Spline Technique for Second Type of Fredholm Integral Equations

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Abstract— *Integral equation plays very important tools in a developed science like numerous troubles in an engineering and mechanical sciences . a cubic spline method renowned to begin with one powerful role to solve many functional equations like integral equations ,ordinary and partial differential equation .In our paper, a spline method is used to solve the problem linear integral equation of Fredholm. Integral function is turned to polynomial by basis of cubic spline . efficiency test of the proposed method , the comparison is check together with method of power series (least square method) and quadrature(nystrom) method,Laguerre,Legendre and Hermite series method . In MATLAB, all programs are written.*

Keywords-Linear Fredholm integral equation ,cubic spline ,Linear system

1.INTRODUCTION

The topic of integral equation is one of the more importance mathematical role in pure and applied mathematics . Integral equation lead to a very importance tools in developed science like numerous problems in engineering and mechanical science,to more details look at [1].actually, some physical problems are modeled in the form of Fredholem integral equation,like problems as potential theory and Dirichlet problems whose discussed in [1], mathematical problem of radioactive equilibrium[2].to updating work;Malek Negad,N.Aga Zadeh,M.Rabbani employed modified Taylor's series expansion method to solve Fredholem integral equation system of the second type [3]. H.kaneko argues that degenerate nucleus method in most cases include a time – consuming step establishing the linear system which entries are evaluated via computing a number from integrals [5]. Pallop Huabsomboon , Boriboon Nova Prateep , Hideaki kaneko offered a new Taylor – series method to approximating a solution of the Fredholm equation of the second type[4].E.Babolion,A.Shahsavaran offered a computational to solve nonlinear fredholem integral equation of the second type whose is depend on the use of Haar wavelets [6]. Jalil Rashidinia , Esmail Bobolina, Zahara Mahmoodi have been shown that approximation to Fredholem integral equation of the first and second type can be got on by using certain easy numerical quadrature rules and collocation spline [7].Habeeb. K. Kadhim argue transforming initial value problem into Volterra integral equation and transforming boundary value problem into Fredholem integral equation vice versa depended on several condition [8].Farshid Mirzaee,Sima Siroozfar argue developed Simpson's quadrature rule to solve linear Fredholem integral equation of the second type [9].Salih Yalcinbas And Muge Aynigul used a orthogonal series(Hermite series)as assumption to solve it in the second type of the Friedholm integral equation[10].Salih Yalcinbas,Tugee Akkaya,Muge Aynigul used a orthogonal series(Laguerre series)as assumption to solve it in the second type of the Friedholm integral equation[11].in our paper, the suggested method(natural cubic spline method)to solve Fredholm integral equation is presented.The integral function is turned to polynomial by basis of cubic spline.suggested method is *strengthened* via complete solution by many example;a comparisons with power series ,quadrature method,Chebyshev

,Laguerre ,Legendre,Hermite series method is being tested for proficiency.

2.MTHODS OF FREDHOLM INTEGRAL EQUATION SOLUTION

In this paper ,we will present a method for solving the Fredholm integral equation of second type with a continuous kernel,where we assume that the unknown function under integration sign converges to the functions of cubic spline of the third degree

2.1 Spline Basis formula to solve a linear integral equation of Fredholm of second type .

We know that $x_{j-1} \le x \le x_j$, the natural cubic spline s(x) is given by

$$s(x) = \frac{m_{j-1}(x_j-x)^3}{6h} + \frac{m_j(x-x_{j-1})^3}{6h} + (y_{j-1} - \frac{h^2}{6} m_{j-1}) \frac{(x_j-x)}{h} + (y_j - \frac{h^2}{6} m_j) \frac{(x-x_{j-1})}{h}, \quad (2.1)\ . \ [12]$$

where $m_j = \acute{s}(x_j)$, $y_j = y(x_j)$ and $x_j = x_0 + jh$ j = 0, 1, …, n

If we are now an approximate the integral term in equation

$$y(x) + \int_c^d k(x,t)y(t)dt = f(x), \quad y(t) \approx s(t) , \ (2.2)$$

using Eq (2.1) , will get

$$y(x_i) + \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} k(x_i,t)[m_{j-1} \frac{(t_j-t)^3}{6h} + m_j \frac{(t-t_{j-1})^3}{6h} + (y_{j-1} - \frac{h^2}{6} m_{j-1}) \frac{(t_j-t)}{h} + (y_j - \frac{h^2}{6} m_j) \frac{(t-t_{j-1})}{h}] dt = f(x_i); \quad (2.3)$$

i=0,1,2,…,n

put $q = \frac{(t-t_{j-1})}{h}$, dt=hdq,the Eq (2.3) will simplify to

$$y_i + h \sum_{j=1}^{n} \int_0^1 k(x_i, t_{j-1} + qh)[m_{j-1} \frac{(1-q)^3}{6h} h^3 + m_j \frac{q^3h^2}{6} + (y_{j-1} - \frac{h^2}{6} m_{j-1})(1-q) + (y_j - \frac{h^2}{6} m_j) q] dq = f_i , \quad (2.4)$$

its follow

$$y_i + m_0[\int_0^1 \frac{h^3}{6} k(x_i, t_0 + qh)(3q^2 - q^3 - 2q)dq + \frac{h^3}{6} \int_0^1 [\sum_{j=1}^{n-1} m_j ((k(x_i, t_j + qh)(3q^2 - q^3 - 2q) + k(x_i, t_{j-1} + qh) (q^3 - q))] dq + \frac{h^3}{6} m_n \int_0^1 k(x_i, t_{n-1} + qh)(q^3 - q)dq + hy_0 \int_0^1 k(x_i, t_0 + qh)(1-q)dq + h\sum_{j=1}^{n-1} y_j \int_0^1 [k(x_i, t_j + qh)(1-q) + k(x_i, t_{j-1} + qh) q] dq + hy_n \int_0^1 k(x_i, t_{n-1} + qh)q dq = f_i ; \quad (2.5)$$

i=0,1,..,n

In Eq. (2.5); **will assume:**

$$I_{io} = \int_0^1 \frac{h^3}{6} k(x_i, t_0 + qh)(3q^2 - q^3 - 2q)dq ;$$

$$I_{ij} = \int_0^1 \frac{h^3}{6} [k(x_i, t_j + qh)(3q^2 - q^3 - 2q) + k(x_i, t_{j-1} + qh) (q^3 - q)] dq; \quad j = 1 , 2 , \dots , n-1.$$

$$I_{in} = \int_0^1 \frac{h^3}{6} k(x_i, t_{n-1} + qh) (q^3 - q) dq;$$

$$g_{i0} = \int_0^1 hk(x_i, t_0 + qh) (1-q) dq;$$

$$g_{ij} = \int_0^1 h[k(x_i, t_j + qh) (1-q) + k(x_i, t_{j-1} + qh) q] dq; \quad j=1,2,\dots,n-1$$

$$g_{in} = h \int_0^1 k(x_i, t_{n-1} + qh) q dq, \quad (2.6).$$

Will get

$$m_0 I_{i0} + \sum_{j=1}^{n-1} m_j I_{ij} + m_n I_{in} + (g_{i0} + \delta_{i0})y_0 + \sum_{j=1}^{n-1}(g_{ij} + \delta_{ij})y_j + (g_{in} + \delta_{in})y_n = f_i, \quad (2.7)$$

An Eq (2.7) i=0,1,2,…,n is linear system where

$$\delta_{i0} = \begin{cases} 0 & i \neq 0 \\ 1 & i = 0 \end{cases}$$

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \quad ; \quad i=1,2 ,\dots, n-1 \quad , \quad j=1,2 ,\dots, n-1$$

$$\delta_{in} = \begin{cases} 0 & i \neq n \\ 1 & i = n \end{cases}$$

We get (n+1) equation in (2n+2) unknown variable (m_0 , m_1 , … , m_n, y_0 , y_2 , … ,y_n) along with the relationship .

$$\frac{h}{6} m_{j-1} + \frac{2}{3} hm_j + \frac{h}{6} m_{j+1} = \frac{y_{j-1} - 2y_j + y_{j+1}}{h} , \quad (2.8)$$

j= 1, 2, …, n–1 and $m_n = m_0 = 0$ is linear system

$$\begin{array}{l} \mathbf{m_0 = 0} \\ \frac{h}{6} \mathbf{m_0} + \frac{2}{3} \mathbf{hm_1} + \frac{h}{6} \mathbf{m_2} - \frac{y_0}{h} + \frac{2y_1}{h} - \frac{y_2}{h} = \mathbf{0} \\ \frac{h}{6} \mathbf{m_1} + \frac{2}{3} \mathbf{hm_2} + \frac{h}{6} \mathbf{m_3} - \frac{y_1}{h} + \frac{2y_2}{h} - \frac{y_3}{h} = \mathbf{0} \\ \vdots \qquad \cdots \qquad \vdots \qquad \cdots \qquad \vdots \\ \frac{h}{6} \mathbf{m_{n-2}} + \frac{2}{3} \mathbf{hm_{n-1}} + \frac{h}{6} \mathbf{m_n} - \frac{y_{n-2}}{h} + \frac{2y_{n-1}}{h} - \frac{y_n}{h} = \mathbf{0} \\ \mathbf{m_n = 0} \end{array} \quad ,(2.9)$$

Of the linear systems (2.7) , (2.9) will get (2n+2) equation in (2n+2) unknown variables (m_0 , m_1 , … , m_n, y_0 , y_2 , … ,y_n), of matrix notation:

$$A = \begin{bmatrix} I_{00} & I_{01} & \dots & I_{0n} & 1+g_{00} & g_{01} & \dots & g_{0n} \\ I_{10} & I_{11} & \dots & I_{1n} & g_{10} & 1+g_{11} & \dots & g_{1n} \\ \vdots & & & & & & & \\ I_{n0} & I_{n1} & \dots & I_{nn} & g_{n0} & g_{n1} & \dots & g_{nn}+1 \\ 1 & 0 & \dots & 0 & 0 & 0 & \dots & \\ 0 & & & & & & & \\ \frac{h}{6} & \frac{2h}{3} & \frac{h}{6} & 0 \dots 0 & \frac{-1}{h} & \frac{2}{h} & \frac{-1}{h} & 0 .. \ 0 \\ 0 & \frac{h}{6} & \frac{2h}{3} & \dots 0 & 0 & \frac{-1}{h} & \frac{2}{h} \ \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & \frac{h}{6} & 0 & 0 & \dots & \frac{-1}{h} \\ 0 & \dots \ 0 & & 1 & 0 & 0 & \dots & 0 \end{bmatrix}$$

$$B=\begin{bmatrix} m_0 \\ m \\ \vdots \\ m_n \\ y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}, \quad C=\begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}; \qquad (2.10)$$

AB=C is Linear system has size $(2n+2) \times (2n+2)$

The solution to the system is given as $B = \{m_0, m_1, \dots, m_n, y_0, y_2, \dots, y_n\}^t$ we set $z = \{m_0, m_1, \dots, m_n\}$

$y = \{y_0, y_2, \dots, y_n\}$ and have

$\boldsymbol{x_j} = \{x_0 = c,\ x_1,\ x_j = x_0 + jh \dots, x_n = d\}$,

put in (2.1) at $j = 1, 2, 3, \dots, n$

$$s_j(x) = m_{j-1}\frac{(x_j - x)}{6h} + m_j\frac{(x - x_{j-1})^3}{6h} + \left(y_j - \frac{h^2}{6}m_j\right)\frac{(x_j - x)}{6h} + \left(y_j - \frac{h^2}{6}m_j\right)\frac{(x - x_{j-1})}{h}, \qquad (2.11)$$

on $x_{j-1} \le x < x_j$, $y(x) \approx s_j(x)$; $\forall\ x_{j-1} \le x \le x_j$ $j=1,2,..,n$ be an approximate solution

3.PROPOSED METHOD FOR SOLVING FREDHOLM INTEGRAL EQUATION

3-1 Example : The proposed method can be applied on integral equation : $y(x) - 4\int_0^1 x^2 t^2\, y(t)dt = x$, take $n = 2$, exact solution is$(x+5x^2)$

solution:- $h = \frac{1-0}{2} = \frac{1}{2}$, $[0,1] \to x_i = 0, \frac{1}{2}, 1$.

Choose n = 2 in system AB=C, (2.10):

$$A = \begin{bmatrix} I_{00} & I_{01} & I_{02} & g_{00}+1 & g_{01} & g_{02} \\ I_{10} & I_{11} & I_{12} & g_{10} & g_{11}+1 & g_{12} \\ I_{20} & I_{21} & I_{22} & g_{20} & g_{21} & g_{22}+1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{h}{6} & \frac{2h}{3} & \frac{h}{6} & \frac{-1}{h} & \frac{2}{h} & \frac{-1}{h} \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \qquad (3.1)$$

Compute from(2.6): $[I_{i0}, I_{i1}, I_{i2}, g_{i0}, g_{01}, g_{i2}]$ i=0,1,2 (3.2)

$$I_{i0} = \int_0^1 \frac{h^3}{6}k(x_i, t_0 + qh)(3q^2 - q^3 - 2q)dq$$

$$I_{i0} = \int_0^1 -\frac{1}{192}i^2\ q^2(3q^2 - q^3 - 2q)\,dq = \frac{1}{2880}i^2$$

$$I_{i1} = \int_0^1 -\frac{i^2}{48}\left(\frac{1}{2}+\frac{q}{2}\right)^2\left(3q^2 - q3 - 2q\right) - \frac{i^2p^2}{192}(q^3 - q)dq = \frac{19}{5760}i^2$$

$$I_{i2} = \int_0^1 \frac{-i^2}{48}\left(\frac{1}{2}+\frac{q}{2}\right)^2 (q^3 - q)dq = \frac{1}{320}i^2$$

$$g_{i0} = \int_0^1 \frac{-1}{8}i^2\ (1-q)\ q^2\,dq = \frac{-1}{96}i^2$$

$$g_{i1} = \int_0^1 \left(\frac{-1}{2}i^2(1-q)\left(\frac{1}{2}+\frac{q}{2}\right)^2 - \frac{1}{8}i^2q^3\right)dq = \frac{-7}{48}i^2$$

$$g_{i2} = \int_0^1 \frac{-i^2}{2}\left(\frac{1}{2}+\frac{q}{2}\right)^2 q\,dq = \frac{-17}{96}i^2$$

Hence, $\left[\frac{1}{2880}i^2, \frac{19}{5760}i^2, \frac{1}{320}i^2, \frac{-1}{96}i^2, \frac{-7}{48}i^2, \frac{-17}{96}i^2\right]$

i = 0 , 1 , 2.

Then, the linear system (2.10) given as follows

$$A=\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0.0003 & 0.0033 & 0.0031 & -0.0104 & 0.8542 & -0.1771 \\ 0.0014 & 0.0132 & 0.0125 & -0.0417 & -0.5833 & 0.2917 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0.0833 & 0.333 & 0.0833 & -2 & 4 & -2 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

;

(i) $C = \begin{bmatrix} 0 \\ 0.500 \\ 1.00 \\ 0 \\ 0 \\ 0 \end{bmatrix}$; $B = \begin{bmatrix} m_0 \\ m_1 \\ m_2 \\ y_0 \\ y_1 \\ y_2 \end{bmatrix}$,

$B = A^{-1}C$;A,nonsingular.

(ii)

Ǘ= [0 16.1798 0 0 1.8483 6.3933]

(iii)

V $[0,1] = (0, \frac{1}{2}, 1)$

y = [0 1.8433 6.3933]

Vi

z = [0 16.1798 0]

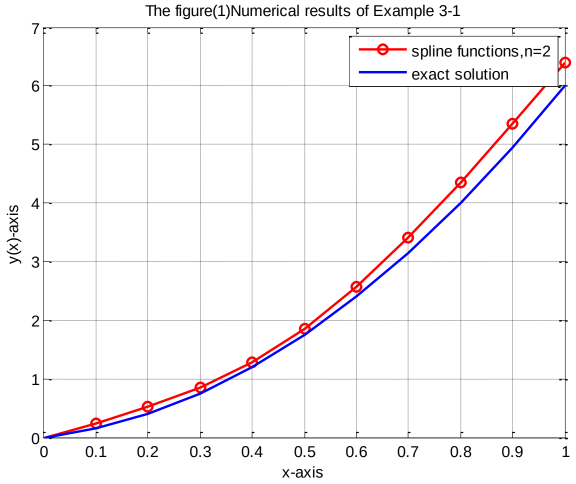
Vii

$j = 1$, $s_1(x) = \frac{480}{89}x^3 + \frac{209}{89}x$, at $[0, \frac{1}{2}]$;

$j = 2\ ,\quad s_2(x) = \frac{-480}{89}\,(x-1)^3 + \frac{929}{89}\,x - \frac{360}{89},\quad \text{at } [\frac{1}{2}\ , 1]$

;

The figure(1)Numerical results of Example 3-1



s_1, s_2 Represent an approximate solution of Fredholm integral equation

4.IMPLEMENTATION AND COMPARISON

Table(I,II) The result method of power series, quadrature and And proposed method(cubic spline method)of Fredholm integral equation of second kind are tabulated below

4-1:EXAMPLE $y(x) = cos(x) + \int_0^1 xt\, y(t)dt\ ,$ **(4.1)**

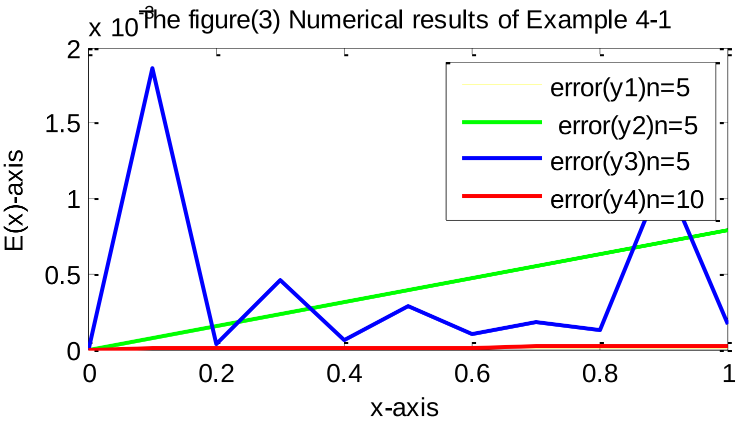
x_i	*Exact solution y(x)=cos(x)+ 1.5(sin(1)+cos(1)-1)x*	*Power series method n=5* *y₁(xi) [14]*	*Nystrom(quadrature) Method n=5* *y₂(xi) [13]*	*Proposed method(cubic spline) n=5* *y₃(xi)*	*Proposed method n=10, y_4 (xi)*
0.0	**1.000000000000000**	**1.000001333974069**	**1.000000000000000**	**1.000000000000000**	**1.000000000000000**
0.1	**1.052270158879431**	**1.052269639811209**	**1.052190698687935**	**1.050406760536225**	**1.052268161284116**
0.2	**1.094598565044052**	**1.094598798826402**	**1.094439644661061**	**1.094565616527490**	**1.094594569853421**
0.3	**1.127134469929822**	**1.127134853167674**	**1.126896089355334**	**1.127592547763846**	**1.127128477143875**
0.4	**1.150124968408507**	**1.150124855812257**	**1.149807127642523**	**1.150059071375382**	**1.150116978027244**
0.5	**1.163912529897400**	**1.163912122225156**	**1.163515228939920**	**1.163623117500203**	**1.163902541920821**
0.6	**1.168931576518111**	**1.168931482017702**	**1.168454815369135**	**1.168832730968425**	**1.168919590946217**
0.7	**1.165704142494326**	**1.165704530606112**	**1.165147921153855**	**1.165887474051243**	**1.165690159327117**
0.8	**1.154834658158409**	**1.154834880870050**	**1.154198976626441**	**1.154702864092161**	**1.154818677395883**
0.9	**1.137003910683313**	**1.137003414811185**	**1.136288768959850**	**1.135848893428353**	**1.136985932325472**
1	**1.112962241882194**	**1.112963535211748**	**1.112167639967234**	**1.112797499299384**	**1.112942265929037**

Table (I)

Xi	*Error*=$\lvert y - y_1 \rvert$	*Error*=$\lvert y - y_2 \rvert$	*Error*=$\lvert y - y_3 \rvert$	**Error**=$\lvert y - y_4 \rvert$
0.0	**0.133397406898439 *10^-5**	**0. 00000000 *10^-4**	**0.00000000**	**0 .00000000**
0.1	**0.051906822262637 *10^-5**	**0.79460191495961 *10^-4**	**18.63398343206*10^-4**	**0.19975953156059*10^-5**
0.2	**0.023378234970117*10^-5**	**1.58920382991923*10^-4**	**0.3294851656199249*10^-4**	**0.39951906314339 *10^-5**
0.3	**0.038323785123140*10^-5**	**2.38380574487884*10^-4**	**4.580778340239622*10^-4**	**0.59927859468178*10^-5**
0.4	**0.011259624943527 *10^-5**	**3.17840765983846*10^-4**	**0.6589703312509521*10^-4**	**0.79903812630899 *10^-5**
0.5	**0.040767224374072 *10^-5**	**3.97300957479807 *10^-4**	**2.894123971970419*10^-4**	**0.99879765786959 *10^-5**
0.6	**0.009450040927561 *10^-5**	**4.76761148975768 *10^-4**	**0.9884554968597747*10^-4**	**1.19855718940798 *10^-5**
0.7	**0.038811178537657*10^-5**	**5.56221340471730*10^-4**	**1.833315569170768*10^-4**	**1.39831672099078*10^-5**
0.8	**0.022271164112908 *10^-5**	**6.35681531967691*10^-4**	**1.317940662479700*10^-4**	**1.59807625259578 *10^-5**
0.9	**0.049587212824775 *10^-5**	**7.15141723463431 *10^-4**	**11.55017254960*10^-4**	**1.79783578415638 *10^-5**
1	**0.129332955389749*10^-5**	**7.94601914959392*10^-4**	**1.647425828099625*10^-4**	**1.9759531573918*10^-5**

Table (II)

Table(II) The error Comparison for proposed method, Quadrature and Power series method



Table(III,IV), the result and comparison to solve Fredholm integral equation by proposed method and three method

4-2 Example $u(x) = \frac{3}{2}e^{x} - \frac{1}{2}e^{x+2} + \int_0^1 e^{x+t}u(t)dt,$ (4.2)

x_i	***Exact solution*** ***n=6*** ***u(x)=e^x***	***Hermite method*** ***n=6 [10]*** ***u_1***	***Laguerre method*** ***n=6 [11]*** ***u_2***	***Legendre method*** ***n=6 [15]*** ***u_3***	***Cubic spline*** ***n=6*** ***(Proposed method)*** ***u_4***	**Proposed method n=10** **u_5**
0	**1.000000000000000**	**1.000000062462004**	**1.000062131858971**	**1.000062131858971**	**0.999589956349680**	**0.999909970631758**
0.10	**1.105170918075648**	**1.105170987106838**	**1.105239584379175**	**1.105239584379175**	**1.105785869192006**	**1.105071420236096**
0.20	**1.221402758160170**	**1.221402834451405**	**1.221478643579310**	**1.221478643579310**	**1.220615882196547**	**1.221292796041484**
0.30	**1.349858807576003**	**1.349858891889217**	**1.349942631734262**	**1.349942631734262**	**1.349177723467666**	**1.349737280640343**
0.40	**1.491824697641270**	**1.491824790793648**	**1.491917045376024**	**1.491917045376024**	**1.491251228760411**	**1.491690389606216**
0.50	**1.648721270700128**	**1.648721373400686**	**1.648822056070347**	**1.648822056070347**	**1.648045223011933**	**1.648572837365722**
0.60	**1.822118800390509**	**1.822118912441763**	**1.822226011255518**	**1.822226011255518**	**1.821612577902422**	**1.821954756186047**
0.70	**2.013752707470477**	**2.013752824943349**	**2.013859935143275**	**2.013859935143275**	**2.012544647493757**	**2.013571410586428**
0.80	**2.225540928492468**	**2.225541035609981**	**2.225633029681848**	**2.225633029681848**	**2.223834104673138**	**2.225340564448681**
0.90	**2.459603111156950**	**2.459603160207402**	**2.459649175581134**	**2.459649175581134**	**2.461513218651399**	**2.459381674642727**
1.00	**2.718281828459046**	**2.718281695362504**	**2.718224433400008**	**2.718224433400008**	**2.717167214255506**	**2.718037103263328**

TABLE(III)

TABLE(III)

Proposed method n=14 u_6
0.999966980911756
1.105067406950768
1.221369403582849
1.349815231577158
1.491775091528035
1.648666831426999
1.822057783200825
2.013688978866774
2.225486426133030
2.459339758173853
2.718192073271477

TABLE (IV)

TABLE(IV) The error Comparison proposed method, Hermite series ,Legendre series and Laguerre series methods

x_i	error= $\|u-u_1\|$ Hermite method	error= $\|u-u_2\|$ Laguerre Series method	error= $\|u-u_3\|$ Legendre method	error= $\|u-u_4\|$ Proposed method Cubic spline method	Proposed method(n=10) Error= $\|u-u_5\|$	Proposed method(n=14) Error= $\|u-u_6\|$
0.0	**0.0621*10^-3**	**0.0621*10^-3**	**0.06213185*10^-3**	**4.100436503200466*10^-4**	**9.002936824209495*10^-5**	**3.3019088244046*10^-5**
0.1	**0.0687*10^-3**	**0.0686*10^-3**	**0.068666303*10^-3**	**6.149511163582311*10^-4**	**9.949783955143232*10^-5**	**1.0351112487971*10^-4**
0.2	**0.0759 *10^-3**	**0.0758*10^-3**	**0.07588541914*10^-3**	**7.868759636229239*10^-4**	**1.099621186855249*10^-4**	**3.3354577320787*10^-5**
0.3	**0.0838*10^-3**	**0.0838*10^-3**	**0.08382415825*10^-3**	**6.810841083371155*10^-4**	**1.215269356606985*10^-4**	**4.9606113235345*10^-5**
0.4	**0.0923*10^-3**	**0.0923*10^-3**	**0.09234773475*10^-3**	**5.734688808594068*10^-4**	**1.343080350544312*10^-4**	**4 .9606113235345*10^-5**
0.5	**0.1008*10^-3**	**0.1007*10^-3**	**0.10078537021*10^-3**	**6.760476881952915*10^-4**	**1.484333344063948*10^-4**	**5.4439273129203*10^-5**
0.6	**0.1072 *10^-3**	**0.10721*10^-3**	**0.1072108650*10^-3**	**5.062224880869604*10^-4**	**1.640442044614776*10^-4**	**6.1017189683987*10^-5**
0.7	**0.1072*10^-3**	**0.1072*10^-3**	**0.10722767279*10^-3**	**0.001208059976720**	**1.812968840484430*10^-4**	**6.3728603702628*10^-5**
0.8	**0.0921*10^-3**	**0.09210*10^-3**	**0.092101189379*10-3**	**0.001706823819330**	**2.003640437866494*10^-4**	**5.4502359438046*10^-5**
0.9	**0.0461*10^-3**	**0.0460644241*10^-3**	**0.04606442418*10^-3**	**0.001910107494449**	**2.214365142232389*10^-4**	**2.6335298309687*10^-4**
1.0	**0.0574*10^-3**	**0.0573950590*10^-3**	**0.05739505903*10^-3**	**0.001114614203539**	**2.447251957171659*10^-4**	**8.9755187568485*10^-5**

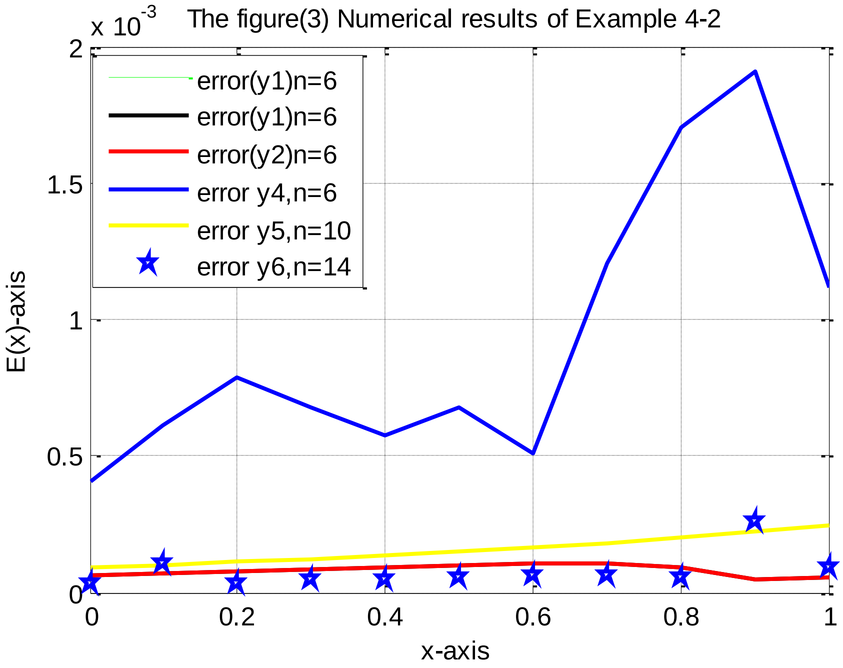


Table (V,VI) The result and comparison to solve Fredholm integral equation by proposed method and three methods

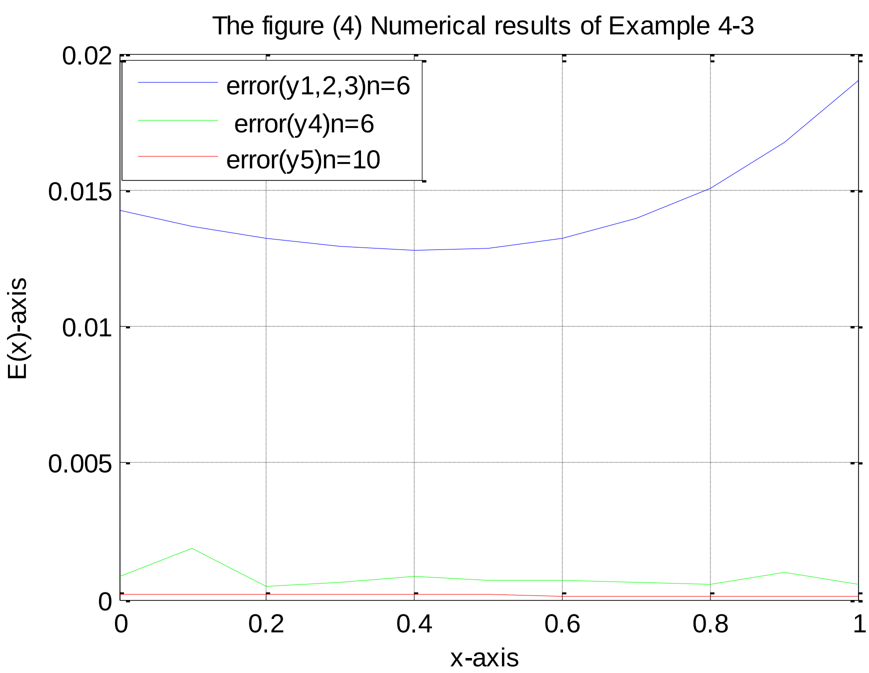
4-3 Example $\boldsymbol{y(x) = e^{-x} + (e^{-(x+1)} - 1)/(x+1) + \int_0^1 e^{-xt} y(t)dt}$, *(4.3)* *n=6*

x_i	Exact solution $y(x) = e^{-x}$	Hermite method y_1 [10]	Laguerre method y_2 [11]	Legendre method y_3 [15]	*Proposed method(n=6)* y_4	*Proposed method(n=10)* y_5
0	1.0000000000000000	1.014244839994024	1.014244839994024	1.014244839994024	1.000853839052898	1.000185837878875
0.1	0.904837418035960	0.918505011195214	0.918505011195214	0.918505011195214	0.906729702697537	0.905015444114686
0.2	0.818730753077982	0.831941477135035	0.831941477135035	0.831941477135035	0.819222569142787	0.818901473216434
0.3	0.740818220681718	0.753717830373594	0.753717830373594	0.753717830373594	0.741430149064003	0.740982103410917
0.4	0.670320046035639	0.683087589314705	0.683087589314705	0.683087589314705	0.671131867810740	0.670477525497076
0.5	0.606530659712633	0.619387780364968	0.619387780364968	0.619387780364968	0.607226089852042	0.606682138356829
0.6	0.548811636094027	0.562033582396085	0.562033582396085	0.562033582396085	0.549495152038271	0.548957487158587
0.7	0.496585303791410	0.510514033510390	0.510514033510390	0.510514033510390	0.497183357033947	0.496725873577076
0.8	0.449328964117222	0.464388800109605	0.464388800109605	0.464388800109605	0.449845738217441	0.449464574078845
0.9	0.406569659740599	0.423286008266821	0.423286008266821	0.423286008266821	0.407563394088538	0.406700608408517
1.0	0.367879441171442	0.386901137401696	0.386901137401696	0.386901137401696	0.368459643666753	0.368006005917422

Table (V)

x_i	Error = $\lvert y - y_i \rvert$ i=1,2,3 Hermite,laguerre,legendre series methods	Error = $\lvert y - y_4 \rvert$ n=6 proposed method	Error = $\lvert y - y_5 \rvert$ $n=10$ proposed method
0	0.014244839994024	8.538390528980422*10^-4	1.85837878875050 *10^-4
0.1	0.013667593159255	1.892284661578*10^-3	1.78026078726967 *10^-4
0.2	0.013210724057053	4.918160648051950*10^-4	1.70720138452207*10^-4
0.3	0.012899609691876	6.119283822850763*10^-4	1.63882729199538 *10^-4
0.4	0.012767543279065	8.118217751006407*10^-4	1.57479461436516 *10^-4
0.5	0.012857120652335	6.954301394085372*10^-4	1.51478644195180*10^-4
0.6	0.013221946302058	6.835159442445393*10^-4	1.45851064560332 *10^-4
0.7	0.013928729718980	5.980532425374774*10^-4	1.40569785666456 *10^-4
0.8	0.015059835992384	5.167741002194415*10^-4	1.35609961623429*10^-4
0.9	0.016716348526222	9.937343479388683*10^-4	1.30948667918129 *10^-4
1	0.019021696230254	5.802024953106555*10^-4	1.26564745979163*10^-4

TABLE(VI) The error Comparison between proposed method, Hermite ,Legendre and Laguerre methods.



5. CONCLUSIONS

In our paper we provided some analytical and approximation methods to solve integral equation of Fredholm .natural Cubic spline foundation have used to approximation the integral function and convert it in to polynomial . The result show in table (I),(II), (III),(IV),(V),(VI), competence the efficiency of the proposed method(cubic spline method) in comparison it with gather the power series methods , Hermite ,Legendre and Laguerre series methods . The result was satisfactory in comparison the suggested method and many examples of complete solutions in example (4-1),(4-2),(4-3),.

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7.REFERENCES

[1] E. Babolian and A. Hajikandi, The Approximate Solution of a Class of Fredholm Integral Equations with a Weakly Singular Kernel, Journal of Computational and Applied Mathematics 235, 1148 – 1159, (2011).

[2] M. Hol, A Symptotic Solution of Class of Strongly Singular Integral Equations, Society for Industrial and Applied Mathematics, 716-725 (1990).

[3] K. Maleknejad, N. Aghazadeh, M. Rabbani, Numerical solution of second kind Fredholm integral equations system by using a Taylor-series expansion method , Applied Mathematics and Computation 175 (2006) 1229–1234

[4] Kanekoa, projection method for solving fredholm integral equation of the second kind Applied numerical Mathematics 5 (1989) 333-344 North- Holland

[5]pallop huabsomboon,boriboon Noaprateep,Hideaki ,Kaneko, on Taylor-series expansion methods for the second kind integral equation ,Journal of computational-1472 and applied mathematics 234(2010)1466

[6] E. Babolian , A. Shahsavaran , Numerical solution of nonlinear Fredholm integral equations of the second kind using Haar wavelets, Journal of Computational and Applied Mathematics 225 (2009) 87–95

[7] Jalil Rashidinia, Esmail Babolian, Zahra Mahmoodi, Spline Collocation for Fredholm Integral Equations, Mathematical Sciences ,Vol. 5, No. 2 (2011) 147-158

[8] Habeeb Khudhur Kadhim, Reducing Initial Value Problem and Boundary Value Problem to Volterra and Fredholm Integral Equation and Solution of Initial Value Problem,international Jouranl and magazine of engineering ,technology, management and Resarch ,volume No:2(2015),Issue No:9(September)

[9] Farshid Mirzaee, Sima Piroozfar, Numerical solution of linear Fredholm integral equations via modified Simpson's quadrature rule, Journal of King Saud University (Science) (2011) 23,7–10

[10] S.Yalcinbas,M .Aynigul,Hermite series solution of linear Fredholm integral equation ,Mathematical and Computational Appllications,Vol.16,No.2,pp,497506,2011

[11]S.Yalcinbas,T.Akkaya,M.Aynigul,Laguerre series solution of fredholm integral equation,Erciyes University Journal of science and technology;26,2,131-142,2010

[12] S.S.sastry, introductory methods of numerical analysis ,Fifth edition by PHI Learning Private Limited, New Delhi (2012)

[13] N. Qatanani and Njood Asad A. Rihan , Numerical Treatment of The Fredholm Integral Equations of the Second Kind Applied Mathematics and Computation, 175, (2013)

[14] Adil. AL-Rammahi, Power Series Form for Solving Linear Fredholm Integral Equations of Second Order via Banach Fixed Point Theorem ,world Academy of science ,Engineering and Technology,International Journal of Mathematical,Computational science and Engineering Vol:8 No:1,2014

[15] S.Yalcinbas,M .Aynigul,T.Akkaya,The Legendre series solution of Fredholm Integral equations,Math.and Comput.App.,15,3,371-381,2010

[16] R.Kanwal, Linear Integral Equations, Theory and Techniques, Academic Press, INC, New York, (1971)

[17] E. Kreyszig, Introductory, Functional Analysis and Applications, John Wiley and Sons, (1978).

[18] J. Jerry, "Introduction to Integral Equations with Applications", Marcel Dekker, INC, 1985.

[19] Jerri, Abdul J. Integral and discrete transforms with applications and error analysis , by MARCEL DEKKER, INC. (1992)

[20] K. Atkinson and W. Han, Theoretical Numerical Analysis: A Functional Analysis Framework, 2nd edition, Springer-Verlag. New York, (2005)

[21] Abdul-Majid Wazwaz, Linear and Nonlinear Integral Equations: Methods and Applications, Springer Heidelberg, Dordrechi London, (2011)