The Cyclic Decomposition of $\mathrm{cf}(Q_{28} \times C_p)/\overline{\mathrm{R}}(Q_{28} \times C_p)$

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DOI : http://dx.doi.org/10.31642/JoKMC/2018/090105

Received Jan. 27, 2022. Accepted for publication Feb. 28, 2022

Abstract*— *In this paper, we propose the cyclic decomposition of the factor group $\boldsymbol{cf(Q_{28} \times C_p), Z)/\overline{R}Q_{28} \times C_p)}$, and the group $\boldsymbol{cf(Q_{28} \times C_p)}$ is Z-valued class functions of the direct product group $\boldsymbol{(Q_{28} \times C_p)}$ under the operation of addition, and $\boldsymbol{R(Q_{28} \times C_p)}$is the subgroup of the generalized characters of the group $\boldsymbol{cf(Q_{28} \times C_p, Z)}$.Then $\boldsymbol{cf(Q_{28} \times C_p)/(Q_{28} \times C_p)}$ is an abelian factor group denoted by $\boldsymbol{K(Q_{28} \times C_p)}$ when $\boldsymbol{Q_{28}}$ the quaternion group $\boldsymbol{|Q_{28}| = 56}$, and$\boldsymbol{|C_p| = p}$. Also, we find the rational valued characters table of the group $\boldsymbol{(Q_{28} \times C_p)}$ when $\boldsymbol{p}$ is prime numbers is given as following:

	$\boldsymbol{\equiv^* (Q_{28} \times C_p) = \equiv^* (Q_{28}) \otimes \equiv^* (C_p)}$	*(1)*

and find the cyclic decomposition of group $\boldsymbol{(Q_{28} \times C_p)}$ in this paper and prove that

	$\boldsymbol{K(Q_{28} \times C_p) = \oplus_{i=1}^{2} [K(Q_{28})] \oplus_{i=1}^{8} K(C_p)}$	*(2)*

***Keywords*— matrix representation, character tables, Quaternion groups, the cyclic decomposition of group)**

I. INTRODUCTION

Γ -conjugate contains all elements of a finite group that generate equivalent conjugate cyclic subgroups in G, which defines an equivalence relation on G; its classes are called Γ-classes."The number of $\Gamma -$ classes of G is equal to the rank of $\mathrm{cf(G.Z)}$ is $\mathrm{cf(G.Z)} \cap \mathrm{R(G)} = \overline{\mathrm{R}}\mathrm{G}$ which is a normal subgroup of $\mathrm{cf(G.Z)}$, then, $\mathrm{cf(G.Z)}/\overline{\mathrm{R}}\mathrm{G})$ is a finite abelian factor group denoted by K(G)".

"$\forall \gamma \in \overline{R}\boldsymbol{Q_{28} \times C_p}$) then $\gamma = v1\theta 1 + v2\theta 2 + \cdots + vr\theta r$,"where r is the number of $\Gamma -$ classes. v1. v2. ... . vr $\in$ Z and , when χi is an irreducible character of the group G and σ is any element in Galios group $Gal\ (Q(\chi i)/Q)$".

"In 1982, M. S. Kirdar [4] studied the $K(C_n)$. In 1994, H. H. Abass [2] studied the $K(D_n)$ and found $\equiv^*(D_n)$. In 1995, N. R. Mahmood [6] studied the factor group $K(Q_{2m})$ and foun $\equiv^*(Q_{2m}$.

"The aim of this paper to find" $\equiv^*$ $\boldsymbol{(Q_{28} \times C_p)}$ "and to collect the cyclic decomposition of the group" $\mathrm{K}(\boldsymbol{Q_{28} \times C_p})$, when p is prime numbers

The group G which is $|G| \neq \infty$, the conjugacy classes of it are finite and a finite number of deferent k-irreducible representations, the characterr of a representation is constant on a conjugacy class CL a. $(1 \leq a \leq k)$, $\equiv(G)$ is the table of the values of the characters , Let $T_a: G_a \to GL(n.K)$ and $T_b: G_b \to GL(m.K)$ are two matrix representations of the groups G_a and G_b. χ_a and χ_b be two characters of T_a and T_b respectively, then the character of $T_a \otimes T_b$ is $\chi_a \chi_b$.

II. BASIC CONCEPTS

Definition (2-1):[1] The character whose values are in Z, which is θ (g)∈ Z. ∀ g ∈ G
Is called A rational valued character θ of G

Proposition (2-2):[4]

$$\theta_i = \sum_{\sigma \in Gal(Q(\chi_i)/Q)} \sigma(\chi_i)$$

θ_i is the rational valued characters

form basis for $\overline{R}(G)$, where χ_i are the irreducible characters of G and their numbers are equal to the number of all distinct Γ- classes of G.

Proposition (2-3): [4]

The rational valued characters table of the cyclic group C_p of the rank 2 where p is a prime number which is denoted by ($\equiv^*$ (C_p)), is given as follows:

Table 1. The Rational Valued Characters Table Of The Cyclic Group $\mathbf{C_p}$

$\Gamma -$ **classes**	$[I]$	$[r]$
θ_1	1	1
θ_2	$p-1$	-1

where its rank 2 represents the number of all distinct Γ-classes.

Definition(2-4): [6]

The generalized Quaternion Group Q_{2m} of order 4m, the Q_{2m} generator bay x and y satisfies $x^m = y^2, y^1x^m\ y^{-1} = x^{-m}$ which implies $x^m = y^4 = 1$
then∀ d ∈Q_{2m} can be expressed uniquely in the form
$Q_{2m} = \{d = x^h y^k \mid 0 \leq h \leq 2m-1, k = 0,1\}$

Lemma (2.5) [6]

The rational valued characters table of the group Q_{2m} in table(1)

	Γ-classes of C2m			
	x2t	x2t+1	[y]	[xky]
θ1	1 1 … 1 1	1 1 … 1 1	1	1
θ2	1 1 … 1 1	-1 -1 … -1 -1	-1	1
:			0	0
:	≡*(C2m)		0	0
			: :	: :
θl -1			0	0
θl			0	0
θl +1	1 … 1	1 … 1	-1	-1
θl +2	1 … 1	-1 … -1	1	-1

table(1)

Where 0 ≤ t ≤ m-1, l is the number of Γ-classes of C_{2m}, θj such that 1 ≤ j ≤ l+2

III. THE MAIN RESULTS

We study the rational valued characters table of ($\boldsymbol{Q_{28}} \times \boldsymbol{C_p}$) which denoted bay$\equiv^*$ ($Q_{28} \times C_p$) . and find the cyclic decomposition of $K(\boldsymbol{Q_{28}} \times \boldsymbol{C_p})$ the group of $\boldsymbol{Q_{28}} \times \boldsymbol{C_p}$) is the direct product of $\boldsymbol{Q_{28}}$ and C_p
$|(\boldsymbol{Q_{28}} \times \boldsymbol{C_p})| = 56p$

Theorem(3,1)

when $n = p, m = 2q$, where p, and q are prime numbers, then $\equiv^*$ ($Q_{28} \times C_p$) has the following form

$$\equiv^* (Q_{28} \times C_p) = \equiv^* (Q_{28}) \otimes \equiv^* (C_p)$$

Proof

For each element $g_{dh} \in (Q_{4q} \times C_p)$, we have $g_{dh} = g_d g_h$ such that $g_d \in Q_{4q}, g_h \in C_{p,} = \prec r \succ$ Then $g_d = x^t y^k, 0 \leq t \leq 4q, k = 0,1$, and each irreducible character of $(Q_{28} \times C_p)$ is $\chi_{ij} = \chi_i \chi'_j$ where χ_i is an irreducible character of Q_{28}, χ'_j is an irreducible character of C_p,

$$\theta_{ij}=\theta_{ij(g_{dh})}= [\![\sum_{\sigma g\ Gal(Q(\chi_{ij}(g_{khl}))/Q)} \sigma(\chi_{ij}(g_{dh}))]\!]$$

$$= [\![\sum_{\sigma g\ Gal(Q(\chi_i(g_d))/Q)} \sigma(\chi_i(g_d)) [\sum_{\sigma g\ Gal(Q(\acute{\chi}_j(g_h))/Q)} \sigma(\acute{\chi}_j(g_h))]]\!]$$

1) If j = 1, then for all $g_h \in C_p$ such that $\theta'_j(g_h) = \sum \chi'_j(g_h) = 1$. Then

$$\theta_{ij(g_{dh})}= [\![\sum_{\sigma g\ Gal(Q(\chi_{ij}(g_{kh}))/Q)} \sigma(\chi_{ij}(g_{dh}))]\!]$$

$$= [\![\sum_{\sigma g\ Gal(Q(\chi_i(g_d))/Q)} \sigma(\chi_i(g_d)) [1]]\!]$$

$$= \theta_i(g_d) * \theta'_j(g_h)$$

If j = 2 ,3, …, p, and g_h is the identity of C_p that $\theta'_2(g_h) = \sum_{i=2}^{p} \chi'_2(g_h) = \sum_{k=2}^{p} 1 = \underbrace{1 + 1 + \cdots + 1}_{(p-1)\ \text{times}}) = (p - 1)$ Then

$$\theta_{ij(g_{dh})}= [\![\sum_{\sigma g\ Gal(Q(\chi_{ij}(dh))/Q)} \sigma(\chi_{ij}(g_{dh}))]\!]$$

$$= [\![\sum_{\sigma g\ Gal(Q(\chi_i(g_d))/Q)} \sigma(\chi_i(g_d)) [p - 1]]\!]$$

$$= \theta_i(g_d)(P - 1) = \theta_i(g_d) * \theta'_j(g_h)$$

If j = 2,3, …, p, and g_h is not identity of C_p that $\theta'_2(g_h) = \sum_{i=2}^{p} \chi'_2(g_h) = (\varepsilon + \varepsilon^2 + \varepsilon^2 + \cdots + \varepsilon^{p-1})'_= - 1$

Then

$$\theta_{ij(g_{dh})}= [\![\sum_{\sigma g\ Gal(Q(\chi_{ij}(dh))/Q)} \sigma(\chi_{ij}(g_{dh}))]\!]$$

$$= [\![\sum_{\sigma g\ Gal(Q(\chi_i(g_d))/Q)} \sigma(\chi_i(g_d)) [-1]]\!]$$

$$= \theta_i(g_d)(-1) = \theta_i(g_d) * \theta_i(g_d)$$

Where $\theta_{ij}(g_{dh})$, $\theta_i(g_d)$, and $\theta_i(g_d)$ are the rational valued characters of the groups $(28 \times C_p)$, (28) and (C_p) respectively

Then we get $(\theta_{ij}(g_{dh}) = \theta_i(g_d) * \theta'_j(g_h)$ for all i,j, $g_{dh} \in (Q_{28} \times C_p)$, $g_d \in Q_{28}$,$g_h \in C_p$

$\equiv^* (Q_{28} \times C_p) = \equiv^* (Q_{28}) \otimes \equiv^* (C_p)$ □

To calculate the $\equiv^* (Q_{28} \times C_p)$

$$\theta_{11} = \psi_{11}, \theta_{12} = \sum_{i=2}^{p} \psi_{1i}, \theta_{21} = \psi_{41}, \theta_{22} = \sum_{i=2}^{p} \psi_{4i},$$

$$\theta_{71} = \psi_{21}, \theta_{72} = \sum_{i=2}^{p} \psi_{2i}, \theta_{81} = \psi_{31}, \theta_{82} = \sum_{i=2}^{p} \psi_{3i},$$

$$\theta_{51} = \chi_{51}, \theta_{52} = \sum_{i=2}^{p} \chi_{5i},$$

sine's $(Gal(\chi_{1i})/Q) = \{ \sigma_{1i}, \sigma_{3i}, \dots, \sigma_{2q-1\,i} \}$

$\sigma_{1i}(\chi_{1i}) = \chi_{1i}$, $\sigma_{3i}(\chi_{1i}) = \chi_{3i}$, …, $\sigma_{2q-1\,i}(\chi_{1\,i}) = \chi_{2q-1\,i}$

and i = 1 , 2 , 3 , … , p

$\theta_{6i} = \sigma_{1i}(\chi_{11}) + \sigma_{3i}(\chi_{1i}) + \cdots + \sigma_{2q-1\,i}(\chi_{1\,i})$, $\theta_{61} = \chi_{11} + \chi_{3i} + \cdots + \chi_{2q-1\,1}$, $= \theta_{61} = \chi_{11} + \chi_{3i} + \cdots + \chi_{2q-1\,1}$, $\theta_{62} = \sum_{i=2}^{p} \sigma_{1i}(\chi_{11}) + \sigma_{3i}(\chi_{1i}) + \cdots + \sigma_{2q-1\,i}(\chi_{1\,i})$

$\theta_{4\,i} = \sigma_{2i}(\chi_{1i}) + \sigma_{6i}(\chi_{1i}) + \cdots + \sigma_{2q-2\,i}(\chi_{1\,i})$, $+ \cdots + \chi_{2q-2\,i}$, $=$, , $\theta_{41} = \chi_{2i} + \chi_{6i} + \cdots + \chi_{2q-2\,i}$, $\theta_{4\,2} = \sum_{i=2}^{p} \sigma_{2i}(\chi_{1i}) + \sigma_{6i}(\chi_{1i}) + \cdots + \sigma_{2q-4\,i}(\chi_{1\,i})$

$\theta_{3\,i} = \sigma_{4i}(\chi_{1i}) + \sigma_{8i}(\chi_{1i}) + \cdots + \sigma_{2q-4\,i}(\chi_{1\,i})$, $= \theta_{31} = \chi_{4i} + \chi_{8i} + \cdots + \chi_{2q-4\,i}$, , $\theta_{32} = \sum_{i=2}^{p} \sigma_{4i}(\chi_{1i}) + \sigma_{8i}(\chi_{1i}) + \cdots + \sigma_{2q-4\,i}(\chi_{1\,i})$

We can write $\boldsymbol{Q_{28}} \times \boldsymbol{C_p}$) as follows:

$$\equiv^* Q_{28} \times C_p) = \equiv^* \boldsymbol{Q_{28}}) \otimes \equiv^* (\boldsymbol{C_p})$$

Γ-classes |	1	p-1	6	6 (p-1)	6	6 (p-1)	1	p-1	6	(6 (p-1)	6	6 (p-1)	14	14p-14	14	14p-14
θ_{11}	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
θ_{12}	p-1	-1	p-1	-1	p-1	-1	p-1	-1	p-1	-1	p-1	-1	p-1	-1	p-1	-1
θ_{21}	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	1	1
θ_{22}	p-1	-1	p-1	-1	p-1	-1	P-1	-1	1-p	1	1-p	-1	1-p	1	p-1	-1
θ_{31}	6	6	-1	-1	-1	-1	6	6	-1	-1	6	6	0	0	0	0
θ_{32}	6p-6	-6	1-p	1	1-p	1	6p-6	-6	-6	1	6p-6	1-q	0	0	0	0
θ_{41}	6	6	-1	-1	-1	-1	6	6	1	1	-6	-6	0	0	0	0
θ_{42}	6p-6	-6	1-p	1	1-p	1	6p-6	-6	p-1	-1	6-6p	6	0	0	0	0
θ_{51}	2	2	-2	-2	2	2	-2	-2	0	0	0	0	0	0	0	0
θ_{52}	2p-2	-2	2-2p	2	2p-2	- 2	2-2p	2	0	0	0	0	0	0	0	0
θ_{61}	12	12	2	2	-2	-2	-12	-12	0	0	0	0	0	0	0	0
θ_{62}	12 p-12	-12	12	-2	-12	2	12-12 p	12	0	0	0	0	0	0	0	0
θ_{71}	1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1
θ_{72}	p-1	-1	p-1	-1	p-1	-1	p-1	-1	p-1	-1	p-1	-1	1-p	1	1-p	1
Θ_{81}	1	1	1	1	1	1	1	1	-1	-1	-1	-1	1	1	-1	-1
Θ_{82}	p-1	-1	p-1	-1	p-1	-1	p-1	-1	1-p	1	1-p	1	p-1	-1	1-p	1

Theorem (3,2) If $m = 14$, p is prime numbers and $p \neq 7$, then the cyclic decomposition of $K(Q_{28} \times C_p)$ is:

$$K(Q_{28.} \times C_p) = \oplus_{i=1}^{2} K(Q_{28.}) \oplus_{i=1}^{8} C_p$$

Proof:

Let A and B be matrices defined as follows :

$$A = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & & \ddots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 1 & 0 \\ 0 & -1 & \cdots & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & & \ddots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 1 & \cdots & 1\ 0 & \cdots & 0 & 1 & 0 \\ 1 & \cdots & 1\ 0 & \cdots & 0 & 0 & 1 \end{bmatrix}$$

When the dimension of $\equiv^* (C_{28})$. equal to f then the dimension of A and B are equal to f+2 We will define two matrices L_p and W_p where

$$L_p = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \equiv^* (C_p) = \begin{bmatrix} 1 & 1 \\ p-1 & -1 \end{bmatrix}, \quad W_p = \begin{bmatrix} 1 & 0 \\ p-1 & -1 \end{bmatrix}$$

$$L_p . \equiv^* (C_p) . W_p = \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}$$

we get: E is a matrix of degree $\equiv^* (C_{2m})$, and if we denote c_{ij} as the

elements of $\equiv^* (C_m)$ which defined by:

$$(A \times L_p). \equiv^* (Q_{28} \times C_p). (B \times W_p)$$

the invariant factors of the matrix $(A \times L_p). \equiv^* (Q_{2m} \times C_p). (B \times W_p)$ will be the same invariant factors of the following matrix :

$$(A \times L_p). \equiv^* (Q_{28} \times C_p). (B \times W_p)$$

$$= \begin{bmatrix} \begin{bmatrix} -pE & \begin{matrix} 0 & 0 \\ 0 & 2 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ -2p & 0 \\ 0 & -2p \end{matrix} \\ \begin{matrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{matrix} & \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} E & \begin{matrix} 0 & 0 \\ 0 & -1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 2 & 0 \\ 0 & 2 \end{matrix} \\ \begin{matrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{matrix} & \end{bmatrix} \end{bmatrix}$$

Then
$$E_q = \begin{bmatrix} -2 & -2 & -2 & -2 & -2 & -2 \\ 1 & -6 & 1 & -6 & 1 & -6 \\ 1 & 1 & -1 & -1 & -1 & -1 \\ -1 & 0 & 1 & -6 & 1 & -6 \\ 0 & 0 & 2 & 2 & -2 & -2 \\ 0 & 0 & -2 & -2 & 12 & -12 \end{bmatrix}$$

Such that
$$L_q = \begin{bmatrix} -1 & -2 & -2 & -2 & -2 & -2 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 & -1 \end{bmatrix}$$

$$W_q = \begin{bmatrix} 12 & 2 & -6 & -1 & 6 & 1 \\ 2 & 0 & -1 & 0 & 1 & 0 \\ 6 & 1 & 6 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 6 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$L_q . E_q . W_q = diag\{56,4,14,2,7,1\}, 2,2$$

$$L_q.(-p\, E_q).W_q = diag\{-56p, -4p, -14p, -2p, -7p, -p\}, 2p, 2p$$

$$K(Q_{28.} \times C_p) = \oplus_{i=1}^{8'} C_p \oplus_{i=1}^{6} C_7 \oplus_{i=1}^{2} C_8 \oplus_{i=1}^{2} C_4 \oplus_{i=1}^{8} C_2)$$

$$K(Q_{28} \times C_p) = \oplus_{i=1}^{2} K(C_{28.}) \oplus_{i=1}^{8} C_p \oplus_{i=1}^{2} C_8 \oplus_{i=1}^{4} C_2 / \oplus_{i=1}^{2} C_4)$$

$$K(\boldsymbol{Q_{28}} \times \boldsymbol{C_p}) = \oplus_{i=1}^{2} K(Q_{28.}) \oplus_{i=1}^{8} K(C_p)$$

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