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Perturbations of Weyl's Theorems for Unbounded Class-A Operators

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Abstract— The aim of this article is to continue study more spectral properties for class-$\boldsymbol{A_N}$ operators. The Search has been focused on define new properties for closed linear operators. These new properties posed on class –A_N, A. The properties are defined and proved for A. So, the research achieve the goal. Also, the connection between the properties are established.

Keywords. Hyponormal operator, Posinormal operator, Fredholm operators, Fredholm operator, Weyl's theorems

I. INTRODUCTION AND DEFINITION

In 1909, H. Weyl tasted the spectrum of all compact perturbations of self-adjoint operators in Hilbert space and found that their intersections consisted of points in a spectrum of finite multiplicity that were not isolated eigenvalues. The bounded linear operators that satisfy this property are said to satisfy Weyl's theorem [1]. Subsequently, Berkani and Weyl introduced some variants of Weyl's theorem, this study is commonly known as the Weyl type's theories including the a-Weyl's theorem. In 2004, researchers presented the Browder's theorem and a-Browder theorem as generalizations of a-Weyl's theorem [2]. Some spectrum properties has been studied for classes of operators that are bounded (see [3], [4], [5] and [6]).

Recently, many investigators start to study the operators which are unbounded in the infinity Hilbert space. They examined the Hyponormal operator, Posinormal operator and class-A operators [7], [8] and [9] respectively. In Hilbert spaces, the theory of hyponormal operators has advanced significantly [10]. Analysts suggested applying the concept of hyponormality of unbounded operators in various cases. It turns out that limited and unbounded hyponormal operators have several characteristics. Newly, the researchers worked in many spaces of unbounded linear operators, including L^2 (see [11]), L^p (see [12]), Pick space (see [13]) and other investigations can be located in [14].

Throughout this work, $\mathcal{H}$ represents infinitely complex Hilbert space, and $C(\mathcal{H})$ is the set of all linear closed operates on $\mathcal{H}$. For this purpose, we need the following preliminaries.

For a $A \in C(\mathcal{H})$, define $N(A)$ as the null space of A, D(A) dnoted as the domain and $R(A)$ as range of A. $\alpha(A)$ is the dimension of the kernel of A and $B(A)$ is the codimention of the range of A. The upper semi Fredholm operator is defined if $R(A)$ is closed and $\alpha(A)$ is finite while we say that A is lower semi Fredholm operator if $B(A)$ is finite. Define the set $\Lambda_+(\mathcal{H}) = \{A \in C(\mathcal{H}) : \alpha(A) < \infty \textit{ and } R(A) \textit{ is close}\}$ and $\Lambda_-(\mathcal{H}) = \{A \in C(\mathcal{H}) : B(A) < \infty\}$. A Fredholm operator is denoted by $\Lambda(\mathcal{H}) = \Lambda_+(\mathcal{H}) \cap \Lambda_-(\mathcal{H})$.The *index* of A is defined as $ind\ (A) = \alpha(A) - B(A)$.

Recall that a closed linear operator A is defined as Weyl operator if $A \in \Lambda(\mathcal{H})$ and in d$A = 0$, while $\sigma_w\ (A) = \{\eta \in \mathbb{C} : A - \eta I$ is not weyl$\}$ is used to define the Weyl spectrum of A. In addition we have the following notations:

$$\Lambda_+^-(\mathcal{H}) = \{A \in C(\mathcal{H}) : A \in \Lambda_+(\mathcal{H}),\ ind(A) \leqslant 0\}$$
$$\Lambda_-^+(\mathcal{H}) = \{A \in C(\mathcal{H}) : A \in \Lambda_-(\mathcal{H}),\ ind\ (A) \geqslant 0\}$$

The B-Fredholm operators can be defined as follows: $\Delta(A) = \{k \in \mathbb{N} : \forall\, l \in \mathbb{N}, l \geqslant k \Rightarrow R(A^k) \cap N(A) \subseteq R(A^l) \cap N(A)\}$.

The degree of stable iteration of A is denoted by dis(A) and defined by $\text{dis}(A) = \inf \Delta(A)$ and $\text{dis}(A) = \infty$, when $\Delta(A) = \emptyset$. For $A \in C(\mathcal{H})$, we say A is a B-Fredholm, if there exists $k \in \Delta(A)$ such that A is upper semi B-Fredholm $B\Lambda_+$ and lower semi B-Fredholm $B\Lambda_-$ satisfying $\dim\{\mathcal{N}(A) \cap \mathcal{R}(A^k)\} < \infty$ and $\text{codim}\{\mathcal{R}(A) + \mathcal{N}(A^k)\} < \infty$.

The *index* of A is:

$$\text{ind}(A) = \dim\{\mathcal{N}(A) \cap \mathcal{R}(A^k)\} - \text{codim}\{\mathcal{R}(A) + \mathcal{N}(A^k)\}$$

Recall that the ascent $M := \mathcal{M}(A)$ of a linear operator A is the smallest positive integer n such that $(A^n) = Ker(A^{n+1})$, if $\mathcal{M}(A) = \emptyset$ we put $\mathcal{M}(A) = \infty$. Yet, the descent $N := \aleph(A)$ of an operator A is the smallest positive $m \in \mathbb{N}$ such that $R^m(A) = R^{m+1}(A)$, and if $\aleph(A) = \emptyset$ we put $\aleph(A) = \infty$.

We call $A \in C(\mathcal{H})$ as $B - weyl$ if it's B-$Fredholm$ operator with $index$ 0 and $\sigma_{BW}(A)$ is used to symbolize the B-$weyl$ spectrum of A and defined by $\sigma_{\text{BW}}\ (A) = \{\eta \in \mathbb{C} : A - \eta I$ is not B-wey $\}$.

Once can define the class of all upper semi-Browder operators as $\mathcal{B}_+(\mathcal{H}) := \{A \in \Lambda_+(\mathcal{H}) : \aleph(A) < \infty\}$, while the class of all lower semi-Browder operators is define by $\mathcal{B}_-(\mathcal{H}) := \{A \in \Lambda_-(\mathcal{H}) : q(A) < \infty\}$.

The class of all Browder operators is defined by $\mathcal{B}(\mathcal{H}) = \mathcal{B}_+(\mathcal{H}) \cap \mathcal{B}_-(\mathcal{H})$. Obviously, if A is Weyl if A is Browder, since if $\mathcal{M}(A)$ and $\aleph(A)$ are finite implies, for a Fredholm operator with index zero.

We can define the upper semi-Browder spectrum, the lower semi-Browder and the Browder spectrum of $A \in C(\mathcal{H})$ by: $\sigma_{\text{ub}}(A) := \{\eta \in \mathbb{C}: \eta I - A \notin \mathcal{B}_+(\mathcal{H})\}, \sigma_{\text{lb}}(A) := \{\eta \in \mathbb{C}: \eta I - A \notin \mathcal{B}_-(\mathcal{H})\}, \sigma_{\text{b}}(A) := \{\eta \in \mathbb{C}: \eta I - A \notin \mathcal{B}(\mathcal{H})\}$ respectively.

Most significant particularity for an operator that belongs to $C(\mathcal{H})$ is the single value extension property which denoted by SVEP: Assume $A: \mathrm{D}(A) \to \mathcal{H}$ be a closed linear operator operates on $\mathcal{H}$, and let $\eta_0 \in \mathbb{C}$. Then A has SVEP at η_0 if $g = 0$ is the only solution to $(\eta I - A)g(\eta) = 0$ where g is holomorphic in a neighborhood of η_0. Moreover, we say A has the SVEP if it has it at every $\eta_0 \in \mathbb{C}$.

In the sequel, we simplify $\sigma(A), \sigma_a(A)$, and $\rho(A)$ to represent the spectrum, approximate spectrum and the resolvent set of $A \in C(\mathcal{H})$, respectively. Also, let iso $\sigma(A)$ and iso $\sigma_a(A)$ be the isolated points of $\sigma(A)$ and $\sigma_a(A)$, respectively. Also, we need to define the following concepts: $\pi(A) = \{\eta \in \text{ iso } \sigma(\mathrm{A}): 0 < \mathcal{M}(A - \eta I) = \aleph(A - \eta I) < \infty\}$, $\pi_o(A) = \{\eta \in \pi(A): \alpha(A - \eta I) < \infty\}$, $\pi^a(A) = \{\eta \in \sigma_a(A): M = \mathcal{M}(A - \eta I) < \infty, \mathcal{R}(A - \eta I)^{m+1} \text{ is closed }\}$, $\pi_o^a(A) = \{\eta \in \pi^a(A): \alpha(A - \eta I) < \infty\}$, $E(A) = \{\eta \in \text{ iso } \sigma(A): 0 < \alpha(A - \eta I)\}$, $E_o(A) = \{\eta \in E(A): \alpha(A - \eta I) < \infty\}$ and $E_o^a(A) = \{\eta \in \text{ iso } \sigma_a(A): 0 < \alpha(A - \eta I) < \infty\}$.

An operator $A \in C(\mathcal{H})$ with $\mathcal{D}(A) = \mathcal{D}(A^*)$ is define as class- A operator if $|A|^2 \leq |A^2|$, where $|A| = (A^*A)^{\frac{1}{2}}$ [5]. Moreover, for A belong to class-A operator, then $\mathcal{M}(A - \eta I)$ is less than or equal 1 for every $\eta \in \mathbb{C}$ [5].

A Polaroid operator define if every isolated point of the spectrum is a pole of $\rho(A)$. A class $\mathcal{N}$ of operators defines as $\mathcal{N} = \{A \in C(\mathcal{H}) : \sigma(A|_V - \eta I) = \{0\}\ i.e., (A|_V - \eta I) := 0, \forall$ invariant subspace V of $\mathcal{H}\}$ and denote by class- $A_{\mathcal{N}}$[5]. In every spectrum of an operator $A \in$ class- $A_{\mathcal{N}}$ every η is isolated point if and only if η is a simple pole of $\rho(A)$. This property means that every A in class-$A_{\mathcal{N}}$ is Polaroid [5]. Finally, let $A \in C(\mathcal{H})$, then A satisfies:

(gw) Property; if $\mathrm{E(A)} = \sigma_a(\mathrm{A}) \setminus \sigma_{B\Lambda_+^-}(\mathrm{A})$,

(b) Property; if $\pi_0(A) = \sigma_a(\mathrm{A}) \setminus \sigma_{\Lambda_+^-}(\mathrm{A})$,

(gb) Property; if $\pi(\mathrm{T}) = \sigma_a(\mathrm{T}) \setminus \sigma_{B\Lambda_+^-}(\mathrm{A})$.

In this paper, we continue the study of Weyl's theorems for the unbounded class- $A_{\mathcal{N}}$ operators, we first use property (gb) to examine property (b), and show that prepare (gb) and property (gw) are equal. Then start exploring the new spectral properties defined for the constrained operator. Starting with the (ao) and (asz) properties, for every A that belongs to class A, indicate that A has them. Finally, we resumed our objectives, writing the last two properties (0) and (sz), proving them with A, and establishing a connection between these properties.

II. APPROXIMATE POINT VARIANTS OF WEYL'S THEOREMS.

Now, we look at some approximation versions of the $A \in$ class-$A_{\mathcal{N}}$ and its adjoint operators, introduced as a variant of Weyl's classical theorem.

Remark 2.1. Let $A \in C(\mathcal{H})$ with $\alpha(\mathbf{A}) < \infty$. If $A \in B\Lambda_+(\mathcal{H})$ then $A \in \Lambda_+(\mathcal{H})$.

Theorem 2.2. If A is a class-A_N operator with $\alpha(A) < \infty$, then A possess property (b) if A possess property (gb)

Proof: Suppose A possess property(gb), then,$\sigma_a(A)\setminus \sigma_{B\Lambda_+}(A) = \pi(A)$, let $\eta \in \dot{\sigma_a}(A)\setminus\sigma_{\Lambda_+}(A)$, then, $\alpha(A - \eta I) < \infty$. Now, since A possess (gb) property we have η is a pole of $\rho(A)$ then η is isolated in spectrum of A, since $A \in$ class A_N then A is poleriod i.e. $\eta \in \pi^\circ(A)$. For the reverse inclusion, let $\eta \in \pi_0(A)$, then, $\eta \in \pi(A)$, since A possess (gb) property then $\eta \in \sigma_a(A)\setminus\sigma_{B\Lambda_+}(A)$, by Remark 2.1, we have $\eta \notin \sigma_{\Lambda_+}(A)$. Thus A possess (b) property.

Proposition 2.3. Let $A \in$ class-A_N. Then A admits property (gw) if and only if A possesses (gb) property.

Proof: As consequence of [5]-Lemma1, we have $\pi(A) = E(A)$, and then the proof is completed.

Theorem 2.4. Let $A \in$ class- A_N, and A^* has SVEP at η. If A posess property (gw) then A satisfies generelized Weyl's theorem.

Proof: Let $\eta \in \sigma(A)\setminus\sigma_{BW}(A)$, since $\sigma_{B\Lambda_+^-}(A) \subseteq \sigma_{BW}(A)$, then, $\eta \notin \sigma_{B\Lambda_+^-}(A)$. In addition, A^* has SVEP at η, then $\sigma(A) = \sigma_\mathrm{a}(A)$ so that $\eta \in \sigma_a(A) \setminus \sigma_{B\Lambda_+^-}(A)$. But, A admits (gw) property, thus $\eta \in E(A)$. Now, let $\eta \in E(T) = \{\eta \in \text{iso}\,\sigma(A): 0 < \alpha(A - \eta \mathrm{I})\}$. Since A has (gw) property with A^* has SVEP with $\eta \notin \sigma_{B\Lambda_+^-}(A)$, thus $\eta \notin \sigma_{BW}(A)$ so that A satisfies generalized Weyl's theorem.

III. NEW SPECTRAL PROPERTIES FOR UNBOUNDED CLASS-A_N

In this section, we start to define new spectral properties for the unbounded class-A_N, these properties were define to the bounded case [11], we relate them to some variant of Weyl's theorems and established the relation between given ideas.

Definition 3.1. An operator $A \in C(\mathcal{H})$ is said to have property (ao) if $\sigma(A)\setminus\sigma_{\Lambda_+^-}(A) = \pi_a(A)$.

Remark 3.2. If $A \in$ class-A_N and A^* has SVEP then A satisfies (ao) property.

Proof: since $A \in$ class-A_N and A^* has SVEP, then A satisfies a-Browder's theorem by [Theorem 6, 5]. Now, let $\eta \in \sigma_a(A)\setminus\sigma_{\Lambda_+^-}(A) = \pi_a^0(A)$but $\sigma_a(A) \subseteq \sigma(A)$ and $\pi_a^0 \subseteq \pi_a$ thus, $\eta \in \sigma(A)\setminus\sigma_{\Lambda_+^-}(A) = \pi_a(A)$ then, A possess (ao) property.

Definition 3.3. An operator $A \in C(\mathcal{H})$ is possess property (asz) if $\sigma(A)\setminus\sigma_{\Lambda_+^-}(A) = \pi(A)$.

Remark 3.4. If A or A^* is a class-A_N operator with $\alpha(A) < \infty$, then A satisfies (asz) property.

Proof: since A or A^* is a class-A_N operator then A satisfies generalized Weyl's theorem and generalized Browder's theorem and these theorems are equivalent [5]: $\sigma(A)\backslash \sigma_{BW}(A) = E(A)$ by Remark 2.1. We get A-ηI is upper semi-Fredholm operator, then $\lambda \notin \sigma_{\Lambda_+^-}$(A). Since, A satisfies generalized Weyl's theorem and $\eta \in E(A)$ then $\eta \in$ iso$\sigma(A)$, also, $E(A) = \pi(A)$, then, $\eta \in \sigma(A) \mid \sigma_{\Lambda_+^-}(A) = \pi(A)$ thus, A satisfies (asz) property.

Definition 3.5. An operator $A \in C(\mathcal{H})$ is said to satisfies (0) property if $\sigma(A) \setminus \sigma_{\Lambda_+^-}(A) = E_a(A)$

Proposition 3.6. If $A \in$ class-A_N and A^* has SVEP, then A own property (o)

Proof: since $A \in$ class-A_N and since A^* has SVEP then A satisfies a-Weyl's theorem [5] : $\sigma_a(A)\backslash\sigma_{\Lambda_+^-}(A) = E_a^0(A)$. Since $\sigma_a(A) \subseteq \sigma(A)$ and for all $\eta \in E_a^0(A)$ we have $\eta \in E_a(A)$, from all of that, we get A own property (0).

Definition 3.7. An operator $A \in C(H)$ is said to possess property (sz) if $\sigma(A) \setminus \sigma_{\Lambda_+^-}(A) = E(A)$

Proposition 3.8. If $A \in$ class-A_N then it has the property(sz).

Proof: since $A \in$ class-A_N, then A satisfies generalized Weyl's theorem: $\sigma(A)\backslash\sigma_{BW}(A) = E(T)$ [5] and by Remark 2.1. σ_{BW} (A) $\subseteq \sigma_{\Lambda_+^-}$(A) then $\eta \notin \sigma_{\Lambda_+^-}$(A)thus A satisfies (sz) property.

Remark 3.9. Let $A \in$ class-A_N and A^* has SVEP with $\sigma(A) = \sigma_a(A)$. Then the properties (0) and (sz) are equivalent.

Proof: suppose A has (o) property then $\sigma(A) \setminus \sigma_{\Lambda_+^-}(A) = E_a(A)$, since by assumption $\sigma(A) = \sigma_a(A)$, then $E(A) = E_a(A)$, thus A possess (sz). Conversely, we have the similar conclusion.

IV. CONCLUSION

Spectral properties are defined in large for many classes of bounded operators, also, the study of Weyl's theorems were limited to the bounded case. In this paper we state some of these properties to the operators that are unbounded and belong to class AN. Moreover, we established the relation and equivalent among these properties.

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REFERENCES

[1] Weyl, H.: Über beschränkte quadatische Formen, deren Differenz Vollstetig ist. Rend. Circ. Mat. Palermo 1909 27, 373-392.

[2] Berkani, M. and Arroud,A.: Generalized Weyl's theorem and hyponormal operators. J. Aust. Math. Soc. 2004 76 (2), 291-302.

[3] Rashid, Z. A. ., & Ahmed, B. A. . Some Properties of Algebraically Paranormal Operator. Iraqi Journal ofScience, (2022),63(8),3587–3591.

[4] Khaleefah, S.A. and Ahmed, B.A.,Spectrum of Soft Compact Linear Operator with Properties. In Journal of Physics: Conference Series, 2020, Vol. 1530, No. 1, p. 012107.

[5] Radhi I. Mohammed, Fatin A. Ahmed, Exponential Function of a bounded Linear Operator on a Hilbert Space Baghdad Science Journal, 2014, Vol.11(3), p. 1267-1273

[6] Gupta, A., Mamtani, K.: Weyl type theorems for unbounded hyponormal operators. Kyungpook Math. J. 2005 55(3), 531-540.

[7] Gupta, A., Mamtani, K.: Weyl type theorems for unbounded posinormal operators. Journal of Contemporary Mathematical Analysis, 2017 52 (4), 191-197.

[8] Gupta, A., Mamtani, K.: Weyl type theorems for unbounded class- A operators.

[9] Janas, Jan. "On unbounded hyponormal operators." Arkiv for Matematik 27, no. 1 (1989): 273-281.

[10] Budzynski, Piotr, Zenon Jablonski, Il Bong Jung, and Jan Stochel. "Unbounded Weighted Composition Operators in L^2-Spaces." (2018).

[11] Liu, Xiao-Feng, and Yousef Estaremi. "Basic properties of unbounded weighted conditional type operators." Filomat 35, no. 2 (2021): 367-379.

[12] Jury, Michael T., and Robert TW Martin. "Unbounded multipliers of complete Pick spaces." Integral Equations and Operator Theory 94, no. 2 (2022): 1-28.

[13] Mortad, Mohammed Hichem. "Unbounded Nonnormal Operators." In Counterexamples in Operator Theory, pp. 457-464. Birkhauser, Cham, 2022. African Mathematical Union and Springer-Verlag Berlin Heidelberg, 2017 DOI 10.1007/s13370-017-0479-7

[14] Buthainah A.A. Ahmed, Zaman A., New Properties and Their Relation to type Weyl-Browder Theorems, IJSR, 2008 2, 1502 – 1506.