On some differential subordination theorems of multivalent functions involving integral operator

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DOI : http://dx.doi.org/10.31642/JoKMC/2018/090102
Received Nov. 21, 2021. Accepted for publication Feb. 21, 2022

Abstract— *This paper is concerned with the differential subordination properties of the multivalent functions involving by generalized integral operator. We investigate some applications of a differential subordination results for a normalized multivalent analytic function* $\left(\frac{I_p^{m+2}f(z)}{z^p}\right)^{\sigma}, \sigma > 0$ *and* I_p^m *is a generalized integral operator on the class of all analytic* $p-valent$ *functions. Some of our results generalized previously known results.*

Keywords — Analytic functions; p–valent; integral operator; differential subordination.

I. INTRODUCTION

Let $N(p)$ be the class of all analytic p-valent function in the open unit disk $U = \{z: z \in C, |z| < 1\}$ and have the form:

$$f(z) = z^p + \sum_{t=p+1}^{\infty} a_t z^t, \quad (a_t \geq 0, p \in N = \{1,2,3, \ldots .\}). \quad (1.1)$$

Let f, g be analytic functions in U. If there exists a function $I(z)$ analytic in U (Schwarz function), such that $I(0) = 0$, $|I(z)| < 1$ and satisfies $f(z) = g\big(I(z)\big)$, then f is said to be subordinate to g in U *and* written as $f \prec g$ or $f(z) \prec g(z)$, $(z \in U)$.

In particular, if w is univalent function in U, then $f \prec w$ if $f(0) = w(0)$, and $f(U) \subset w(U)$ ([4,13]).

We will denote the set of all analytic injective functions h on $\underline{U}$ / $E(h)$ by Q, where

$$E(h) = \{\xi \in \partial U \;\; : h(z) = \infty\}.$$

and $h'(\xi) \neq 0$ for $\xi \in \partial U \setminus E(h)$ (see [14]).

In [7] the authors consider the following problem

$$\Phi\,(\eta(z), z\eta'(z), z^2\eta''(z); z) \prec k(z) \quad (1.2)$$

where $\Phi: C^3 \times U \to C$, and k a univalent function in U. They set the necessary conditions on admissible function Φ such that the problem (1.2) implies $\eta(z) \prec \omega(z)$, where $\omega \in Q$ and for all functions $\eta(z) \in H[a,n]$ that satisfy the problem (1.2). Additionally, they derived the required conditions such that ω is the smallest function, in this case the function ω is said to be the best dominant $(b.d)$ of the subordination (1.2).

In [8,9] Miller and Mocanu investigated the dual problem

$$k(z) \prec \Delta(\eta(z), z\eta'(z), z^2\eta''(z); z) \quad (1.3)$$

where $\Delta : C^3 \times U \to C$, and $k \in H$ with $\omega \in H[a,n]$, and they set the necessary conditions on Δ such that the problem (1.3) implies $\omega(z) \prec \eta(z)$, for all functions $\eta \in Q$ that satisfy the superordination (1.3) They also derived the required conditions such that the function ω is the largest function, in this case the function ω is said to be the best subordinate $(b.s)$ of the superordination (1.3).

II. MATHEMATICAL TOOLS:

In this section, we will mention and lemmas necessary to present our main results.

Lemma (2.1) [4]: Let ω be univalent function in U, $\xi \in C^* \setminus \{0\}$ and let

$$Re\left\{1 + \frac{z\omega''(z)}{\omega'(z)}\right\} > max\left\{0, -Re\left(\frac{1}{\xi}\right)\right\}. \quad (2.1)$$

If $\eta(z)$ is analytic function in U, with $\eta(0) = \omega(0)$ and

$$\eta(z) + \zeta z\eta'(z) \prec \omega(z) + \zeta z\omega'(z), \quad (2.2)$$

then $\eta(z) \prec \omega(z)$, and $\omega(z)$ is the b.d.

Lemma (2.2) [5]: Let $\omega(z)$ be a univalent in U, and ϑ, χ are analytic functions in a domain D containing $\omega(U)$ *with* $\chi(w) \neq 0$ when $w \in \omega(U)$. Assume $Q(z) = z\acute{\omega}(z)\chi(\omega(z))$ and $h(z) = \vartheta(\omega(z)) + Q(z)$. Assume that

1. Q is $starlike$ univalent in U.
2. $Re\left\{\frac{z\acute{h}(z)}{Q(z)}\right\} > 0$ for $z \in U$.

If η is analytic with $\eta(0) = \omega(0), \eta(U) \subseteq D$ and

$$\vartheta(\eta(z)) + z\acute{\eta}(z)\chi(\eta(z)) \prec \vartheta(\omega(z)) + z\acute{\omega}(z)\chi(\omega(z)), \quad (2.3)$$

then $\eta \prec \omega$, and $\omega(z)$ is the b.d.

Lemma (2.3) [2]: Let $\omega(z)$ be convex in $U, \omega(0) = a$ and $\xi \in C, Re(\xi) > 0$.

If $\eta \in H[a,1]$ and $\eta(z) + \xi z\acute{\eta}(z)$ is univalent in U then

$$\omega(z) + \xi z\acute{\omega}(z) \prec \eta(z) + \xi z\acute{\eta}(z), \quad (2.4)$$

Implies $\omega(z) \prec \eta(z)$ and $\omega(z)$ is the b. s.

Lemma (2.4) [3]: Let $\omega(z)$ be convex is univalent in the unit disk U and ϑ, χ are analytic functions in a domain D containing $\omega(U)$. Assume that

1. $Re\left\{\frac{\vartheta(\omega(z))}{\chi(\omega(z))}\right\} > 0, for \ z \in U.$
2. $z\acute{\omega}(z)\chi(\omega(z))$ is starlike univalent in U.

If $\eta(z) \in H[\omega(0),1] \cap Q$, with $\eta(U) \subseteq D$, and $\vartheta(\eta(z)) + z\acute{\eta}(z)\varphi(\eta(z))$ is univalent in U, and

$$\vartheta(\omega(z)) + z\acute{\omega}(z)\chi(\omega(z)) \prec \vartheta(\eta(z)) + z\acute{\eta}(z)\chi(\eta(z)), \quad (2.5)$$

then $\omega(z) \prec \eta(z)$, and $\omega(z)$ is the b. s.

Now we define the following integral operator

$$I_p^m(\lambda,\alpha,\beta,\mu): N(p) \to N(p),$$

$m \in N_0, \lambda,\alpha,\beta,\mu \in R, \lambda+\beta \neq 0$ as follows:

$$I_p^0(\lambda,\alpha,\beta,\mu)f(z) = f(z)$$

$$I_p^1(\lambda,\alpha,\beta,\mu)f(z) = I_p^1(\lambda,\alpha,\beta,\mu)f(z) = \left(\frac{p+\alpha\mu}{\lambda+\beta}\right) z^{p-\left(\frac{p+\alpha\mu}{\lambda+\beta}\right)} \int_0^z s^{\left(\frac{p+\alpha\mu}{\lambda+\beta}\right)-(p+1)} f(s)ds$$

$$I_p^2(\lambda,\alpha,\beta,\mu)f(z) = \left(\frac{p+\alpha\mu}{\lambda+\beta}\right) z^{p-\left(\frac{p+\alpha\mu}{\lambda+\beta}\right)} \int_0^z s^{\left(\frac{p+\alpha\mu}{\lambda+\beta}\right)-(p+1)} I_p^1(\lambda,\alpha,\beta,\mu)f(s)ds$$

and, in general

$$I_p^m(\lambda,\alpha,\beta,\mu)f(z) = \left(\frac{p+\alpha\mu}{\lambda+\beta}\right) z^{p-\left(\frac{p+\alpha\mu}{\lambda+\beta}\right)} \int_0^z s^{\left(\frac{p+\alpha\mu}{\lambda+\beta}\right)-(p+1)} I_p^{m-1}(\lambda,\alpha,\beta,\mu)(s)ds$$

$$(f(z) \in N(p); m \in N_0; z \in U) \quad (2.6)$$

For a function $f \in N(p)$, we can see that

$$I_p^m(\lambda,\alpha,\beta,\mu)f(z) = z^p + \sum_{t=p+1}^{\infty} \left(\frac{p+\alpha\mu}{p+\alpha\mu+(\lambda+\beta)(k-p)}\right)^m a_t z^t,$$

$(m \in N_0)$. (2.7)

From (2.7), its clearly that

$$(\lambda+\beta)z\left(I_p^{m+2}f(z)\right)' = (\alpha\mu+p)\left(I_p^{m+1}f(z)\right) - \left(\alpha\mu + p(1-(\lambda+\beta))\right)\left(I_p^{m+2}f(z)\right). \quad (2.8)$$

Special cases:

1- $I^m(\lambda,0,0,0)f(z) = I_\lambda^{-m}f(z)$ ([12])
2- $I_1^\alpha(1,1,0,1)f(z) = I^\alpha f(z)$ ([6]).
3- $I_p^m(1,1,0,1)f(z) = I_p^m f(z)$ ([14]).
4- $I_1^m(1,1,0,1)f(z) = D^m f(z)$ ([11]).
5- $I_1^m(1,1,0,1)f(z) = I^m f(z)$ ([5]).
6- $I_1^m(1,0,0,0)f(z) = I^m f(z)$ ([13]).

Also, we note that:

1. $I_p^m(1,0,0,0)f(z) = J_p^m f(z)$

$$= \left\{f(z): J_p^m f(z) = z^p + \sum_{k=n+p}^{\infty} \left(\frac{p}{k}\right)^m a_k z^k, m \in N_0, z \in U\right\}.$$

2. $I_p^m(1,l,0,1)f(z) = J_p^m(l)f(z)$

$$= \left\{f(z): J_p^m(l)f(z) = z^p + \sum_{k=n+p}^{\infty} \left(\frac{p+l}{k+l}\right)^m a_k z^k, m \in N_0, l > 0, z \in U\right\}.$$

3. $I_p^m(\lambda,0,0,0)f(z) = J_{p,\lambda}^m f(z)$

$$= \left\{f(z): J_{p,\lambda}^m f(z) = z^p + \sum_{k=n+p}^{\infty} \left(\frac{p}{k+\lambda(k-p)}\right)^m a_k z^k, m \in N_0, \lambda \geq 0, z \in U\right\}.$$

4. $I_p^m(\lambda,\alpha\delta,0,1)f(z) = I_p^m(\lambda,\alpha,\delta)f(z)$ [1]

$$\left\{f(z): J_{p,\lambda}^m f(z) = z^p + \sum_{k=n+p}^{\infty} \left(\frac{p+\alpha\delta}{p+\alpha\delta+\lambda(k-p)}\right)^m a_k z^k, m \in N_0, \lambda \geq 0, z \in U\right\}$$

In this work, we will determine some properties on the admissible functions defined with operator $I_p^m(\lambda,\alpha,\beta,\mu)$. Our results are a generalization of some results of [1].

III. MAIN RESULTS:

Throughout this study, we will assume that $\geq 0, \lambda > 0, \mu > 1; m \in N_0 = N \cup \{0\}, p \in N; z \in U$, unless otherwise stated.

Theorem (3.1): Let $\omega(z)$ be univalent in U with $\omega(0) = 0, \gamma > 0$ and suppose that

$$Re\left\{1 + \frac{z\acute{\omega}(z)}{\acute{\omega}(z)}\right\} > max\left\{0, -Re\left(\frac{\sigma(\alpha\mu + p)}{\beta + \lambda}\right)\right\}, \quad (3.1)$$

If $f \in N(p)$ satisfies the subordination

$$\left(\frac{I_p^{m+2} f(z)}{z^p}\right)^{\sigma} + \left(\frac{I_p^{m+2} f(z)}{z^p}\right)^{\sigma}\left(\frac{I_p^{m+1} f(z)}{I_p^{m+2} f(z)} - 1\right) \prec \omega(z) + \frac{\beta + \lambda}{\sigma(\alpha\mu + p)} z\acute{\omega}(z), \quad (3.2)$$

then

$$\left(\frac{I_p^{m+2} f(z)}{z^p}\right)^{\sigma} \prec \omega(z)$$

and $\omega(z)$ is the b. d.

Proof: Let us consider the analytic $p - valent$ function

$$\eta(z) = \left(\frac{I_p^{m+2} f(z)}{z^p}\right)^{\sigma}, \sigma > 0 , \quad z \in U \quad (3.3)$$

By differentiating the function (3.3) with respect to variable, we get that

$$\frac{z\acute{\eta}(z)}{\eta(z)} = \frac{\sigma(\alpha + p)}{\lambda + \beta}\left(\frac{I_p^{m+1} f(z)}{I_p^{m+2} f(z)} - 1\right), \quad (3.4)$$

that is

$$\frac{\lambda + \beta}{\sigma(\alpha\mu + p)} z\acute{\eta}(z) = \left(\frac{I_p^{m+2} f(z)}{z^p}\right)^{\sigma}\left(\frac{I_p^{m+1} f(z)}{I_p^{m+2} f(z)} - 1\right)$$

Therefore,

$$\eta(z) + \frac{\lambda + \beta}{\sigma(\delta\alpha\mu + p)} z\acute{\eta}(z) \prec \omega(z) + \frac{\lambda + \beta}{\sigma(\delta\alpha\mu + p)} z\acute{\omega}(z). \quad (3.5)$$

That is equivalent to the subordination (3.2)

Using Lemma (2.1), with $\xi = \frac{\lambda+\beta}{\sigma(\alpha\mu+p)}$. This completed the proof of Theorem (3.1).

If we take the convex function $(z) = \frac{1+Cz}{1+Dz}$, in the theorem (3.1), we get the following result.

Corollary (3.2): Let $C, D \in C, C \neq D, |C| < 1, \ \sigma > 0 \ and \ Re(\xi) > 0$. If $f(z) \in N(p)$ satisfies the following subordination

$$\left(\frac{I_p^{m+2} f(z)}{z^p}\right)^{\sigma} + \left(\frac{I_p^{m+2} f(z)}{z^p}\right)^{\sigma}\left(\frac{I_p^{m+1} f(z)}{I_p^{m+2} f(z)} - 1\right) \prec \frac{1 + Cz}{1 + Dz} + \frac{\lambda + \beta}{\sigma(\alpha\mu + p)} \frac{(C - D)z}{(1 + Dz)^2}$$

Then

$$\left(\frac{I_p^{m+2} f(z)}{z^p}\right)^{\sigma} \prec \frac{Cz + 1}{Dz + 1}$$

and $\frac{Cz+1}{Dz+1}$ is the b.d.

Setting $m = 0 \ in \ theorem$ (3.1), we have the next result:

Corollary (3.3): Let $\omega(z)$ be univalent function in the unit disk U, with $\omega(0) = 1, \beta \in C^*, \sigma > 0$, and assume that (3.1) $holds$. If $f(z) \in N(p)$ satisfies the subordination

$$(1 - \beta)\left(\frac{I_p^2 f(z)}{z^p}\right)^{\sigma} + \beta\left(\frac{I_p^2 f(z)}{z^p}\right)^{\sigma}\left(\frac{I_p^1 f(z)}{I_p^2 f(z)} - 1\right) \prec \omega(z) + \frac{\lambda + \beta}{\sigma(\alpha\mu + p)} z\acute{\omega}(z),$$

then

$$\left(\frac{I_p^2 f(z)}{z^p}\right)^{\sigma} \prec \omega(z).$$

and $\omega(z)$ is the b. d.

Setting $\lambda = \mu = \alpha = 1$in the theorem (3.1), we get the following result .

Corollary (3.4): Let $\omega(z)$ be univalent function in the unit disk U, with $\omega(0) = 1, \beta \in C^*, \sigma > 0$, and assume that (3.1) $holds$. If $f(z) \in N(p)$ satisfies the following subordination

$$\left(\frac{I_p^{m+2} f(z)}{z^p}\right)^{\sigma} + \left(\frac{I_p^{m+2} f(z)}{z^p}\right)^{\sigma}\left(\frac{I_p^{m+1} f(z)}{I_p^2 f(z)} - 1\right) \prec \omega(z) + \frac{1 + \beta}{\sigma(\mu + p)} z\acute{\omega}(z),$$

then

$$\left(\frac{I_p^{m+2} f(z)}{z^p}\right)^{\sigma} \prec \omega(z).$$

and $\omega(z)$ is the b. d.

Theorem (3.5): Let $\omega(z)$ be univalent function in the unit disk U, with $\omega(0) = 1$ and $\omega(z) \neq 0$ for all $z \in U$, let $\sigma, \lambda \in C^*, \ f \in N(p)$ and assume that f and q satisfy the following conditions:

$$\frac{I_p^{m+2} f(z)}{z^p} \neq 0, \quad (3.6)$$

and $Re\left\{1 + \frac{z\acute{\omega}(z)}{\acute{\omega}(z)} - \frac{z\acute{\omega}(z)}{\omega(z)}\right\} > 0, \quad (z \in U) \quad (3.7)$

If

$$\frac{I_p^{m+1} f(z)}{I_p^{m+2} f(z)} \prec 1 + \frac{(\lambda + \beta)z\acute{\omega}(z)}{\sigma(\alpha\mu + p)\omega(z)}, \quad (3.8)$$

then

$$\left(\frac{I_p^{m+2} f(z)}{z^p}\right)^{\sigma} \prec \omega(z)$$

and $\omega(z)$ is the b.d of (3.6).

proof: Let

$$\eta(z) = \left(\frac{I_p^{m+2} f(z)}{z^p}\right)^{\sigma}, \quad z \in U \quad (3.9)$$

Then the function $\eta(z)$ is analytic in U. By differentiating (3.9) with respect to the variable z, we get that

$$\frac{z\acute{\eta}(z)}{\eta(z)} = \frac{\sigma(\alpha\mu + p)}{\lambda + \beta}\left(\frac{I_p^{m+1} f(z)}{I_p^{m+2} f(z)} - 1\right), \quad (3.10)$$

To prove our theorem, we shall use Lemma (2.2). in this lemma we consider

$$\vartheta(w) = 1 \text{ and } \chi(w) = \frac{\lambda+\beta}{\sigma(\alpha\mu+p)w}$$
then ϑ is analytic function in C and $\chi(w) \neq 0$ *is analytic* in C^*. Also, if *we* let
$$\mu(z) = z\acute{\omega}(z)\chi(\omega(z)) = \frac{(\lambda+\beta)z\acute{\omega}(z)}{\sigma(\alpha\mu+p)\omega(z)}$$
and
$$h(z) = \vartheta(q(z)) + \mu(z) = 1 + \frac{(\lambda+\beta)z\acute{\omega}(z)}{\sigma(\alpha\mu+p)\omega(z)}$$
from (3.7), we see that $\mu(z)$ is a starlike function in U. We also have
$$Re\left\{\frac{z\acute{h}(z)}{\mu(z)}\right\} = Re\left\{1 + \frac{z\acute{\acute{\omega}}(z)}{\acute{\omega}(z)} - \frac{z\acute{\omega}(z)}{\omega(z)}\right\} > 0, \quad (z \in U)$$
and then, by applying Lemma (2.2) we see that the subordination (3.6) implies
$$\eta(z) \prec \omega(z)$$
and the function $\omega(z)$ is the b. d of (3.8).

Seething $\omega(z) = \frac{Sz+1}{Tz+1}$ $(-1 \le T < S \le 1)$ in Theorem (3.5), it is observably that the assumption (3.5) holds, hence we obtain the following result.

Corollary (3.6): Let $\sigma \in C^*$. Let $f(z) \in N(p)$ and assume that
$$\frac{I_p^{m+2}f(z)}{z^p} \neq 0, \quad (z \in U).$$
If

This paper has been retracted

$$\frac{I_p^{m+1}f(z)}{I_p^{m+2}f(z)} \prec 1 + \frac{(\lambda+\beta)z(S-T)}{\sigma(\alpha\mu+p)(1+Sz)(1+Tz)}$$
then
$$\left(\frac{I_p^{m+2}f(z)}{z^p}\right)^{\sigma} \prec \frac{Sz+1}{Tz+1}$$
and $\omega(z) = \frac{Sz+1}{Tz+1}$ is the b. d.

In particular, set $\omega(z) = \frac{1+z}{1-z}$ in theorem (3.5), it is observably that the assumption (3.5) *holds*. Therefore, we obtain the following result.

Corollary (3.7): Let $\sigma \in C^*$, $f(z) \in N(p)$ *and* assume that
$$\frac{I_p^{m+2}f(z)}{z^p} \neq 0, \quad (z \in U).$$
If
$$\frac{I_p^{m+1}f(z)}{I_p^{m+2}f(z)} \prec 1 + \frac{2(\lambda+\beta)z}{\sigma(\alpha\mu+p)(1-z)(1+z)}$$
then
$$\left(\frac{I_p^{m+2}f(z)}{z^p}\right)^{\sigma} \prec \frac{1+z}{1-z} \quad \text{and } \omega(z) = \frac{1+z}{1-z} \text{ is the b. d.}$$

IV. CONCLUSIONS.

The theory of analytic functions has piqued the attention of several researchers. A review of previous research reveals a wide range of studies on various forms of multivalent analytic functions. The interplay of geometric structures is crucial in complicated analysis. The differential subordination properties of multivalent functions employing the generalized integral operator have been the focus of this work. For a normalized multivalent analytic function, we studied some applications of differential subordination results. We proved under different conditions that the function $\frac{Sz+1}{Tz+1}$ $(-1 \le T < S \le 1)$ is best dominant of the function $\left(\frac{I_p^{m+2}f(z)}{z^p}\right)^{\sigma}$.

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