Weiner Topological Index for Neutrosophic Graph based on Strong Domination Set and Number

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Abstract — *Different topological indices will always be valuable in a variety of disciplines, including chemistry, economics, electronics, business studies, social sciences and medicine. According to the dominating set and number, the Wiener index for neutrosophic graphs (NG) will be studied in considerable detail in this research. Additionally, the Average Wiener index for an NG has been defined, and several associated theorems have been introduced*.

Keywords—: single valued NG, Domination set, topological index, strong path)

I. Introduction

Zadeh [1] originally put forward the theory of fuzzy sets in 1965, while Rosenfeld [2] first established the idea of fuzzy graphs in 1975.Since then, numerous studies on the characteristics and uses of fuzzy graphs have been conducted. Calculating distance-based topological indices and degree-based indices in fuzzy graphs was one of these issues.
These indicators assist by giving each graph a numerical value so that we may compare graphs with the same number of vertices using an appropriate standard. Atanassov [3] then put out the notion of intuitionistic fuzzy sets after that. Finally, new sets known as neutrosophic sets were created with the broadening of fuzzy theory by Smarandache [4] in 1995.Researchers attempted to offer additional mathematical ideas into this sector by providing this hypothesis. One of these was the idea of graphs, from which the new idea of neutrosophic graphs emerged. Neutrosophic graphs contain a variety of properties and uses that have recently been suggested by theorists in this area.
A topological index is a numerical measurement for a molecule's structural graph. The topological indices are typically well-known in chemistry, but Harold, who invented the graph structure in his theory and built a sizable branch of chemical graph theory, has a mathematical background. Paraffin-type alkanes' characteristics can be discovered using Wiener use indices
The Wiener index W(G) was initially defined by Harold Wiener in 1947 as " he sum of the distance between all the couple of vertices in a graph G " [5]. In comparison to traditional models, neutrosophic graphs offer systems that are more precise and flexible. Additionally, it has applications outside of the field of chemistry, including computer science, networking, human traffic, and internet routing. Studying dominance inside graph and neutrosophic graphs may be expanding at a rapid rate due to its numerous and varied uses in these fields.Proffesor. Samapathkumar and L. Pushpa Latha introduced the notion of strong domination in graphs [6]There are several kinds of domination depending on the type, such as strong dominance in neutrosophic graphs, which inspired us to discover the Wiener index because of the work done on domination on neutrosophic graphs and its potential application. If the number is large, it is challenging to determine the Wiener index using the standard methods.

II. Preliminary

Definition2.1 [7] A graph $G_N = (A, B)$ is a single-valued Neutrosophic graph(SVNG), where A is a neutrosophic set(NS) on V and, B is a(NS) on E, which satisfy the following

$$T_B(u,v) \leq \min(T_A(u), T_A(v))$$
$$I_B(u,v) \leq \min(I_A(u), I_A(v))$$
$$F_B(u,v) \geq \min(F_A(u), F_A(u))$$

Where u and v are two vertices of G_N, and $(u, v) \in E$ is an edge of G_N.

Definition2.2[8] **:** In a neutrosophic graph $G_N = (A, B)$, **a path** P of length n is a sequence of distinct

vertices $(u_0, u_1, \ldots, u_n$ such that
$T_B(u_i, u_{i+1})$, $I_B(u_i, u_{i+1})$, $F_B(u_i, u_{i+1})$ > 0, i = 1, 2,…,n

Definition2.3[9] **:** if $G_N = (A, B)$ be a neutrosophic graph on a simple underline graph $G_N^* = (V, E)$ then

1) $O_N = (\sum T_A(u), \sum I_A(u), \sum F_A(u)), \forall u \in V$ is called a neutrosophic order of G_N
2) $S_N = (\sum T_B(u,v), \sum I_B(u,v), \sum F_B(u,v)) \forall\, e = uv \in E$ is called a neutrosophic size of G_N
3) $Score(z) = \frac{1+T_A(Z)+I_A(Z)-F_A(Z)}{3}$

*Definition 2.4 ([10]***).** An edge $(v_i, v_j)$is categorized in a neutrosophic graph $G_N = (A, B)$ as:

1- $\propto -strong\ \ if$

$$T_B(v_i, v_j) > CONN_{T(G)-(v_i,v_j)}(v_i, v_j),$$
$$I_B(v_i, v_j) > CONN_{I(G)-(v_i,v_j)}(v_i, v_j)\ \ and$$
$$F_B(v_i, v_j) < CONN_{F(G)-(v_i,v_j)}(v_i, v_j)$$

2- $\beta - strong\ \ if$

$$T_B(v_i, v_j) = CONN_{T(G)-(v_i,v_j)}(v_i, v_j),$$
$$I_B(v_i, v_j) = CONN_{T(G)-(v_i,v_j)}(v_i, v_j)\ and$$
$$F_B(v_i, v_j) = CONN_{F(G)-(v_i,v_j)}(v_i, v_j)$$

3- $\delta - strong\ \ if$

$$T_B(v_i, v_j) < CONN_{T(G)-(v_i,v_j)}(v_i, v_j),$$
$$I_B(v_i, v_j) < CONN_{T(G)-(v_i,v_j)}(v_i, v_j)\ \ and$$
$$F_B(v_i, v_j) > CONN_{F(G)-(v_i,v_j)}(v_i, v_j)$$

Definition 2.5 **.***L*(*P*) is defined as the sum of the weights of the edges in *P;*

$$L(P) = (\sum_{i=1}^{m} T_B(u_i, u_{i+1}), \sum_{i=1}^{m} I_B(u_i, u_{i+1}), \sum_{i=1}^{m} F_B(u_i, u_{i+1}))$$

For any two vertices *u* and *v* in G_N.
let $p = \{P_i : P_i\ is\ a\ u - v\ \ path\ i = 1,2,3, \ldots n\}$.
The sum distance between *u* and *v* is defined as $\boldsymbol{d_s(u,v)} = \min\{L(p_i) : p_i \in P, i = 1,2, \ldots\}$
Let G_N be a connected neutrosophic graph, the Wiener index $W(G_N)$ of *G* is defined by

$$W(G_N) = \sum_{u,v \in V(G)} d_s(u,v)$$

The average distance $\mu(G_N)$ between the vertices of *G* is defined by

$$\mu(G) = \frac{W(G_N)}{C_2^n}$$

The Wiener index of neutrosophic path *P* is defined by

$$W(P) = \sum_{i=1}^{n-1} i(n-i)(T_B(u_i, u_{i+1}), I_B(u_i, u_{i+1}), F_B(u_i, u_{i+1}))$$

Let $G_N = (A, B)$ be a neutrosophic graph on *V*, let $u, v \in V$,then u is said to dominates v in G_N if edge e=uv be strong edge.

A set *D* of vertices of G_N is strong dominating set of G_N if every vertex of $V(G_N)$ - *D* is a strong neighbor of some vertex in *D*. The weight of a strong dominating set *D* is defined as

$$\omega(G) = (\sum_{u \in D} T_B(u,v), \sum_{u \in D} I_B(u,v), \sum_{u \in D} F_B(u,v)),$$

where $T_B(u,v)$, $I_B(u,v)$ are the minimum of theo(truth and indeterminacy)memberships values (weights)respectively , and $F_B(u,v)$ is the maximum of the false membership values of the strong edge's incident on *u*. The strong domination number of a neutrosophic graph G_N defined as the minimum weight of strong dominating sets of G_N and it is denoted by $\gamma_s(G_N)$ or simply γ_s.

III. WIENER INDEX OF NEUTROSOPHIC GRAPH BY USING STRONG DOMINATION

*Definition 3.1***.** *Let G_N be neutrosophic graph;*
1*: If G_N is not strong neutrosophic cycle. Then*

$$(\sum_{u,v \in V} T_B(u,v), \sum_{u,v \in V} I_B(u,v), \sum_{u,v \in V} F_B(u,v)) \geq \gamma_s.$$

2*: If G_N is a strong neutrosophic cycle, Then*

$$(\sum_{u,v \in V} T_B(u,v), \sum_{u,v \in V} I_B(u,v), \sum_{u,v \in V} F_B(u,v)) \leq \gamma_s.$$

Theorem 3.2. for any neutrosophic path p,
$W(P) \leq 2m * q - (m+1) * Sc(\gamma_s(P)).\ where\ m = |E|$ *and q=Score(*S_N*),* $S_N\ is\ a\ neutrosophic\ size\ of\ G_N$

Proof: Let *G* be a neutrosophic path with length *L*.Then the sum strong distance between *u* and *w*
defined as $d_s(u,w) = \min\{L(Pi);\ Pi \in P;\ i = 1, 2, \ldots\}$.
So $d_s(u,w) \leq L(P) =$
$(\sum_{i=1}^{m} T_B(u_i, u_{i+1}), \sum_{i=1}^{m} I_B(u_i, u_{i+1}), \sum_{i=1}^{m} F_B(u_i, u_{i+1})) \leq q.$
Taken sum

$$\sum_{i=1}^{m} d_s(u,v) \leq mq \ldots\ldots\ldots\ldots..(1)$$

Let *D* be strong dominating set of *P*.
Then $\gamma_s(p) \leq \frac{q}{2}$

$$\Rightarrow (m+2) \leq \frac{(m+2)q}{2} \leq mq \ldots\ldots\ldots..(2)$$

From (1) and (2),we get
$\sum_{i=1}^{m} d_s(u,v) \leq 2mq - (m+2)\gamma_s \ldots\ldots\ldots\ldots(3)$
By triangle equality $d_s(u,v) \leq d_s(u,\omega) + d_s(\omega,v)$ we have

$$\sum_{i=1}^{m} d_s(u,v) \leq \sum_{i=1}^{m} d_s(u,\omega) + \sum_{i=1}^{m} d_s(\omega,v) \ldots\ldots\ldots\ldots..(4)$$

Now we assume wv be a strong edge with minimum wight adjacent to at all vertices in strong dominating set of G. Then $d_s(\omega,v) = (T_B(\omega,v), I_B(\omega,v), F_B(\omega,v))$.Therefore,

$$\sum_{i=1}^{m} d_s(u,\omega) = \gamma_s \ldots\ldots\ldots\ldots..(5)$$

Substitute from (3) and (5) in (4)*;* we obtain

$$W(P) \leq 2m * q - (m+1) * Sc(\gamma_s(P)).$$

Example:3.2.1 consider the G_N = (A,B) be path neutrosophic graph, with $v(1,1,0)$ for all $v \in V(G)$ gevin in the Fig. 1*:*

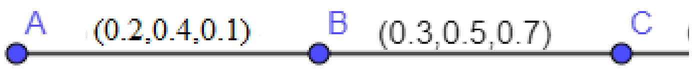

Figure1: neutrosophic path

In figure 1; the vertex subsets $D_{s1} = \{b,d\}$ $D_{s2} = \{a,d\}$. $D_{s3} = \{a,c\}, D_{s4} = \{b,c\}$ *are strong dominating set of G. Hence* γ_s = *min*$\{\gamma_{s1}, \gamma_{s2}, \gamma_{s3}, \gamma_{s4}\}$ *so* γ_s (*G*) = 0.3 *and* $m = |E| = 3$.

Since

γ_s =*min*$\{(0.4,0.8,1.2), (0.6,1.2,1.8), (0.5,0.9,1.3), (0.3,0.5.0.7)\}$ $= (0.6,1.2,1.8)$

Then the Wiener index of G

$$W(P) \leq 2 * 3 * (0.6,1.2,1) - (3+1) * (0.3,0.5,0.7)$$
=*1.6-0.13333=1.46667*

Also, the average of a path P is

$$\mu(G) \leq \frac{W(G)}{C_2^n} = \frac{2W(G)}{n(n-1)} = \frac{1.46667}{4*3} = 0.122222$$

Theorem:3.3 *for any strong cycle neutrosophic graph,*

$$W(C_n) \leq \frac{m(m*q - Sc(\gamma_s))}{8} + \frac{Sc(\gamma_s)}{2}, m = |E|$$

Proof: Let *G* be a strong cycle neutrosophic graph, the maximum length of geodesic in $Cn = \frac{m}{2}$

$$So\ d_s(u,w) = \frac{mq1}{4}\ for\ u,\omega \in P_{\frac{m}{2}}\ Therefore, q_1 = \frac{q}{2}$$

$Taken\ the\ sum, we\ have$

$$\sum_{i=1}^{m} d_s(u,\omega) = \frac{m^2q}{16} \dots \dots \dots (1)$$

Since $\gamma_s \leq \frac{q}{2}$, *then* $\frac{m}{8}\gamma_s \leq \frac{mq}{16} \leq \frac{m^2q}{16} \dots \dots \dots .(2)$

From (1) and (2) , we obtain,

$$\sum_{(u,w)\in p_{\frac{m}{2}}} d_s(u,\omega) \leq \frac{m(mq-\gamma_s)}{8} \dots \dots \dots (3)$$

Since $d_s(u,v) \leq d_s(u,w) + d_s(w,v)$,

we have $\sum_{i=1}^{m} d_s(u,v) \leq \sum_{i=1}^{m} d_s(u,\omega) + \sum_{i=1}^{m} d_s(\omega,v)$

We assume wv be strong edge with minimum wight adjacent to at all vertices in dominating set *Ds* of $P_{\frac{m}{2}}$, Then $d_s(\omega,v) = \mu(\omega,v)$ which implies

$$\sum_{i=1}^{m} d_s(\omega,v) = \frac{\gamma_s}{2} \dots \dots \dots (5)$$

Substitute from (5) and (3) in (4)*;* we obtain

$$W(C_n) \leq \frac{m(m*q - Sc(\gamma_s))}{8} + \frac{Sc(\gamma_s)}{2}$$

Example 3.3.1 Consider the strong cycle neutrosophic graph G_N = (*A*,*B*), *with* $v(1,1,0)$ *for all* $v \in V(G)$,*and all edges are* ∝-*strong gevin in the Fig.* 2.

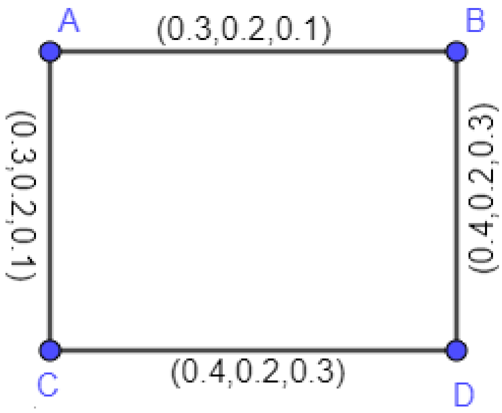

Figure 2

In Fig. 2*: we have the strong dominating set* $D = \{a,b\}, q = (1.4,0.8,0.8)\gamma(G) = (0.6,0.4,0.2)$ *and* $m = |E|$. *Then the Wiener index of G*

$$W(C_n) \leq \frac{m(m*q - Sc(\gamma_s))}{8} + \frac{Sc(\gamma_{s)}}{2}$$
$$= \frac{4(4 * Sc(1.4,0.8,0.8) - Sc(0.6,0.4,0.2)}{8} + \frac{Sc(0.6,0.4,0.2)}{2}$$
$$= \frac{4(0.46667-0.2666)}{8} + \frac{0.8}{2} = 0.8 + 0.4 = 1.2$$

Also, the average of a strong neutrosophic cycle C is

$$\mu(G) = \frac{W(G)}{C_2^n} = \frac{2W(G)}{n(n-1)} = \frac{2*1.2}{4*3} = 0.2$$

Theorem 3.4. For any neutrosophic graph G has not strong cycle neutrosophic graph,

$$W(G) \leq 2m_sSc(q_s) - (m_s + 1)Sc(\gamma_s).$$

Where m_s *is the number of strong edges in G, and* q_s *is the sum of strong edge*

Proof. Let *G* be a neutrosophic graph, with *n* vertices and *m* edges has not strong cycle and has at least one δ-edge. Then *G* has a path join v_1 and v_n we obtained it after delete δ-edge. So, the path in *G* is strong path has not δ-edge. And has m_s strong edge. Therefore, by Theorem (3.2),

$$W(G) \leq 2m_sSc(q_s) - (m_s + 1)Sc(\gamma_s).$$

Example 3.4.1 Consider the neutrosophic graph G $_N$= (*A*,*B*), *with v*(1,1,0) *for all* $v \in V$ (*G*) *and all edges are* δ-*strong except* (*cd*) *is* δ-*edge given in the Fig.* 3.

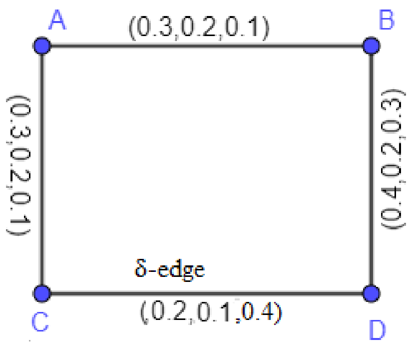

Figure 3 : neutrosophic graph with δ-edge

In Fig. 3, we have, $q = (1.2,0.7,0.9)$; $q_s = (1,0.6,0.5)$ $, \gamma_s(G) = (0.6,0.4,0.2)$ *and* $m = |E| = 4$; $m_s = 3$.

Then the Wiener index of G

$$W(G) \leq 2m_s Sc(q_s) - (m_s + 1)Sc(\gamma_s).$$

=2*3*Sc(1,0.6,0.5)-(3+1)*Sc(0.6,0.4,0.2)

=1.8-1.06667=0.73333

Also, the average of G is

$$\mu(G) = \frac{W(G)}{C_2^n} = \frac{2W(G)}{n(n-1)} = \frac{2*0.73333}{4*3} = 0.12222$$

Theorem 3.5. For any neutrosophic graph G without δ *−edge and contain k strong cycle neutrosophic graph, such that k ≥2.*

$$W(G) \leq \frac{m(mSc(q)-Sc(\gamma_s))}{8K} \quad \text{Where } m = |E|$$

Proof. Let G be a neutrosophic graph without *without* δ *−edge* and contain $K \geq 2$ strong cycles. Then $G = \cup_{i=2}^{k} C_i$

By Theorem (3,4); $W(G) \leq \frac{m(mq-\gamma_s)}{8}$

Since $m=\sum_{i=1}^{k} \mu_i$*.hence* $W(G) \leq \frac{m(mq-\gamma_s)}{8K} + \frac{\gamma_s}{2k} \approx \frac{m(mq-Sc(\gamma_s))}{8K}$ *because* $\frac{\gamma_s}{2k}$ *is very small .*

Example 3.5.1 Consider the neutrosophic graph G with v(1,1,0) for all v ∈ V (G) given in the Fig. 4.

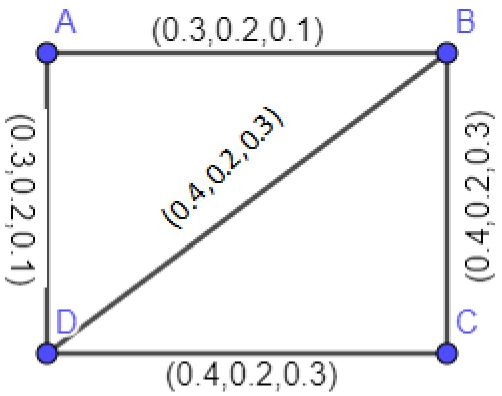

Figure 4 : neutrosophic graph without δ-edge

From Fig. 4. we have, $q = (1.8,1,1.1)$; $\gamma_s(G) = 0.3,0.2,0.1)$ *and* $m = |E| = 5, K = 2$.

Then $W(G) \leq \frac{m(mSc(q)-Sc(\gamma_s))}{8K} = \frac{5(5*0.566667-0.46667)}{8*2} = 0.739583$

Also, the average of a neutrosophic graph G is

$$\mu(G) = \frac{W(G)}{C_2^n} = \frac{2W(G)}{n(n-1)} = \frac{2*0.739583}{4*3} = 0.123264$$

Theorem 3.6. Let G be a neutrosophic graph without δ *−edge, contains strong cycle neutrosophic graph and strong neutrosophic path. Then*

$$W(G) \leq \frac{m(mq-Sc(\gamma_s))}{8K} + \frac{q}{2}$$

Proof. By Theorem (3.3) and Theorem (3:2), we have

$$W(G) \leq \frac{m(mq-\gamma_s)}{8K} + \frac{q}{2} + 2mq - (m+1)\gamma_s$$

There exists a vertex u which is a common strong cycle and strong path. Let *u* belong to a

strong cycle. Therefore, the path contains $(V(P) - \{u\})$, also the path *P* contains *m* edge such

$m = m_c + m_p, q = q_c + q_p$ then $2m_p q_p - (m_p + 1)\gamma_s(p) \leq \frac{q}{2}$. hence

$$W(G) \leq \frac{m(mq-Sc(\gamma_s))}{8K} + \frac{q}{2}$$

Example 3.6.1. Consider the neutrosophic graph G with v(1,1,0) for all v ∈ V (G) given in the Fig. 5.

From Fig. 5*; we have,* $q = (1.7,1,0.9)$, $\gamma_s(G) = (0.4,0.2,0.1)$ *and* $m = |E| = 5$.

Then

$(G) \leq \frac{m(mSc(q)-Sc(\gamma_s))}{8K} + \frac{Sc(q)}{2} = \frac{5(5*Sc(1.7,1,0.9)-Sc(0.4,0.2,0.1))}{8} + \frac{Sc(1.7,1,0.9)}{2} = 1.771 + 0.3 = 2.071$

Also, the average of a strong neutrosophic cycle C is

$$\mu(G) = \frac{W(G)}{C_2^n} = \frac{2W(G)}{n(n-1)} = \frac{2*2.071}{5*4} = 0.2071$$

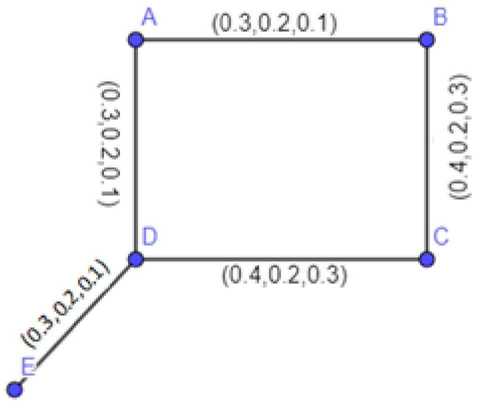

Figure 5: neutrosophic graph without δ-edge with strong cycle

Theorem 3.7. For any neutrosophic graph G, with at least one δ-edge and has strong neutrosophic cycle. Then $W(G) \leq \frac{m_s(m_s Sc(q_s) - Sc(\gamma_s))}{8}$

Where m_s *is the number of strong edges of G:*

Proof. Let G be a neutrosophic graph with δ*-edge* and G has strong cycle. Then G contains strong neutrosophic cycle and neutrosophic subgraph obtained by deleting δ*-edge*. So, by Theorems (3.3); (3.2) and Theorem (3.6), we have

$$W(G) \leq \frac{m_s(m_s q_s - \gamma_s)}{8} + 2m_H q - (m_H + 1)\gamma_s$$

Hence

$$W(G) \leq \frac{m_s(m_s Sc(q_s) - Sc(\gamma_s))}{8}$$

Example 3.7.1 Consider the neutrosophic graph G, with

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