, [Theorem 2.11 (ii)].

Remark 2.14 $A^{bs} \subset A^b$, **in any space.**

Remark 2.15 [5] **In any space** X, $A \cap A^b = \phi$ **iff** A **is an open set.**

Theorem 2.16 [5] $(A \cup B)^b \subset A^b \cup B^b$ **and** $(A \cap B)^b \subset A^b \cup B^b$, **in any space.**

Theorem 2.17 [4] **If** Y **a subspace of** X **then** $A^{b_Y} \subset Y \cap A^b$, **for any subset** A **of** Y.

3 Main Results

Definition 3.1 A space (X, τ) **is called** ZT_1 **space if and only if for any distinct points** x, y **in** X, **there exists** $U, V \in \tau$ **such that** $x \in U, y \notin U, x \notin V, y \in V$, **and** $U^b \cap V^b = \phi$.

Example 3.2 The usual space $(\mathbb{R}, \tau_u)$ **is** ZT_1 **space:**
Let $x, y \in \mathbb{R}$ **with** $x < y$ **and let** $d = d(x, y) = |x - y|$. **put** $U = (x-1, x+\frac{d}{4}), V = (y - \frac{d}{2}, y+1)$.
Note that $U, V \in \tau_u$, $x \in U$, $y \notin U$ **and** $y \in V$, $x \notin V$.
Now, $U^\circ = U$, $U^c = \mathbb{R} \setminus (x-1, x+\frac{d}{4})$ **then** $U^e = \mathbb{R} \setminus [x-1, x+\frac{d}{4}]$ **so** $U^b = \{x-1, x+\frac{d}{4}\}$ **similarity,** $V^b = \{y - \frac{d}{2}, y+1\}$, **so** $U^b \cap V^b = \phi$. **Then** $(\mathbb{R}, \tau_u)$ **is** ZT_1 **space.**

Example 3.3 The discrete space of more than one point, (X, τ_d), **is** ZT_1 **space:**
For any $a, b \in X, a \neq b$, $U = \{a\}$, $V = \{b\}$ **are open sets,** $a \in U, b \notin U$, $b \in V$ **and** $a \notin V$. **Since** $U^b = \phi$ **and** $V^b = \phi$, **then** $U^b \cap V^b = \phi$. **Hence** (X, τ_d) **is** ZT_1 **space.**

Example 3.4 Let X **be infinite set. The Co-finite space** (X, τ_c) **is** ZT_1 **space :**
$\forall x, y \in X$, $x \neq y$, $U = X \setminus \{y\}$ **and** $V = X \setminus \{x\}$ **are open sets such that** $x \in U, y \notin U$ **and** $y \in V, x \notin V$. **Now**
$U^\circ = X \setminus \{y\}$, $U^c = \{y\}$, **so** $U^e = \phi$. **Then** $U^b = \{y\}$ **similarity,** $V^b = \{x\}$ **then** $U^b \cap V^b = \phi$.
Hence (X, τ_c) **is** ZT_1 **space.**

Example 3.5 The indiscrete space of more than one point, (X, τ_{ind}) **is not** ZT_1.

Example 3.6 Let $X = \{a, b, c, d\}, \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. (X, τ) **is not** ZT_1 **space, for** $a, c \in X, a \neq c$ **cannot find open set contains** c **but not** a.

Theorem 3.7 The property of being ZT_1 **is a topological property.**

Proof: Suppose that $X \cong Y$, **where** X **is** ZT_1 **space. Then there exists a homeomorphism** $f : X \to Y$. **For any** $y_1, y_2 \in Y, y_1 \neq y_2$ **we have** $f^{-1}(y_1), f^{-1}(y_1) \in X$ **and** $f^{-1}(y_1) \neq f^{-1}(y_2)$ **for** f **is one to one. Then there exist** $U, V \in \tau$, $f^{-1}(y_1) \in U, f^{-1}(y_2) \notin U$,$f^{-1}(y_2) \in V, f^{-1}(y_1) \notin V$ **and** $U^b \cap V^b = \phi$. **Now,**
$y_1 \in f(U)$, $y_2 \notin f(U)$ **and** $y_2 \in f(V)$, $y_1 \notin f(V)$. **Since** f **is open function, then** $f(U)$ **and** $f(V)$ **are open in** Y. **Now,** $(f(U))^b \cap (f(V))^b = f(U^b) \cap f(V^b)$, **[Theorem 2.4],** $= f(U^b \cap V^b) = f(\phi) = \phi$. **Hence** (Y, τ^*) **is** ZT_1 **space.** □

Theorem 3.8 If X **and** Y **are** ZT_1 **spaces, then** $X \times Y$ **is also** ZT_1 **space.**

Proof: Let (X, τ) **and** (Y, τ^*) **be** ZT_1 **spaces. To prove** $(X \times Y, \tau_{pro})$ **is also** ZT_1 **space, let** (x_1, y_1), $(x_2, y_2) \in X \times Y$, $(x_1, y_1) \neq (x_2, y_2)$. **Suppose that** $x_1 \neq x_2$. **Since** X **is a** ZT_1 **space,** $x_1 \neq x_2$, **then there exist** $U_1, U_2 \in \tau$ **such that** $x_1 \in U_1, x_2 \notin U_1, x_2 \in U_2, x_1 \notin U_2$ **and** ${U_1}^b \cap {U_2}^b = \phi$.
$U_1 \times Y$ **and** $U_2 \times Y$ **are open sets in** $X \times Y$

and $(x_1, y_1) \in U_1 \times Y, (x_2, y_2) \notin U_1 \times Y$, $(x_2, y_2) \in U_2 \times Y, (x_1, y_1) \notin U_2 \times Y$.

Now, $(U_1 \times Y)^b \cap (U_2 \times Y)^b = \phi$
$(U_1 \times Y)^b \cap (U_2 \times Y)^b = ((U_1{}^b \times \overline{Y}) \cup (\overline{U_1} \times Y^b)) \cap ((U_2{}^b \times \overline{Y}) \cup (\overline{U_2} \times Y^b))$, [Theorem 2.3]
$= ((U_1{}^b \times \overline{Y}) \cup (\overline{U_1} \times \phi)) \cap ((U_2{}^b \times \overline{Y}) \cup (\overline{U_2} \times \phi))$, [Remark 2.10]
$= ((U_1{}^b \times Y) \cup \phi) \cap ((U_2{}^b \times Y) \cup \phi)$
$= (U_1{}^b \times Y) \cap (U_2{}^b \times Y) = (U_1{}^b \cap (U_2{}^b) \times (Y \cap Y) = \phi \times Y = \phi$
Hence $X \times Y$ is a ZT_1 space.

$\square$

Theorem 3.9 An open subspace of a ZT_1 space is a ZT_1 space.

Proof: Let (X, τ) be a ZT_1 space and (Y, τ_y) be an open subspace of (X, τ). For any $x, y \in Y$, $x \neq y$ we have $x, y \in X$. Since (X, τ) is ZT_1 space then, there exists open sets U and V such that $x \in U, y \notin U$, $y \in V$, $x \notin V$, and $U^b \cap V^b = \phi$.
Note that
$x \in (U \cap Y), y \notin (U \cap Y) \in \tau_y$, $y \in (V \cap Y)$, $x \notin (V \cap Y) \in \tau_y$. Now,
$(U \cap Y)^{b_y} \subset Y \cap (U \cap Y)^b$, [Theorem 2.17]
$\subset Y \cap (U^b \cup Y^b)$, [Theorem 2.16]
$= (Y \cap U^b) \cup (Y \cap Y^b)$, [Remark 2.15]
$= (Y \cap U^b) \cup \phi$
$= (Y \cap U^b)$
Similarity, $(V \cap Y)^{b_y} \subset (Y \cap V^b)$
So, $(U \cap Y)^{b_y} \cap (V \cap Y)^{b_y} \subset (Y \cap U^b) \cap (Y \cap V^b)$
$= (U^b \cap V^b) \cap Y = \phi \cap Y = \phi$.
Hence, (Y, τ_y) is a ZT_1 space.

$\square$

Remark 3.10 The continuous image of ZT_1 needs not be a ZT_1 space :
$f : (\mathbb{Z}, \tau_d) \to (\mathbb{R}, \tau_{ind})$, $f(x) = x$ is continuous function and, $(\mathbb{Z}, \tau_d)$ is ZT_1 space but $f(\mathbb{Z}) = \mathbb{Z}$ with the relative indiscrete topology is not ZT_1 space.

Definition 3.11 A space (X, τ) be defined a semi-ZT_1 space if and only if for any distinct points x, y in X, there exist semi-open sets U and V, $x \in U, y \notin U, x \notin V, y \in V$, and $U^{bs} \cap V^{bs} = \phi$.

Example 3.12 The usual space $(\mathbb{R}, \tau_u)$ is semi-ZT_1 space:
Let $x, y \in \mathbb{R}$ with $x < y$ and let $d = d(x, y) = |x - y|$. Put $U = (x-1, x+\frac{d}{4}), V = (y - \frac{d}{2}, y+1)$ are semi-open sets such that $x \in U$, $y \notin U$, $y \in V$, and $x \notin V$.
Since U is semi-clopen, then $U^{bs} = \phi$, [Theorem 2.13].
similarity, $V^{bs} = \phi$, so $U^{bs} \cap V^{bs} = \phi$. Then $(\mathbb{R}, \tau_u)$ is a semi-ZT_1 space.

Example 3.13 The discrete space of more than one point, (X, τ_d), is a semi-ZT_1 space.

Example 3.14 Let X be infinite set. The Co-finite space (X, τ_c), is a semi-ZT_1 space.

Example 3.15 The indiscrete space of more than one point, (X, τ_{ind}), is not a semi-ZT_1 space.

Example 3.16 Let $X = \{a, b, c, d\}, \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. (X, τ) is a semi-ZT_1 space.

Example 3.17 Let $X = \{a, b\}, \tau = \{X, \phi, \{a\}\}$
$a, b \in X, a \neq b$. $a \in \{a\}$ but can not find semi-open set containing b but not a so (X, τ) is not semi-ZT_1 space.

Theorem 3.18 The property of being semi-ZT_1 space is a semi-topological property.

Proof: Suppose that X is semi-homomorphic to Y, where X is a semi-ZT_1 space. Then there exists a semi-homeomorphism $f : X \to Y$. For any $y_1, y_2 \in Y, y_1 \neq y_2$ we have $f^{-1}(y_1), f^{-1}(y_1) \in X$ and $f^{-1}(y_1) \neq f^{-1}(y_2)$ for f is one to one. Since X is semi-ZT_1 space then there exist U, V semi-open sets such that $f^{-1}(y_1) \in U$, $f^{-1}(y_2) \notin U$, $f^{-1}(y_2) \in V$, $f^{-1}(y_1) \notin V$ and $U^{bs} \cap V^{bs} = \phi$. Now, $y_1 \in f(U)$, $y_2 \notin f(U)$ and $y_2 \in f(V)$, $y_1 \notin f(V)$. Since f pre-semi-open then $f(U)$ and $f(V)$ are semi-open in Y. Now, $(f(U))^{bs} \cap (f(V))^{bs} = f(U^{bs}) \cap f(V^{bs})$, [Theorem 2.12 (i)], $= f(U^{bs} \cap V^{bs}) = f(\phi) = \phi$. Hence (Y, τ^*) is semi-ZT_1 space.

□

Theorem 3.19 The product space of two semi-ZT_1 spaces is semi-ZT_1 space.

Proof: Let (X, τ) and (Y, τ^*) be semi-ZT_1 spaces. To prove $(X \times Y, \tau_{pro})$ is semi-ZT_1 space, let (x_1, y_1), $(x_2, y_2) \in X \times Y$, $(x_1, y_1) \neq (x_2, y_2)$. Suppose that $x_1 \neq x_2$. Since X is a semi-ZT_1 space, $x_1 \neq x_2$, then there exist semi-open sets U_1, U_2 such that
$x_1 \in U_1, x_2 \notin U_1, x_2 \in U_2, x_1 \notin U_2$ and ${U_1}^{bs} \cap {U_2}^{bs} = \phi$.
$U_1 \times Y, U_2 \times Y$ are semi-open sets in $X \times Y$, [Theorem 2.8], such that $(x_1, y_1) \in U_1 \times Y$, $(x_2, y_2) \notin U_1 \times Y$, $(x_2, y_2) \in U_2 \times Y$ and $(x_1, y_1) \notin U_2 \times Y$
Now,
$(U_1 \times Y)^{bs} = (\overline{U_1 \times Y}^{s} \cap (\overline{(U_1 \times Y)^{c}}^{s})$, [Remark 2.6]
$= (\overline{U_1 \times Y}^{s} \cap (\overline{(U_1{}^{c} \times Y) \cup (X \times Y^{c})}^{s})$
$= (\overline{U_1 \times Y}^{s} \cap (\overline{(U_1{}^{c} \times Y) \cup (X \times \phi)}^{s})$
$= \overline{(U_1 \times Y)}^{s} \cap \overline{(U_1{}^{c} \times Y)}^{s} \subset (\overline{U_1}^{s} \times \overline{Y}^{s}) \cap (\overline{U_1{}^{c}}^{s} \times \overline{Y}^{s})$, [Theorem 2.5]
$= (\overline{U_1}^{s} \times Y) \cap (\overline{U_1{}^{c}}^{s} \times Y) = (\overline{U_1}^{s} \cap \overline{U_1{}^{c}}^{s}) \times (Y \cap Y) = (\overline{U_1}^{s} \cap \overline{U_1{}^{c}}^{s}) \times Y = {U_1}^{bs} \times Y$.
Similarity, $(U_2 \times Y)^{bs} \subset {U_2}^{bs} \times Y$.

So $(U_1 \times Y)^{bs} \cap (U_2 \times Y)^{bs} \subset ({U_1}^{bs} \times Y) \cap ({U_2}^{bs} \times Y) = ({U_1}^{bs} \cap {U_2}^{bs}) \times Y = \phi \times Y = \phi$. Hence the product space of two semi-ZT_1 spaces is a semi-ZT_1 space.

□

Remark 3.20 The semi continuous image of semi-ZT_1 needs not be a semi-ZT_1 space. $f : (\mathbb{Z}, \tau_d) \to (\mathbb{R}, \tau_{ind})$, $f(x) = x$ is semi continuous function and, $(\mathbb{Z}, \tau_d)$ is semi-ZT_1 space but $f(\mathbb{Z}) = \mathbb{Z}$ with the relative indiscrete topology is not a semi-ZT_1 space.

Theorem 3.21 The property of being semi-ZT_1 is not hereditary property
For Example 3.16 is semi-ZT_1 space but $Y = \{c, d\}$ as a subspace of $(X, \mathtt{t})$ with the indiscrete topology is not semi-ZT_1 space.

Theorem 3.22 Let (X, τ) be a ZT_1 space then it is a semi-ZT_1 space.

Proof: Let x and y are distinct points in X, where X is a ZT_1 space, then there are $U, V \in \tau$ such that $x \in U, y \notin U, x \notin V, y \in V$, and $U^b \cap V^b = \phi$. U and V are semi-open sets for every open set is semi-open set and $U^{bs} \subset U^b$, $V^{bs} \subset V^b$, [Remark 2.14], and $U^b \cap V^b = \phi$, then $U^{bs} \cap V^{bs} = \phi$. Hence, (X, τ) is a semi-ZT_1 space.

Remark 3.23 The converse of Theorem 3.22 is not true in general, for Example 3.16 is semi-ZT_1 space but not ZT_1 space.

Remark 3.21 Let (X, τ) be a space and if X is a ZT_1 space then it is a T_1 space and if X is a semi-ZT_1 space then it is a semi-T_1 space.

4 Conclusion

After we have been introduced a ZT_1 space and a semi-ZT_1 space, we have the following diagram.

$$\begin{array}{ccc} \mathbf{ZT_1}\text{ space} & \Rightarrow & \text{semi-}\mathbf{ZT_1}\text{ space} \\ \Downarrow & & \Downarrow \\ \mathbf{T_1}\text{ space} & \Rightarrow & \text{semi-}\mathbf{T_1}\text{ space} \end{array}$$

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