A Principally $\mathrm{Rad_g}$-Lifting Modules

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Abstract—*In this article we present a new class of modules which is named as a principally $\boldsymbol{Rad_g}$-lifting modules. This class termed by Principally $\boldsymbol{Rad_g}$-lifting in this work which defined as, a module $\boldsymbol{\mathcal{M}}$ is called Principally $\boldsymbol{Rad_g}$-lifting if for every cyclic submodule $\boldsymbol{\Upsilon}$ of $\boldsymbol{\mathcal{M}}$ with $\boldsymbol{Rad_g(\mathcal{M})} \subseteq \boldsymbol{\Upsilon}$, there is a decomposition $\boldsymbol{\mathcal{M} = \aleph \oplus \beta}$ such that $\aleph \leq \boldsymbol{\Upsilon}$ and $\boldsymbol{\Upsilon} \cap \boldsymbol{\beta}$ is g-small in $\boldsymbol{\beta}$. Thus, a ring $\mathfrak{R}$ is called Principally $\boldsymbol{Rad_g}$-lifting if it is a principally $\boldsymbol{Rad_g}$-lifting as $\mathfrak{R}$-module. We determined it is structure. Several characterizations, properties, and instances are described of these modules'.$\boldsymbol{\mathcal{M}}$*

Keywords—component; *Pricipally semi simple*, *Principally $\boldsymbol{Rad_g}$-lifting, principally g-lifting*, *principally generalized hollow*.

1. INTRODUCTION

In this article, all rings $\mathfrak{R}$are associative with unity, and all modules are left unitary. We will go through some of the key definitions that we will require in our work. Let μ be a module and β a submodule of μ, denoted by $\beta \leq \mathcal{M}$. Also, we refer to the direct summand β of $\mathcal{M}$ by $\beta \leq^{\oplus} \mathcal{M}$. Principally semisimple module is a module in which all its cyclic submodules are direct summand [1]. An essential submodule β in $\mathcal{M}$ symbolized by $\beta \trianglelefteq \mathcal{M}$, is a submodule which satisfying $\beta \cap \Upsilon = 0$ implies $\Upsilon = 0$ for any submodule Υ in $\mathcal{M}$ [2]. As dual, a submodule β of $\mathcal{M}$ is called small in $\mathcal{M}$, denoted by $\beta \ll \mathcal{M}$ if, whenever $\mathcal{M} = \beta + \Upsilon$ for $\Upsilon \leq \mathcal{M}$ implies $\Upsilon = \mathcal{M}$ [1]. Zhou and Zhang [3] recalled that, a submodule Υ of $\mathcal{M}$ is called generalized small, denoted by $\Upsilon \ll_g \mathcal{M}$, if for $\beta \trianglelefteq \mathcal{M}$ with $\mathcal{M} = \Upsilon + \beta$ implies $\beta = \mathcal{M}$. We have found in [4] that if any proper cyclic submodule of $\mathcal{M}$ is g-small, then $\mathcal{M}$ is called principally g-hollow. Again Zhou and Zhang gave the definition of Jacobson generalized radical of an $\mathfrak{R}$-module $\mathcal{M}$ as; $Rad_g(\mathcal{M}) = \sum\{\Upsilon | \Upsilon \ll_g \mathcal{M}\} = \cap\{\Upsilon \trianglelefteq \mathcal{M} | \Upsilon \text{is maximal in } \mathcal{M}\}$. A g-coclosed submodule is defined as a submodule $\beta \leq \mathcal{M}$, if $\Upsilon \leq \beta$ such that $\frac{\beta}{\Upsilon} \ll_g \frac{\mathcal{M}}{\Upsilon}$, then $\Upsilon = \beta$ [5].Let Υ, ω be two submodules of $\mathcal{M}$.Υ is called a supplement of ω in μ if it is minimal in $\mathcal{M} = \Upsilon + \omega$ [6]. If $\mathcal{M} = \Upsilon + \beta$ and $\Upsilon \cap \beta \ll_g \beta$, then β is called a g-supplement of Υ in $\mathcal{M}$ [7]. Let $\mathcal{M}$ be a module, $\Upsilon, \beta \leq \mathcal{M}$ and $\mathcal{M} = \Upsilon + \beta$ such that $\Upsilon \cap \beta \leq Rad_g(\beta)$, then β is called a generalized radical supplement, briefly g-radical supplement of Υ in $\mathcal{M}$

[8]. A module $\mathcal{M}$ is called principally generalized lifting if for every cyclic submodule $\Upsilon \leq \mathcal{M}$, there exists a decomposition $\mathcal{M} = \aleph \oplus \beta$ and $\Upsilon \cap \beta \ll_g \beta$ where $\aleph \leq \Upsilon$ [4]. In [9] Mirza and Ghawi gave a definition of $\mathrm{Rad_g}$-lifting modules as, if for every submodule Υ of μ such that $Rad_g(\mathcal{M}) \subseteq \mathcal{M}$, there exists a decomposition $\mathcal{M} = \aleph \oplus \beta$ and $\Upsilon \cap \beta \ll_g \mathcal{M}$ where $\aleph \leq \Upsilon$. In this paper, we are going to define the concept of a principally $\mathrm{Rad_g}$-lifting modules and give equivalent statements of this definition. Next some of the numerous properties of this class of modules and their relationships to other types of modules also given.

2. DEFINITION OF PRINCIPALLY $\mathrm{Rad_g}$-LIFTING AND SOME CONNECTIONS.

Definition 2.1. A module $\mathcal{M}$ is called principally $\mathrm{Rad_g}$-lifting, briefly P-Rad_g-lifting, if for every cyclic submodule ϖ of $\mathcal{M}$contains$Rad_g(\mathcal{M})$, there is a decomposition $\mathcal{M} = \omega \oplus \beta$ such that $\omega \leq \varpi$ and $\varpi \cap \beta \ll_g \beta$. In other words, for any $a \in \mathcal{M}$such that $a\mathfrak{R} \leq \mathcal{M}$with $Rad_g(\mathcal{M}) \subseteq a\mathfrak{R}$, there are submodules ω, β of $\mathcal{M}$ such that $\mathcal{M} = \omega \oplus \beta$, $\omega \leq a\mathfrak{R}$ and $a\mathfrak{R} \cap \beta \ll_g \beta$.

Remarks and Examples2.2.
(1) A principally semisimple module is P-Rad_g-lifting, so that for any semisimple ring $\mathfrak{R}$, all right $\mathfrak{R}$-modules are P-Rad_g-lifting modules, and each submodule of any principally semisimple module so is P-Rad_g-lifting.
(2) If $\mathcal{M}$ is a principally g-lifting module then $\mathcal{M}$ is P-Rad_g-lifting. Moreover, if $Rad_g(\mathcal{M}) = 0$, clearly, the reverse is true.
(3) If $\mathcal{M}$ is a cyclic module over a PID, then we know that every submodule of $\mathcal{M}$ is also cyclic. Hence, any cyclic P-Rad_g-lifting module over a PID is Rad_g-lifting.
(4) The P-Rad_g-lifting modules not inherited by its submodules. As we see in (3), the $\mathbb{Z}$-module $\mathbb{Q}$ is P-Rad_g-lifting, while $\mathbb{Z} \leq \mathbb{Q}$ as $\mathbb{Z}$-module is not P-Rad_g-lifting.
(5) If $\mathcal{M}$ is a module such that $Rad_g(\mathcal{M})$ is a non-cyclic maximal submodule of $\mathcal{M}$, then $\mathcal{M}$ is a P-Rad_g-lifting module.
Proof. Assume $a \in \mathcal{M}$ such that $Rad_g(\mathcal{M}) \subseteq a\mathfrak{R}$. Since $Rad_g(\mathcal{M})$ is a non-cyclic submodule**,** then $Rad_g(\mathcal{M}) \neq a\mathfrak{R}$, that implies $Rad_g(\mathcal{M}) \subset a\mathfrak{R}$. Since $Rad_g(\mathcal{M})$ is maximal of $\mathcal{M}$, thus $a\mathfrak{R} = \mathcal{M}$, a trivially, $\mathcal{M} = \mathcal{M} \oplus (0)$ such that $\mathcal{M} \leq a\mathfrak{R}$ and $a\mathfrak{R} \cap (0) \ll_g (0)$. Therefore $\mathcal{M}$ is a P-Rad_g-lifting module. □

Proposition 2.3. For a module $\mathcal{M}$, the following statements are equivalent;
(1) $\mathcal{M}$ is a P-Rad_g-lifting module;
(2) for any cyclic submodule ϖ of $\mathcal{M}$ with $Rad_g(\mathcal{M}) \subseteq \varpi$, there is a decomposition $\mathcal{M} = \omega \oplus \beta$ such that $\omega \leq \varpi$ and $\varpi \cap \beta$ is g-small in $\mathcal{M}$;
(3) for any cyclic submodule ϖ of $\mathcal{M}$ has $Rad_g(\mathcal{M})$ can be written as $\varpi = \varrho \oplus G$, where $\varrho$is a direct summand of $\mathcal{M}$ and $G \ll_g \mathcal{M}$;
(4) for any cyclic submodule ϖ of $\mathcal{M}$ has $Rad_g(\mathcal{M})$, there exists a direct summand η of $\mathcal{M}$such that $\eta \leq \varpi$ and $\varpi/\eta \ll_g \mathcal{M}/\eta$;
(5) for any cyclic submodule ϖ of $\mathcal{M}$ has $Rad_g(\mathcal{M})$, ϖ has a g-supplement β in $\mathcal{M}$ such that $\varpi \cap \beta$ is a direct summand of ϖ;
(6) for any cyclic submodule ϖ of $\mathcal{M}$ has $Rad_g(\mathcal{M})$, there is an $e = e^2 \in End(\mathcal{M})$ with $e\mathcal{M} \leq \varpi$ and $(1-e)\varpi$ is g-small in $(1-e)\mathcal{M}$;
(7) for any cyclic submodule ϖ of $\mathcal{M}$ has $Rad_g(\mathcal{M})$, there exists a direct summand ω of $\mathcal{M}$and a g-small submodule β of $\mathcal{M}$ such that $\omega \leq \varpi$ and $\varpi = \omega + \beta$;
(8) for any cyclic submodule ϖ of $\mathcal{M}$ has $Rad_g(\mathcal{M})$, there is a submodule ϱ of $\mathcal{M}$ inside ϖ such that $\mathcal{M} = \varrho \oplus \beta$ and β a g-supplement of ϖ in $\mathcal{M}$;
(9) for each $a \in \mathcal{M}$ with $Rad_g(\mathcal{M}) \subseteq a\mathfrak{R}$, there are principal ideals I and J of $\mathfrak{R}$ such that $a\mathfrak{R} = aI \oplus aJ$, where $aI \leq^{\oplus} \mathcal{M}$ and $aJ \ll_g \mathcal{M}$.
Proof.$(\mathbf{1}) \Longrightarrow (\mathbf{2})$ Clear.
$(\mathbf{2}) \Longrightarrow (\mathbf{3})$ Let ϖ be a cyclic $\varpi \leq \mathcal{M}$with $Rad_g(\mathcal{M}) \subseteq \varpi$. By (2), there is $\mathcal{M} = \omega \oplus \beta$ where $\omega \leq \varpi$ and $\varpi \cap \beta \ll_g \mathcal{M}$. By the modular law, $\varpi = \omega \oplus (\varpi \cap \beta)$witht $\omega \leq^{\oplus} \mathcal{M}$ and $\varpi \cap \beta \ll_g \mathcal{M}$.
$(\mathbf{3}) \Longrightarrow (\mathbf{4})$ Assume ϖ is any cyclic submodule of $\mathcal{M}$ such that $Rad_g(\mathcal{M}) \subseteq \varpi$. By (3),$\varpi = \eta \oplus G$, where $\eta \leq^{\oplus} \mathcal{M}$ and $G \ll_g \mathcal{M}$. Define a natural map $\pi: \mathcal{M} \longrightarrow \mathcal{M}/\eta$. Since $G \ll_g \mathcal{M}$, we deduce that $\pi(G) \ll_g \mathcal{M}/\eta$, i.e., $(\eta + G)/\eta = \varpi/\eta \ll_g \mathcal{M}/\eta$.
$(\mathbf{4}) \Longrightarrow (\mathbf{5})$ Assume that ϖ is a cyclic submodule of $\mathcal{M}$ with $Rad_g(\mathcal{M}) \subseteq \varpi$. By (4), there exists $\eta \leq^{\oplus} \mathcal{M}$ such that $\eta \leq \varpi$and $\varpi/\eta \ll_g \mathcal{M}/\eta$, where $\mathcal{M} = \eta \oplus \beta$ for some $\beta \leq \mathcal{M}$. Therefore, $\mathcal{M} = \varpi + \beta$. By the modular law, $\varpi = \eta \oplus (\varpi \cap \beta)$. So, $\varpi/\eta \cong \varpi \cap \beta$ and $\mathcal{M}/\eta \cong \beta$. Therefore, $\varpi \cap \beta \ll_g \beta$. Hence ϖ has a g-supplement β in $\mathcal{M}$ and $\varpi \cap \beta$ is a direct summand of ϖ.
$(\mathbf{5}) \Longrightarrow (\mathbf{6})$ Let ϖ be a cyclic submodule of $\mathcal{M}$ with $Rad_g(\mathcal{M}) \subseteq \varpi$. By hypothesis, if we assume ϖ has a g-supplement β in $\mathcal{M}$ such that $\varpi \cap \beta \leq^{\oplus} \varpi$, then $\mathcal{M} = \varpi + \beta$ and $\varpi \cap \beta \ll_g \beta$. Also, $\varpi = (\varpi \cap \beta) \oplus \varrho$ for some $\varrho \leq \varpi$. Thus, $\mathcal{M} = \varrho \oplus \beta$. Assume that $e: \mathcal{M} \to \varrho$; $e(h+k) = h$ and $(1-e): \mathcal{M} \longrightarrow \beta$;$(1-e)(h+k) = k$ are projection maps for all $h + k \in \mathcal{M}$. Obviously, $e = e^2$ in $End(\mathcal{M})$. Therefore $e\mathcal{M} \leq \varpi$ and $(1-e)\varpi = \varpi \cap \beta \ll_g \beta = (1-e)\mathcal{M}$.
$(\mathbf{6}) \Longrightarrow (\mathbf{7})$ Suppose that ϖ is a cyclic submodule of $\mathcal{M}$ such that $Rad_g(\mathcal{M}) \subseteq \varpi$. By **(6)**, there is an $e = e^2 \in End(\mathcal{M})$ such that $e\mathcal{M} \leq \varpi$ and $(1-e)\varpi \ll_g (1-e)\mathcal{M}$. We know that $\mathcal{M} = e\mathcal{M} \oplus (1-e)\mathcal{M}$. Thus $\varpi = \varpi \cap \mathcal{M} = \varpi \cap (e\mathcal{M} \oplus (1-e)\mathcal{M}) = e\mathcal{M} \oplus (\varpi \cap (1-e)\mathcal{M}) = e\mathcal{M} \oplus (1-e)\varpi$. We put $\omega = e\mathcal{M}$, $\gamma = (1-e)\varpi$ and $\beta = (1-e)\mathcal{M}$. Therefore, $\omega \leq \varpi$ and $\varpi = \omega + \gamma$ where $\gamma \ll_g \beta$ (so in $\mathcal{M}$), and $\omega \leq^{\oplus} \mathcal{M}$.
$(\mathbf{7}) \Longrightarrow (\mathbf{8})$ Assume that ϖ is a cyclic submodule of $\mathcal{M}$ such that $Rad_g(\mathcal{M}) \subseteq \varpi$. By **(7)**, there exists $\omega \leq^{\oplus} \mathcal{M}$ and $\beta \ll_g \mathcal{M}$ such that ω and $\varpi = \omega + \beta$. Thus, $\mathcal{M} = \omega \oplus \beta$ for some $\beta \leq \mathcal{M}$. Therefore β is a g-supplement of ω in $\mathcal{M}$, so ([4], Lemma 2.3) implies β is a g-supplement of $\varpi = \omega + \beta$ in $\mathcal{M}$.
$(\mathbf{8}) \Longrightarrow (\mathbf{1})$ Clear.
$(\mathbf{6}) \Longrightarrow (\mathbf{9})$ Let $a \in \mathcal{M}$ such that $Rad_g(\mathcal{M}) \subseteq a\mathfrak{R}$. By **(6)**, there is an $e = e^2 \in End(\mathcal{M})$ such that $e\mathcal{M} \leq a\mathfrak{R}$ and $(1-e)a\mathfrak{R} \ll_g (1-e)\mathcal{M}$. Notice that $\mathcal{M} = e\mathcal{M} \oplus (1-e)\mathcal{M}$. Let $r \in \mathfrak{R}$ such that $ar = (1-e)a'$ for some $a' \in \mathcal{M}$, then $a' = ea' + ar \in a\mathfrak{R}$, since $e\mathcal{M} \leq a\mathfrak{R}$, and so $a\mathfrak{R} \cap (1-e)\mathcal{M} \leq (1-e)a\mathfrak{R}$**.** It follows that $a\mathfrak{R} \cap (1-e)\mathcal{M} =$

$(1-e)a\mathfrak{R}$. By the modular law, we have that $a\mathfrak{R} = a\mathfrak{R} \cap (e\mathcal{M} \oplus (1-e)\mathcal{M}) = e\mathcal{M} \oplus (a\mathfrak{R} \cap (1-e)\mathcal{M}) = e\mathcal{M} \oplus (1-e)a\mathfrak{R}$. Put $I = \{s \in \mathfrak{R}: as \in e\mathcal{M}\}$ and $J = \{r \in \mathfrak{R}: ar \in (1-e)a\mathfrak{R}\}$. Then $a\mathfrak{R} = aI \oplus aJ$, where $aI = e\mathcal{M} \leq^{\oplus} \mathcal{M}$ and $aJ = (1-e)a\mathfrak{R} \ll_g (1-e)\mathcal{M}$, hence in $\mathcal{M}$.

(**9**) $\Longrightarrow$ (**1**) By **(9)**, for any cyclic submodule ϖ of $\mathcal{M}$ has $Rad_g(\mathcal{M})$, there exists two ideals I and $J$and $\varpi = I \oplus J$, where $I \leq^{\oplus} \mathcal{M}$ and $J \ll_g \mathcal{M}$. Thus, $\mathcal{M} = I \oplus \beta$ for some $\beta \leq \mathcal{M}$. Hence $\varpi = I \oplus (\varpi \cap \beta)$, by the modular law, and so $\varpi \cap \beta \cong J \ll_g \mathcal{M}$. Since $\varpi \cap \beta \leq \varpi \leq^{\oplus} \mathcal{M}$, then $\varpi \cap \beta \ll_g \beta$, by ([10], Lemma 2.12). Therefore, $\mathcal{M}$ is a P-$\mathrm{Rad_g}$-lifting module. □

Corollary 2.4. Let $\mathcal{M}$ be a P-$\mathrm{Rad_g}$-lifting module.Then for any indecomposable cyclic submodule ϖ of $\mathcal{M}$ has $Rad_g(\mathcal{M})$, either ϖ is a direct summand or g-small.

Proof. As ϖ is a cyclic submodule of $\mathcal{M}$ with $Rad_g(\mathcal{M}) \subseteq \varpi$, so by Proposition 2.3, $\varpi = \varrho \oplus G$ where $\varrho \leq^{\oplus} \mathcal{M}$ and $G \ll_g \mathcal{M}$. Since ϖ is indecomposable, thus either $\varpi = \varrho$ or $\varpi = G$.□

Proposition 2.5. Let $\mathcal{M}$ be a P-$\mathrm{Rad_g}$-lifting module with $Rad_g(\mathcal{M}) \neq \mathcal{M}$ is cyclic. Then there exists a decomposition $\mathcal{M} = \varrho \oplus \omega$ such that ω is a g-supplement of $Rad_g(\mathcal{M})$ in $\mathcal{M}$, $Rad_g(\omega) \ll_g \omega$ and $\varrho$is a g-radical.

Proof. Suppose $Rad_g(\mathcal{M}) \neq \mathcal{M}$ is a cyclic submodule. Since $\mathcal{M}$ is a P-$\mathrm{Rad_g}$-lifting module and$Rad_g(\mathcal{M}) \subseteq Rad_g(\mathcal{M})$, so by Proposition 2.3(8), there is a submodule ϱ of $\mathcal{M}$ in $Rad_g(\mathcal{M})$ such that $\mathcal{M} = \varrho \oplus \omega$ and ω a g-supplement of $Rad_g(\mathcal{M})$ in $\mathcal{M}$, i.e., $\mathcal{M} = Rad_g(\mathcal{M}) + \omega$ and $Rad_g(\mathcal{M}) \cap \omega \ll_g \omega$. Since $\omega \leq^{\oplus} \mathcal{M}$, so it is a g-supplement and so $Rad_g(\mathcal{M}) \cap \omega = Rad_g(\omega)$. from ([10], Lemma 2.12), we get $Rad_g(\omega) \ll_g \omega$. By ([11], Corollary 2.3), $\mathcal{M} = Rad_g(\mathcal{M}) + \omega = Rad_g(\varrho) \oplus \omega$. By the modular law, we have $\varrho = \varrho \cap (Rad_g(\varrho) \oplus \omega) = Rad_g(\varrho) \oplus (\omega \cap \varrho) = Rad_g(\varrho)$. Therefore, ϱ is a g-radical. □

The reverse of above proposition need not be correct, in general, for instance, for the $\mathbb{Z}$-module $\mathbb{Z}$, we have $Rad_g(\mathbb{Z}) = 0 \neq \mathbb{Z}$ is cyclic and a decomposition $\mathbb{Z} = \mathbb{Z} \oplus (0)$ such that $\mathbb{Z}$ is a g-supplement of $Rad_g(\mathbb{Z}) = 0$, $Rad_g(\mathbb{Z}) = 0 \ll_g \mathbb{Z}$ and (0) is a g-radical, while $\mathbb{Z}$ is not P-$\mathrm{Rad_g}$-lifting $\mathbb{Z}$-module.

Proposition 2.6. Let $\mathcal{M}$ be an indecomposable module such that$Rad_g(\mathcal{M}) \neq \mathcal{M}$ is cyclic. If $\mathcal{M}$ is a P-$\mathrm{Rad_g}$-lifting module, then $Rad_g(\mathcal{M}) \ll_g \mathcal{M}$.

Proof. Assume $\mathcal{M}$ is an indecomposable P-$\mathrm{Rad_g}$-lifting module. Since $Rad_g(\mathcal{M}) \subseteq Rad_g(\mathcal{M})$ is cyclic, then by Proposition 2.5, there exists a unique decomposition $\mathcal{M} = \mathcal{M} \oplus 0$ where$\mathcal{M}$ is a g-supplement of $Rad_g(\mathcal{M})$ and 0 is a g-radical. Hence, $Rad_g(\mathcal{M}) = Rad_g(\mathcal{M}) \cap \mathcal{M} \ll_g \mathcal{M}$, as required.□

Corollary 2.7. Let $\mathcal{M}$ be an indecomposable module such that $Rad_g(\mathcal{M})$ is cyclic. If $\mathcal{M}$ is a principally $\mathrm{Rad_g}$-lifting module, then either

(1) $\mathcal{M}$ is a cyclic module, or

(2) $Rad_g(\mathcal{M})$ is a g-small submodule of $\mathcal{M}$.

Proof. Suppose $\mathcal{M}$ is an indecomposable and principally $\mathrm{Rad_g}$-lifting module such that $Rad_g(\mathcal{M}) = a\mathfrak{R}$ for some $a \in \mathcal{M}$. If $\mathcal{M}$ is not cyclic, therefore $\mathcal{M} \neq a\mathfrak{R}$ and so $Rad_g(\mathcal{M}) \neq \mathcal{M}$. That means $Rad_g(\mathcal{M})$ is a proper cyclic submodule of $\mathcal{M}$, so by Proposition 2.6, $Rad_g(\mathcal{M})$ is g-small in $\mathcal{M}$. □

Proposition 2.8. Let $\mathcal{M} = \eta + \omega$ be a P-$\mathrm{Rad_g}$-lifting module such that $\eta \leq \mathcal{M}$ and $\omega \leq^{\oplus} \mathcal{M}$. If $\eta \cap \omega$ is a cyclic submodule of $\mathcal{M}$ such that $Rad_g(\mathcal{M}) \subseteq \eta \cap \omega$, then ω containing a g-supplement of η in $\mathcal{M}$.

Proof. Let $\eta \cap \omega$ be a cyclic submodule of $\mathcal{M}$ and $Rad_g(\mathcal{M}) \subseteq \eta \cap \omega$. Since $\mathcal{M}$ is a P-$\mathrm{Rad_g}$-lifting module, we deduce by Proposition 2.3(3), $\eta \cap \omega = \omega \oplus \beta$ where $\omega \leq^{\oplus} \mathcal{M}$ (hence in ω) and $\beta \ll_g \mathcal{M}$. Write $\omega = \omega \oplus \varpi$, for some $\varpi \leq \omega$. Thus, $\eta \cap \omega = \omega \oplus (\eta \cap \varpi)$. Consider $\pi: \omega \longrightarrow \varpi$ is the natural projection. As $\omega \leq^{\oplus} \mathcal{M}$ and $\beta \ll_g \mathcal{M}$, we have that $\beta \ll_g \omega$ and hence $\pi(\beta) \ll_g \varpi$. But $\eta \cap \varpi = \pi(\omega \oplus (\eta \cap \varpi)) = \pi(\eta \cap \omega) = \pi(\omega \oplus \beta) = \pi(\beta)$, so $\eta \cap \varpi \ll_g \varpi$. Moreover, $\mathcal{M} = \eta + \omega = \eta + \omega + \varpi = \eta + \varpi$. Therefore ω contains ϖ as a g-supplement of η in $\mathcal{M}$. □

Corollary 2.9. Let $\mathcal{M} = \eta + a\mathfrak{R}$ be a P-$\mathrm{Rad_g}$-lifting module over a PID$\mathfrak{R}$ such that $\eta \trianglelefteq \mathcal{M}$ and $a \in \mathcal{M}$. If $Rad_g(\mathcal{M}) \subseteq \eta \cap D$ for each $D \leq^{\oplus} a\mathfrak{R}$, then $a\mathfrak{R}$ containing a g-supplement of η in $\mathcal{M}$.

Proof. Let $\mathcal{M} = \eta + a\mathfrak{R}$ and $\eta \trianglelefteq \mathcal{M}$ and $a \in \mathcal{M}$. Since $\mathcal{M}$ is a P-$\mathrm{Rad_g}$-lifting module, so by Proposition 2.3(3), we can write $a\mathfrak{R} = \omega \oplus \varpi$, where $\omega \leq^{\oplus} \mathcal{M}$ and $\varpi \ll_g \mathcal{M}$. So $\mathcal{M} = \eta + a\mathfrak{R} = \eta + \omega + \varpi$, and as $\eta \trianglelefteq \mathcal{M}$ implies $\eta + \omega \trianglelefteq \mathcal{M}$, and hence $\mathcal{M} = \eta + \omega$ (because $\varpi \ll_g \mathcal{M}$), where ϖ is cyclic and $\varpi \leq^{\oplus} \mathcal{M}$, thus $\eta \cap \omega$ is a cyclic submodule of $\mathcal{M}$ has $Rad_g(\mathcal{M})$ (by hypothesis), and so by applying Proposition 2.8, ϖ (so that, $a\mathfrak{R}$) containing a g-supplement of ω in $\mathcal{M}$.□

Proposition 2.10. An indecomposable $\mathfrak{R}$-module $\mathcal{M}$ is a P-$\mathrm{Rad_g}$-lifting module if and only if for $a \in \mathcal{M}$ with $Rad_g(\mathcal{M}) \subseteq a\mathfrak{R} \neq \mathcal{M}$, $a\mathfrak{R}$ is g-small in $\mathcal{M}$.

Proof. $\Rightarrow$) Let $Rad_g(\mathcal{M}) \subseteq a\mathfrak{R} \neq \mathcal{M}$ and $a \in \mathcal{M}$. As $\mathcal{M}$ is a P-$\mathrm{Rad_g}$-lifting module, by Proposition 2.3(8), there are submodules ϱ, G of $\mathcal{M}$ such that $G \leq a\mathfrak{R}$, $\mathcal{M} = a\mathfrak{R} + \varrho = G \oplus \varrho$ and $a\mathfrak{R} \cap \varrho \ll_g \varrho$. Now, if $\varrho = 0$ then $a\mathfrak{R} = \mathcal{M}$ which is a contradiction. By assumption, $\varrho = \mathcal{M}$ and $G = 0$. Therefore $a\mathfrak{R} \ll_g \mathcal{M}$.

$\Leftarrow$) Let $a \in \mathcal{M}$ such that $Rad_g(\mathcal{M}) \subseteq a\mathfrak{R}$. If $a\mathfrak{R} = \mathcal{M}$, trivially, there is a decomposition $\mathcal{M} = \mathcal{M} \oplus (0)$ such that $\mathcal{M} \leq a\mathfrak{R}$ and $a\mathfrak{R} \cap (0) = (0) \ll_g (0)$. Suppose $a\mathfrak{R} \neq \mathcal{M}$, by hypothesis, $a\mathfrak{R} \ll_g \mathcal{M}$. Trivially, $\mathcal{M} = (0) \oplus \mathcal{M}$ such that $(0) \leq a\mathfrak{R}$ and $a\mathfrak{R} \cap \mathcal{M} = a\mathfrak{R} \ll_g \mathcal{M}$. From two cases, (1) holds. □

Corollary 2.11. For a uniform $\mathfrak{R}$-module $\mathcal{M}$ is a P-Rad$_g$-lifting module if and only if for $a \in \mathcal{M}$ with $Rad_g(\mathcal{M}) \subseteq a\mathfrak{R} \neq \mathcal{M}$, $a\mathfrak{R}$ is g-small in $\mathcal{M}$.

Proof. From ([12], Lemma 3.2.9] every uniform module is indecomposable. Then by Proposition 2.10 the result comes. □

Proposition 2.12. Let $\mathcal{M}$ be a P-Rad$_g$-lifting module has a cyclic generalized radical. Then $\mathcal{M} = X_1 \oplus X_2$ such that $Rad_g(X_1)$ is g-small in X_1 and $Rad_g(X_2) = X_2$.

Proof. It follows by Proposition 2.3 and ([13], Proposition 3.1.10]). □

Proposition 2.13. Every principally generalized hollow module is P-Rad$_g$-lifting.

Proof. Let $\mathcal{M}$ be a principally generalized hollow module, and ϖ a cyclic submodule of $\mathcal{M}$ with $Rad_g(\mathcal{M}) \subseteq \varpi$. If $\varpi = \mathcal{M}$, then there is a decomposition $\mathcal{M} = \mathcal{M} \oplus (0)$ such that $\mathcal{M} \leq \varpi$and $\varpi \cap (0) \ll_g (0)$. Let $\varpi \subset \mathcal{M}$, by hypothesis, $\varpi \ll_g \mathcal{M}$, then there is a decomposition $\mathcal{M} = (0) \oplus \mathcal{M}$ such that $(0) \leq \varpi$ and $\varpi \cap \mathcal{M} = \varpi \ll_g \mathcal{M}$. Hence $\mathcal{M}$ is a P-Rad$_g$-lifting module.□

As an application of proposition 2.13, For any p and any natural n, the $\mathbb{Z}$-module $\mathbb{Z}/p^n\mathbb{Z} \cong \mathbb{Z}_{P^n}$ is P-Rad$_g$-lifting as a $\mathbb{Z}$-module, because $\mathbb{Z}_{P^n}$ is a generalized hollow module. While $\mathbb{Z}$ as $\mathbb{Z}$-moduleis not P-Rad$_g$-lifting.

The reverse of above Proposition is not correct, in general, instance; in Remarks and Examples 2.2(3) the $\mathbb{Z}$-module $\mathbb{Z}_{24}$ is P-Rad$_g$-lifting, while $\mathbb{Z}_{24}$ is not principally generalized hollow as $\mathbb{Z}$-module, in fact, $3\mathbb{Z}_{24}$ is a proper cyclic submodule which not g-small in $\mathbb{Z}_{24}$.

Also, as application example of Proposition 2.13; since every finitely submodule in $\mathbb{Q}$ as $\mathbb{Z}$-module is small, then all cyclic submodules are g-small in $\mathbb{Q}$ as $\mathbb{Z}$-module, that is $\mathbb{Q}$ as $\mathbb{Z}$-module is principally generalized hollow, so that it is P-Rad$_g$-lifting.

3. SUBMODULES AND DIRECTSUMMANDS

As we see in Remakes and Examples 2.2(6), that P-Rad$_g$-lifting module doesn't inherited by their submodules. Below, give some conditions for it to be inherited by their submodules.

Proposition 3.1. Let $\mathcal{M}$ be a P-Rad$_g$-lifting $\mathfrak{R}$-module. Then, a submodule ϖ of $\mathcal{M}$ with $Rad_g(\mathcal{M}) \subseteq \varpi$ is P-Rad$_g$-lifting if, one of the following cases is hold;

(1) ϖ is a direct summand of $\mathcal{M}$.

(2) ϖ is a cyclic g-coclosed submodule of $\mathcal{M}$.

Proof. (1) Let $a \in \varpi$ with $Rad_g(\varpi) \subseteq a\mathfrak{R}$, where $\varpi \leq^{\oplus} \mathcal{M}$. By ([14], Lemma 3.16), we have $Rad_g(\mathcal{M}) = Rad_g(\varpi)$. Thus, $Rad_g(\mathcal{M}) \subseteq a\mathfrak{R} \leq \mathcal{M}$. As $\mathcal{M}$ is a P-Rad$_g$-lifting module, there is a decomposition $\mathcal{M} = \rho \oplus \acute{\rho}$ such that $\rho \leq a\mathfrak{R}$ and $a\mathfrak{R} \cap \acute{\rho} \ll_g \acute{\rho}$. As $\rho \leq \varpi$, so by modular law, $\varpi = \varpi \cap (\rho \oplus \acute{\rho}) = \rho \oplus (\varpi \cap \acute{\rho})$. Also $a\mathfrak{R} \cap (\varpi \cap \acute{\rho}) = a\mathfrak{R} \cap \acute{\rho} \ll_g \mathcal{M}$. As $a\mathfrak{R} \cap (\varpi \cap \acute{\rho}) \leq \varpi \cap \acute{\rho} \leq^{\oplus} \mathcal{M}$, ([10], Lemma 2.12). implies $a\mathfrak{R} \cap (\varpi \cap \acute{\rho}) \ll_g \varpi \cap \acute{\rho}$. Hence ϖ is P-Rad$_g$-lifting.

(2) Let ϖ be a cyclic g-coclosed submodule of $\mathcal{M}$ with $Rad_g(\mathcal{M}) \subseteq \varpi$. Since $\mathcal{M}$ is a P-Rad$_g$-lifting module, so by Proposition 2.3(4), there exists $\eta \leq^{\oplus} \mathcal{M}$ such that $\eta \leq \varpi$ and $\varpi/\eta \ll_g \mathcal{M}/\eta$. Since ϖ is a g-coclosed submodule of $\mathcal{M}$, $\eta = \varpi$, that means $\varpi \leq^{\oplus} \mathcal{M}$. By **(1)**, ϖ is P-Rad$_g$-lifting. □

Corollary 3.2. If$\mathcal{M}$ is a P-Rad$_g$-lifting $\mathfrak{R}$-module such that $Rad_g(\mathcal{M})$ is a direct summand of $\mathcal{M}$, then $Rad_g(\mathcal{M})$ is P-Rad$_g$-lifting.

Proof. Since $Rad_g(\mathcal{M}) \leq^{\oplus} \mathcal{M}$ and $Rad_g(\mathcal{M}) \subseteq Rad_g(\mathcal{M})$, Proposition 3.1(1) implies $Rad_g(\mathcal{M})$ is P-Rad$_g$-lifting. □

Recall [15] If all submodules of a module $\mathcal{M}$ are fully invariant,then $\mathcal{M}$ is called a duo module. A submodule A of a module $\mathcal{M}$ is called distributive if $A \cap (B + C) = (A \cap B) + (A \cap C)$ or $A + (B \cap C) = (A + B) \cap (A + C)$ for all submodules B, C of $\mathcal{M}$. A module $\mathcal{M}$ is said to be distributive if all submodules of $\mathcal{M}$ are distributive [17].

In general, we expect that the sum of two P-Rad$_g$-lifting module is not P-Rad$_g$-lifting, but we could not find an example to confirm this. However, we now gives a condition that make the class of a principally Rad$_g$-lifting modules is closed under finite direct sums.

Theorem 3.3. Let $\mathcal{M}$ be a duo (or, distributive) $\mathfrak{R}$-module and $\mathcal{M} = \oplus_{i=1}^{n} \mathcal{M}_i$, where $\{\mathcal{M}_i |\ i = 1,2, \dots, n\}$ a finite family of P-Rad$_g$-lifting modules. Then $\mathcal{M}$ is a P-Rad$_g$-lifting $\mathfrak{R}$-module.

Proof. We will prove this in the case when $n = 2$. Let U be any cyclic submodule of a duo $\mathfrak{R}$-module $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ and $Rad_g(\mathcal{M}) \subseteq \eta$. Since η is a fully invariant submodule of $\mathcal{M}$, ([15], Lemma 2.1) implies $\eta = (\mathcal{M}_1 \cap \eta) \oplus (\mathcal{M}_2 \cap \eta)$. We have that $Rad_g(\mathcal{M}_i) \subseteq \mathcal{M}_i \cap \eta$ and $\mathcal{M}_i \cap \eta$ is a cyclic submodule of $\mathcal{M}_i$ for $i = 1,2$. Since $\mathcal{M}_i$ is P-Rad$_g$-lifting, for $i = 1,2$, then there are decompositions $\mathcal{M}_i = \rho_i \oplus \omega_i$ such that $\rho_i \leq \mathcal{M}_i \cap \eta$ and $(\mathcal{M}_i \cap \eta) \cap \omega_i = \eta \cap \omega_i \ll_g \omega_i$. Then $\mathcal{M} = (\rho_1 \oplus \rho_2) \oplus (\omega_1 \oplus \omega_1)$, $\rho_1 \oplus \rho_2 \leq (\mathcal{M}_1 \cap \eta) \oplus (\mathcal{M}_2 \cap \eta) = \eta$ and so $\eta \cap (\omega_1 \oplus \omega_1) = (\eta \cap \omega_1) \oplus (\eta \cap \omega_2) \ll_g \omega_1 \oplus \omega_1$, by ([3], Proposition 2.5(3)). Hence, by mathematical induction, $\mathcal{M}$ is a P-Rad$_g$-lifting $\mathfrak{R}$-module. Similarly, when $\mathcal{M}$ is a distributive $\mathfrak{R}$-module. □

Proposition 3.4. Let $\mathcal{M}$ be a P-Rad$_g$-lifting $\mathfrak{R}$-module and $\gamma \neq 0$ a submodule of $\mathcal{M}$. If $\gamma \cap Rad_g(\mathcal{M}) = 0$, then γ is principally semisimple.

Proof. Let $a \in \gamma$. Since $\mathcal{M}$ is a P-Rad$_g$-lifting $\mathfrak{R}$-module, there is a decomposition $\mathcal{M} = \omega \oplus \beta$ such that $\omega \leq a\mathfrak{R}$ and $a\mathfrak{R} \cap \beta \ll_g \beta$, so in $\mathcal{M}$. It follows that $a\mathfrak{R} \cap \beta \subseteq Rad_g(\mathcal{M})$. By the modular law, we have $\gamma = \gamma \cap \mathcal{M} = \gamma \cap (a\mathfrak{R} + \beta) = a\mathfrak{R} + (\gamma \cap \beta)$. As $a\mathfrak{R} \cap (\gamma \cap \beta) \subseteq \beta \cap Rad_g(\mathcal{M}) = 0$, we get $\gamma = a\mathfrak{R} \oplus (\gamma \cap \beta)$. Therefore, $a\mathfrak{R} \leq^{\oplus} \gamma$ and hence γ is principally semisimple. □

Proposition 3.5. Let $\mathcal{M}$ be an $\mathfrak{R}$-module, consider the following statements:

(1) $\mathcal{M}$ is a principally semisimple $\mathfrak{R}$-module.

(2) $\mathcal{M}$ is a principally g-lifting $\mathfrak{R}$-module.

(3) $\mathcal{M}$ is a principally Rad$_g$-lifting $\mathfrak{R}$-module.

Then (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3). If $Rad_g(\mathcal{M}) = 0$, then (3) $\Longrightarrow$ (1).

Proof. $(\mathbf{1}) \Longrightarrow (\mathbf{2}) \Longrightarrow (\mathbf{3})$ Clear.

$(\mathbf{3}) \Longrightarrow (\mathbf{1})$ If $Rad_g(\mathcal{M}) = 0$, then $\mathcal{M} \cap Rad_g(\mathcal{M}) = 0$ and then $\mathcal{M}$ is a principally semisimple $\mathfrak{R}$-module, by Proposition 3.4. □

Corollary 3.6. Let $\mathcal{M}$ be a P-Rad$_g$-lifting $\mathfrak{R}$-module such that $Rad_g(\mathcal{M}) = 0$, then every nonzero submodule of $\mathcal{M}$ is principally semisimple.

Proof. Directly by Proposition 3.5.

4. FACTOR MODULE OF PRINCIPALLY Rad$_g$-LIFTING

Theorem 4.1. Let $\mathcal{M}$ be aP-Rad$_g$-lifting module and assume $\varpi \leq \mathcal{M}$. If for every direct summand η of $\mathcal{M}$, $(\eta + \varpi)/\varpi$ is a direct summand of $\mathcal{M}/\varpi$. Then $\mathcal{M}/\varpi$ is a P-Rad$_g$-lifting module.

Proof. Let $\varpi \leq a\mathfrak{R} \leq \mathcal{M}$ such that $a \in \mathcal{M}$ and $Rad_g(\mathcal{M}/\varpi) \subseteq a\mathfrak{R}/\varpi$. Consider the natural map $\pi: \mathcal{M} \to \mathcal{M}/\varpi$. From $Rad_g(\mathcal{M}) \subseteq \mathcal{M}$, we deduce that $\pi(Rad_g(\mathcal{M})) \subseteq Rad_g(\mathcal{M}/\varpi)$, i.e., $(Rad_g(\mathcal{M}) + \varpi)/\varpi \subseteq Rad_g(\mathcal{M}/\varpi)$, so$(Rad_g(\mathcal{M}) + \varpi)/\varpi \subseteq x\mathfrak{R}/\varpi$, and hence $Rad_g(\mathcal{M}) \subseteq a\mathfrak{R}$. Since $\mathcal{M}$ is a P-Rad$_g$-lifting module, then by Proposition 2.3(3), there exists $\varrho \leq^{\oplus} \mathcal{M}$ where $\varrho \leq a\mathfrak{R}$ and $a\mathfrak{R}/\varrho \ll_g \mathcal{M}/\varrho$. By the hypothesis, $(\varrho + \varpi)/\varpi \leq^{\oplus} \mathcal{M}/\varpi$. Clearly, $(\varrho + \varpi)/\varpi \leq a\mathfrak{R}/\varpi$. Consider a projection map $\rho: \frac{\mathcal{M}}{\varrho} \to \frac{\mathcal{M}/\varrho}{(\varrho+\varpi)/\varrho}$. Since $a\mathfrak{R}/\varrho \ll_g \mathcal{M}/\varrho$ then $\frac{a\mathfrak{R}}{\varrho+\varpi} \ll_g \frac{\mathcal{M}}{\varrho+\varpi}$, that implies $\frac{a\mathfrak{R}/\varpi}{(\varrho+\varpi)/\varpi} \ll_g \frac{\mathcal{M}/\varpi}{(\varrho+\varpi)/\varpi}$. Therefore $\mathcal{M}/\varpi$ is a P-Rad$_g$-lifting module.□

Theorem 4.2. Let $\mathcal{M}$ be a P-Rad$_g$-lifting module and $\varpi \leq \mathcal{M}$ that satisfies one of the following:

(1) If ϖ is a distributive submodule of $\mathcal{M}$.

(2) If ϖ is a fully invariant submodule of $\mathcal{M}$.

(3) If Y is a submodule of $\mathcal{M}$ has $Rad_g(\mathcal{M})$ such that $\mathcal{M} = \varpi \oplus Y$.

Then $\mathcal{M}/\varpi$ is a P-Rad$_g$-lifting module.

Proof. (1) let $\mathcal{M} = \eta \oplus \grave{\eta}$ for some $\grave{\eta} \leq \mathcal{M}$. By Theorem 11, we prove that $(\eta + \varpi)/\varpi \leq^{\oplus} \mathcal{M}/\varpi$. It is obvious to ensure that $\mathcal{M}/\varpi = ((\eta + \varpi)/\varpi) + ((\grave{\eta} + \varpi)/\varpi)$. Now, as ϖ is a distributive submodule of $\mathcal{M}$, $(\eta + \varpi) \cap (\grave{\eta} + \varpi) = (\eta \cap \grave{\eta}) + \varpi = \varpi$. So $((\eta + \varpi)/\varpi) \cap ((\grave{\eta} + \varpi)/\varpi) = 0$, therefore $\mathcal{M}/\varpi$ is a P-Rad$_g$-lifting module.

(2) Let $\varrho \leq^{\oplus} \mathcal{M}$, then $\mathcal{M} = \varrho \oplus \grave{\varrho}$ for some $\grave{\varrho} \leq \mathcal{M}$. As ϖ is a fully invariant submodule of $\mathcal{M}$, therefore, $\mathcal{M}/\varpi = ((\varrho + \varpi)/\varpi) \oplus ((\varrho' + \varpi)/\varpi)$, by ([16], Lemma 3.3), i.e., $(\eta + \varpi)/\varpi \leq^{\oplus} \mathcal{M}/\varpi$. Hence $\mathcal{M}/\varpi$ is a P-Rad$_g$-lifting module, by Theorem 4.1.

(3) By Proposition 3.1(1), Y is a P-Rad$_g$-lifting module. Thus, $\mathcal{M}/\varpi \cong Y$, and then $\mathcal{M}/\varpi$ is a P-Rad$_g$-lifting module. □

Corollary 4.3. Let $\mathcal{M}$ be a P-Rad$_g$-lifting module, then:

(1) If $\mathcal{M}$ is a distributive (or, duo) module, then every factor module of $\mathcal{M}$ is also P-Rad$_g$-lifting.

(2) If $f: \mathcal{M} \to \grave{\mathcal{M}}$ is a homomorphism has distributive (or, fully invariant) kernel, then $f(\mathcal{M})$ is P-Rad$_g$-lifting. Moreover, if f is an epimorphism, then $\grave{\mathcal{M}}$ is P-Rad$_g$-lifting.

Proof. (1) Clear from Theorem 4.2(1) and (2), respectively.

(2) let $f: \mathcal{M} \to \grave{\mathcal{M}}$ is a homomorphism. By 1st isomorphism theorem, we have that $\mathcal{M}/Kerf \cong f(\mathcal{M})$. From Theorem 4.2(1) or (2), $\mathcal{M}/Kerf$ is a P-Rad$_g$-lifting module. Hence $f(\mathcal{M})$ is P-Rad$_g$-lifting.□

Recall [15] If all direct summands of a module $\mathcal{M}$ are fully invariant, then $\mathcal{M}$ is called a weak duo module.

Proposition 4.4. Let $\mathcal{M}$ be a weak duo module and X a direct summand of $\mathcal{M}$. If $\mathcal{M}$ is a P-Rad$_g$-lifting module then X and $\mathcal{M}/X$ are both P-Rad$_g$-lifting modules.

Proof. Suppose that $\mathcal{M}$ is a weak duo module and $X \leq^{\oplus} \mathcal{M}$, then $\mathcal{M} = X \oplus \eta$ where X, η are fully invariant submodules of $\mathcal{M}$. By Theorem 4.2(2), $\mathcal{M}/X$ and $X \cong \mathcal{M}/\eta$ are P-Rad$_g$-lifting modules.□

Proposition 4.5. Let $\mathcal{M}$ be an $\mathfrak{R}$-module and X a direct summand of $\mathcal{M}$. Then $\mathcal{M}$ is a P-Rad$_g$-lifting module if and only if X and $\mathcal{M}/X$ are both P-Rad$_g$-lifting modules if one of the following conditions hold:

(1) $\mathcal{M}$ is a distributive module.

(2) $\mathcal{M}$ is a duo module.

Proof. (1) let $\mathcal{M}$ be a distributive module and X a direct summand of $\mathcal{M}$, so $\mathcal{M} = X \oplus \beta$ for a submodule β of $\mathcal{M}$. By Corollary 4.3(1), $\mathcal{M}/X$ is P-Rad$_g$-lifting. However, $X \cong \mathcal{M}/\beta$, again by Corollary 4.3(1), X is P-Rad$_g$-lifting.

Conversely, as $\mathcal{M} \cong X \oplus (\mathcal{M}/X)$, the result is included by Theorem 3.3.

(**2**) Since any duo module is weak duo, then the result is follows by Proposition 4.4 and Theorem 3.3.□

Corollary 4.6. Let $\mathcal{M} = \bigoplus_{i=1}^{n} \mathcal{M}_i$ be a duo module. Then, for any $i = 1,2, \ldots, n$, $\mathcal{M}_i$ is a P-$\mathrm{Rad_g}$-lifting module if and only if $\mathcal{M}$ is a P-$\mathrm{Rad_g}$-lifting module.
Proof. It follows directly from Theorem 3.3 and Proposition 4.5(2).□

Proposition 4.7. If $\mathcal{M}$ is a P-$\mathrm{Rad_g}$-lifting module then, $\mathcal{M}/Rad_g(\mathcal{M})$ is principally semisimple.
Proof. Let $a \in \mathcal{M}$ and $Rad_g(\mathcal{M}) \subseteq a\mathfrak{R}$. By hypothesis there is $\mathcal{M} = \eta \oplus \acute{\eta}$ for some $\eta \leq a\mathfrak{R}$ and $a\mathfrak{R} \cap \acute{\eta} \ll_g \acute{\eta}$, so in $\mathcal{M}$. Therefore, $\mathcal{M} = a\mathfrak{R} + \acute{\eta}$ and $a\mathfrak{R} \cap \acute{\eta} \subseteq Rad_g(\mathcal{M})$. It follows that, $\frac{\mathcal{M}}{Rad_g(\mathcal{M})} = \frac{a\mathfrak{R}}{Rad_g(\mathcal{M})} + \frac{\acute{\eta}+Rad_g(\mathcal{M})}{Rad_g(\mathcal{M})}$, and so $\left(\frac{a\mathfrak{R}}{Rad_g(\mathcal{M})}\right) \cap \left(\frac{\acute{\eta}+Rad_g(\mathcal{M})}{Rad_g(\mathcal{M})}\right) = \frac{a\mathfrak{R}\cap(\acute{\eta}+Rad_g(\mathcal{M}))}{Rad_g(\mathcal{M})} = \frac{Rad_g(\mathcal{M})+(a\mathfrak{R}\cap\acute{\eta})}{Rad_g(\mathcal{M})} = 0$, i.e., $a\mathfrak{R}/Rad_g(\mathcal{M}) \leq^{\oplus} \mathcal{M}/Rad_g(\mathcal{M})$. Therefore $\mathcal{M}/Rad_g(\mathcal{M})$ is a principally semisimple module. □

Corollary 4.10. Let $\mathcal{M}$ be a P-$\mathrm{Rad_g}$-lifting module then, $\mathcal{M}/Rad_g(\mathcal{M})$ is a P-$\mathrm{Rad_g}$-lifting module.
Proof. From Propositions 3.5 and 4.7. □

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