Modules in which every surjective endomorphism has an e-small kernel

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Abstract- In this paper we introduce the notion of e-gH modules which is a proper generalization of Hopfian modules and defined as, a module M is called an e-gH if any surjective R-endomorphism g of M has an e-small kernel, a ring R is called an e-gH ring if R is an e-gH as R-module. We give some characterizations and properties of this modules.

Keywords-component: Hopfian module, generalized Hopfian module, e-small submodule, e-gH module.

1. INTRODUCTION

Throughout this paper all modules are unitary right R-modules and R is an associative ring with identity. A nonzero submodule L \subseteq M is said to be essential in M, denoted by L \nsubseteq M, if N \cap L \neq 0 for every nonzero submodule N of M [7]. A submodule S of M is called small, denoted by S \ll M, if S \neq M and for every submodule L \subseteq M with the property M = S + L implies L = M. A submodule E \subseteq M is called e-small, denoted by E \ll_e M, if for every essential submodule S of M with the property M = E + S implies S = M [10]. In 1986, V. A. Hiremath introduced the concept of Hopfian module, defined as a module Mis called Hopfian if for every surjective R-endomorphism of M is an isomorphism [8]. Gorbani and Haghany introduced generalized Hopfian (gh), as a module is called gH, if for every surjective R-endomorphism of M has an small kernel [6]. Now we are represent a new definition of a proper generalized of Hopfian, called, e-gH, defined as, a module M is called, e-gH, if for every surjective R-endomorphism f of M has an e-small kernel (i.e., kerf \ll_e M). In this paper we show many properties and examples of e-gH modules. Also we generalize the notion of e-gH modules to concept of e-gH relative to a module.

2. e-gH AND SOME BASIC PROPERTIES.

Definition 2.1. A nonzero R-module M is said to be an e-gH module if every surjective R-endomorphism g of M has an e-small kernel, i.e., ker g \ll_e M. Moreover, a ring R is called e-gH if, R_{\text{g}} is e-gH.

Remarks and Examples 2.2.
(1) Every gH module is an e-gH module.

Proof Since every small submodule is e-small, then the result is follows. \Box

(2) Every Hopfian module is an e-gH module.

Proof Since every Hopfian module is gH module, and so it is an e-gH module, by (1). \Box

(3) Every Noetherian module is an e-gH module.

Proof It follows directly by ([8], Proposition 6(i)) and (2). \Box

(4) The concept of e-gH modules is a proper generalization of Hopfian modules, as example: consider the 2-prüfer group \mathbb{Z}_2^\infty as a \mathbb{Z}-module. From ([3], p.15) \mathbb{Z}_2^\infty is a hollow \mathbb{Z}-module, and so it is gH as \mathbb{Z}-module. By (1), \mathbb{Z}-module \mathbb{Z}_2^\infty is an e-gH module, but it is not Hopfian, see ([8], Remark 7).\Box

(5) The two rings \mathbb{Q} and \mathbb{Z} are e-gH, because that the only rings homomorphism of them is identity map.

(6) The \mathbb{Z}-module \mathbb{Z} and \mathbb{Q}-module (also \mathbb{Z}-module) \mathbb{Q} are e-gH, in fact, they are Hopfian, see ([9], Examples 1.5(b)).

Theorem 2.3. The following are equivalent for an R-module M.

(1) M is an e-gH module.

(2) If \ E \subseteq M and there is an epimorphism g:M/E \rightarrow M, then E \ll_e M.

(3) If S \ll M (i.e., S is a proper essential submodule of M) and if f \in \text{End}(M) is surjective, then f(S) \neq M.
Proof (1) $\Rightarrow$ (2) Assume $g: M/E \to M$ is an epimorphism. Then $gr \in \mathcal{E}(M)$ is a surjective, where $r: M \to M/E$ is a natural map. By (1), we deduce that $\ker gr \ll_{\epsilon} M$. If $e \in E = \ker r$, then $\pi(e) = 0$ and so $gr(e) = f(0) = 0$, hence $e \in \ker gr$. Therefore $E \leq \ker gr$ and hence $E \ll_{\epsilon} M$. \\
(2) $\Rightarrow$ (3) Let $S \leq M$ and $f: M \to M$ an epimorphism. Suppose $f(S) = M$. By ([4], Lemma 3.18(2)), we have $M = f^{-1}(M) = f^{-1}(f(S)) = S + kerf$. Moreover $\overline{f}: M/kerf \to M$ is an epimorphism, so by (2), $kerf \ll_{\epsilon} M$ and hence $S = M$ that is a contradiction. Hence $f(S) \neq M$. \\
(3) $\Rightarrow$ (1) Let $f \in \mathcal{E}(M)$ and $f$ a surjective. To prove that $kerf \ll_{\epsilon} M$. Assume that $S \leq M$ such that $kerf + S = M$. If $S \neq M$, so by (3), $f(S) \neq M$ and hence $f^{-1}(f(S)) \neq M$ (since if, $f^{-1}(f(S)) = M$, then $f(f^{-1}(f(S))) = f(M) = M$, and so by ([4], Lemma 3.18(3)), $f(S) = f(S) \cap f(M) = M$, therefore $kerf + S \neq M$ which is a contradiction. Then $S = M$ and hence $kerf \ll_{\epsilon} M$. So (1), holds. \\

Corollary 2.4. Let $M$ be a module. Then $M$ is e-gH if and only if $g: M/N \to M$ is a non-epimorphism, for all $N$ non e-small in $M$. \\

Proposition 2.5. Let $M$ be an $R$-module. Then the following are equivalent. \\
(1) $M$ is e-gH. \\
(2) For all epimorphism $\varphi \in \mathcal{E}(M)$, if there exist $C \leq M$ with $\varphi(C) = \varphi(M)$, then $C$ is closed in $M$. \\
Proof (1) $\Rightarrow$ (2) Suppose that $M$ is e-gH and $\varphi \in \mathcal{E}(M)$ be an epimorphism. Assume $\varphi(C) = \varphi(M)$ for some $C \leq M$. Let $S$ be any complement for $C$ in $M$, then we have $C \oplus S \leq M$. It is obvious that $C + S + ker\varphi = M$. Since $C + S \leq M$ and $ker\varphi \ll_{\epsilon} M$, then $C + S = M$, and so $C \oplus S = M$. Therefore $C$ is a direct summand, and hence $C$ is closed in $M$. \\
(2) $\Rightarrow$ (1) Let $\varphi \in \mathcal{E}(M)$ be an epimorphism. Assume $ker\varphi + C = M$ where $C \leq M$, then $\varphi(C) = \varphi(M)$. By (2), $C$ is closed in $M$, and thus $C = M$. Hence $ker\varphi \ll_{\epsilon} M$ and $M$ is e-gH. \\

Proposition 2.6. Let $M$ be an $R$-module such that for any $N \leq M$, $Z_{\mathcal{E}}(M) \subseteq C$. Then the following are equivalent. \\
(1) $M$ is e-gH. \\
(2) For any epimorphism $\varphi \in \mathcal{E}(M)$, if there exist $C \leq M$ with $ker\varphi + C = M$, then $M/C$ is nonsingular. \\
(3) For any epimorphism $\varphi \in \mathcal{E}(M)$, if there exist $C \leq M$ with $ker\varphi + C = M$, then $C$ is closed. \\
Proof (1) $\Rightarrow$ (2) Let $\varphi \in \mathcal{E}(M)$ be an epimorphism. If $ker\varphi + C = M$ for some $C \leq M$. Then $\varphi(C) = \varphi(M)$. From (Proposition 2.5), $C$ is closed in $M$. By our assumption, $Z_{\mathcal{E}}(M) \subseteq C$. So by ([2], Proposition 2.6), $M/C$ nonsingular. \\
(2) $\Rightarrow$ (3) Let $\varphi \in \mathcal{E}(M)$ be an epimorphism. Assume that there exist $C \leq M$ with $ker\varphi + C = M$. Then $\varphi(C) = \varphi(M)$, by (2) $M/C$ is nonsingular. Hence by ([2], Proposition 2.6), $C$ is closed in $M$. \\
(3) $\Rightarrow$ (1) Let $\varphi \in \mathcal{E}(M)$ be an epimorphism. Assume that there exist $C \leq M$ with $\varphi(C) = \varphi(M)$, then $ker\varphi + C = M$. By (3), $C$ is closed in $M$. From (Proposition 2.5), $M$ is e-gH. \\

Proposition 2.7. Every direct summand of an e-gH module is e-gH. \\
Proof Let $M$ be an e-gH module and $N \subseteq^\oplus M$. So, $M = N \oplus K$ for some $K \leq M$. Assume that $f \in \mathcal{E}(N)$ and an epimorphism. Consider $I_K: K \to M$ is an identity map over $K$. Then $f \oplus I_K(M) = f \oplus I_K(N \oplus K) = (f(N) \oplus I_K(K) = N \oplus K = M$, that means $f \oplus I_K$ is an epimorphism. But $M$ is e-gH, then $ker(f \oplus I_K) = kerf \oplus ker_k = kerf \oplus 0 = kerf$ and then $kerf \ll_{\epsilon} M$. Since $kerf \leq N \subseteq^\oplus M$, then $kerf \ll_{\epsilon} N$, by ([5], Lemma 12.1(1)). Therefore $N$ is e-gH. \\

Proposition 2.8. Let $M = M_1 \oplus M_2$ such that $M_1$ and $M_2$ be fully invariant under every surjection of $M$. Then $M$ is e-gH if and only if $M_1$ is e-gH for all $i = 1,2$. \\
Proof “If” part is follows directly by (Proposition 2.7). “Only if” part. Let $f: M \to M$ be an $R$-epimorphism. Then $f_i = f|_{M_i}: M_i \to M_i$ is an $R$-epimorphism for all $i = 1,2$, because $M_1$ and $M_2$ are fully invariant submodules. Since $M_i$ is e-gH, for all $i = 1,2$, then $ker f_i \ll_{\epsilon} M_i$, so $ker f = ker f_1 \oplus ker f_2 = ker f_1 \oplus ker f_2 \ll_{\epsilon} M_1 \oplus M_2 = M$. by (10), Proposition 2.5(3)). Therefore $M = M_1 \oplus M_2$ is an e-gH module. \\

Corollary 2.9. Let $M = \bigoplus_{i=1}^n M_i$ such that $M_i$ be fully invariant under every surjection of $M$ for all $i = 1,2,\ldots,n$. Then $M$ is e-gH if and only if $M_i$ is e-gH for all $i = 1,2,\ldots,n$. \\

Proposition 2.10. If $M = M_1 \oplus M_2$ with $r_{M_1}(M_1) \oplus r_{M_2}(M_2) = R$, then $M$ is e-gH if and only if $M_1$ is e-gH for all $i = 1,2$. \\
Proof “If” part is follows directly by (Proposition 2.7). “Only if” part. Let $f: M \to M$ be an $R$-epimorphism. As $r_{M_1}(M_1) \oplus r_{M_2}(M_2) = R$ and $Imf \leq M_1 \oplus M_2$, then by ([11], Proposition 1.4.2) there exists $N \leq M_1$ and $K \leq M_2$ such that $Imf = N \oplus K$ implies $Imf|_{M_1} \oplus Imf|_{M_2} = N \oplus K$, thus $Imf|_{M_1} \leq M_1$ and $Imf|_{M_2} \leq M_2$. As $M_1$ is e-gH and $f|_{M_1}$ is an $R$-epimorphism for all $i = 1,2$, then $ker f_i \ll_{\epsilon} M_1$. Then $kerf = ker f_1 \oplus ker f_2 = ker f_1 \oplus ker f_2 \ll_{\epsilon} M_1 \oplus M_2 = M$, by ([10], Proposition 2.5(3)). Hence $M = M_1 \oplus M_2$ is an e-gH module. \\

Lemma 2.11. Let $f: M \to M$ be an $R$-isomorphism. If $A$ is not e-small in $M$ then $f^{-1}(A)$ is not e-small in $M$. \\
Proof If $A$ is not e-small in $M$, then there is a proper essential submodule $E$ of $M$ such that $A + E = M$. Then $f^{-1}(A) + f^{-1}(E) = f^{-1}(A + E) = f^{-1}(M) = M$ with $f^{-1}(E)$ is an
essential in $M$. If $f^{-1}(E) = M$, then $E = f(f^{-1}(E)) = f(M) = M$ that is a contradiction. Thus, $f^{-1}(E) \not\subseteq M$ and hence $f^{-1}(A)$ is not $e$-small in $M$.

**Proposition 2.12.** Let $M$ be an $R$-module such that $M/N$ be an $e$-gH $R$-module for any $0 \neq N \subseteq M$. Then $M$ is $e$-gH.

**Proof** If false, then there is an $R$-epimorphism $f \in \text{End}(M)$ such that $\ker f$ is not $e$-small, then $\ker f \neq 0$. From $1$-st isomorphism theorem, there is an $R$-isomorphism $g:M/\ker f \rightarrow M$. Let $\pi:M \rightarrow M/\ker f$ be the natural $R$-epimorphism. It follows that $\pi \circ g:M/\ker f \rightarrow M/\ker f$ is an $R$-epimorphism which $\ker (\pi \circ g) = \pi^{-1}(\ker g) = g^{-1}(\ker f)$ is not $e$-small in $M/\ker f$, by (Lemma 2.11), a contradiction. Hence $M$ is an $e$-gH $R$-module.

**Proposition 2.13.** Let $M$ be a nonsingular and $e$-gH $R$-module with $f \in \text{End}(M)$ is an epimorphism and $M/f(N)$ is singular for all and $N \subseteq M$. Then $f(L) \subseteq \ker f$ if and only if $L \subseteq \ker f$.

**Proof** If $f(L) \subseteq \ker f$, assume $L + K = M$ where $K \subseteq M$. Then $f(L) + f(K) = M$. By hypothesis, $M/(f(K)$ is singular and $M$ is a nonsingular $R$-module, ([17], Proposition 1.21) implies that $f(K) \subseteq M$. As $f(L) \subseteq M$, then $f(K) = M$ and hence $K + f(K) = M$. Since $M$ is an $e$-gH $R$-module, then $K \subseteq M$, and hence $K = M$. Therefore $L \subseteq \ker f$. The converse is follows by ([10], Proposition 2.5(2)).

**Proposition 2.14.** Let $M$ be a module. Consider the following assertions:

1. For any epimorphism $f \in \text{End}(M)$, if $N \subseteq \ker f$ then $f^{-1}(N) \subseteq \ker f$.
2. $M$ is $e$-gH.

Then (1) $\Rightarrow$ (2). If $M$ is a uniform module, then (2) $\Rightarrow$ (1).

**Proof** (1) Suppose (1) holds. If $g \in \text{End}(M)$ is a surjective and $N \subseteq \ker f$, then $f^{-1}(N)$ is not $e$-small kernel, then $g^{-1}(0)$ is not $e$-small in $M$, by assumption, (0) is not $e$-small in $M$, which is a contradiction. Hence $\ker f = \ker f \subseteq M$ and $M$ is $e$-G.

(2) $\Rightarrow$ (1) Assume $f \in \text{End}(M)$ is a surjective and $N \subseteq \ker f$. Let $f^{-1}(N) + K = M$ for some $K \subseteq M$. Thus, $N = f(K) \subseteq M$. Since $M$ is uniform, then $f(K) \subseteq M$. As $N \subseteq \ker f$ then $f(K) = M$. It follows that $K + f(K) = M$. By (2), $\ker f \subseteq M$, hence $K = M$. Therefore, $f^{-1}(N) \subseteq \ker f$.

**Theorem 2.15.** Let $M$ be a quasi-projective uniform module. Then $M$ is $e$-gH if and only if $M/E$ is $e$-gH, for all $E \subseteq M$.

**Proof** The sufficiency is clear by taking $E = 0$. Assume $M$ is $e$-gH, $E \subseteq M$ and $f:M/E \rightarrow M/E$ a surjective. Consider $\pi:M \rightarrow M/E$ is a natural map. Therefore $f \circ \pi:M \rightarrow M/E$ is an homomorphism as $M$ is a quasi-projective module, there is a homomorphism $g:M \rightarrow M$ such that $\pi g = f r$. So $(\text{Im} g + E)/E = \pi(\text{Im} g) = f(\text{Im}(M)) = f(M/E) = M/E$, hence Im $g + E = M$. Since $E \subseteq M$ and $\text{Im} g$ is essential in $M$ (as $M$ is uniform), then $\text{Im} g = M$, i.e., $g$ is an epimorphism. Thus, $\ker g \subseteq \ker f$. As will as, $(\pi g)(E) = f r(E) = E$ imply that $g(E) + E = g(E) + \ker \pi = \pi^{-1}(\ker g(E)) = \pi^{-1}(E) = E$, and then $g(E) \subseteq E$. So $E + \ker g = g^{-1}(g(E)) \subseteq g^{-1}(E)$.

**Theorem 2.16.** Let $M$ be a module such that $N$ is a fully invariant submodule of $M$ and $M/N$ is Hopfian. If $N$ is an $e$-gH module, then so is $M$.

**Proof** Consider a surjective $f \in \text{End}(M)$. Define $g \in \text{End}(M/N)$ by $g(m + N) = f(m) + N$ for all $m + N \in M/N$. Thus $\text{Im} g = g(M/N) = f(M)/N = M/N$, i.e., $g$ is an $R$-epimorphism, so $g$ is an $R$-epimorphism (i.e., $\ker g = N$), since $M/N$ is Hopfian. We conclude that $\ker f + N = \{m + N \mid m \in \ker f\} = \{m + N \mid f(m) = 0\} + \{m + N \mid f(m) = N\} = \{m + N \mid g(m + N) = N\} = \ker g = N$. So $\ker f \subseteq N$. Put $h = f|_N$, then $h$ is also an $R$-epimorphism, because $N$ is fully invariant. Since $N$ is $e$-gH, $\ker h \subseteq N$. It follow that $\ker h = \ker f \cap N = \ker f$, then $\ker f \subseteq N \subseteq M$, hence $\ker f \subseteq M$. Hence $M$ is an $e$-gH module.

**Proposition 2.17.** Let $M_1$ and $M_2$ be an $R$-modules such that $M_1 \cong M_2$. Then $M_1$ is $e$-gH if and only if $M_2$ is $e$-gH.

**Proof** Assume that $M_1$ is an $e$-gH $R$-module such that $M_1 \cong M_2$. Then there is an $R$-isomorphism $g: M_1 \rightarrow M_2$. Notice that $g^{-1}(M_2) \rightarrow M_1$ is an $R$-isomorphism. Let $f \in \text{End}(M_2)$ be an $R$-epimorphism. Put $h = g^{-1} \circ f \circ g$. Then $h \in \text{End}(M_1)$ and $h(M_2) = g^{-1} \circ f(g(M_2)) = g^{-1} \circ f(M_2) = g^{-1} = M_2$. Hence $h$ is a surjective. Therefore $\ker h \subseteq M_1$, as $M_1$ is $e$-gH. By ([10], Proposition 2.5(2)), $(\ker h) \subseteq M_2$. So $(\ker h) = (g^{-1} \circ f \circ g) = g^{-1} \circ f(g^{-1} \circ f) = f^{-1}(0) = \ker f$. Hence $\ker f \subseteq M_2$ and $M_2$ is $e$-gH.

**Proposition 2.18.** Let $M$ be an $R$-module and let $K \subseteq S \subseteq M$ with $M = K + S$ and $M/(K \cap S)$ is $e$-gH. Then $M/K$ and $M/S$ are $e$-gH.

**Proof** If $M = K + S$, it follows that $M/(K \cap S) = K/(K \cap S) \oplus S/(K \cap S)$, thus $K/(K \cap S)$ and $S/(K \cap S)$ are $e$-gH by ([27]). As $K/(K \cap S) \cong M/S$ and $S/(K \cap S) \cong M/K$, so $M/K$ and $M/S$ are $e$-gH, by (Proposition 2.17). The next is a characterization of $e$-gH modules.

**Theorem 2.19.** Let $M$ be a module. Then $M$ is an $e$-gH module if and only if for any proper $N \subseteq M$ with $M/N \cong M$, $N$ is an $e$-small submodule of $M$.

**Proof** Suppose that $M$ is an $e$-gH module. Let $N$ be a proper submodule of $M$ such that $M/N \cong M$. Then there is an isomorphism $\psi: M/N \rightarrow M$. Now, consider the sequence $M \rightarrow M/N \psi$ where $\psi$ is a canonical epimorphism map. Since $M$ is an $e$-gH module and $\psi \circ \pi \in \text{End}(M)$ is a surjective, then $\ker (\psi \circ \pi) \subseteq \ker f$. But, we have that $\ker (\psi \circ \pi) = \pi^{-1}(\ker f) \subseteq \pi^{-1}(N) = \ker f = N$, therefore $N \subseteq M$. Conversely, if $g: M \rightarrow M$ is an epimorphism, $1$-st isomorphism theorem implies that $M/\ker g \cong M$ and $\ker g \neq M$ (if $\ker g =$...
Let $F$ be a property of modules preserved under isomorphism. Assume a module $M$ has the property $F$ and satisfies ACC on proper (non-$e$-small) submodules $N$ with $M/N$ has the property $F$, then $M$ is $e$-GH.

**Proof** Assume that $M$ is not $e$-GH, then by (Theorem 2.19), there exists a proper non-$e$-small submodule $L_1$ of $M$ such that $M/L_1\cong M$. Therefore $M/L_1$ is not $e$-GH and has the property $F$ by hypothesis. Again by (Theorem 2.19), there is a proper non-$e$-small submodule $L_2/L_1$ of $M/L_1$ with $M/L_2\cong (M/L_1)/(L_2/L_1)\cong M/L_1$. Note that $L_2$ is a proper non-$e$-small submodule of $M$. By repeating this argument, we get an ascending chain $L_1\subseteq L_2\subseteq L_3\subseteq\cdots$ of proper (non-$e$-small) submodules of $M$, which is a contradiction. Thus $M$ must be $e$-GH. □

**Corollary 2.21.** If $M$ is a module that has ACC on proper (non-$e$-small) submodules $N$ with $M/N$ is not $e$-GH module, then $M$ is $e$-GH.

**Proof** Let $F$ be the property for not being $e$-GH, and assume $M$ is not $e$-GH. From (Proposition 2.20), $M$ must be $e$-GH. This contradiction proves that $M$ is $e$-GH. □

**Corollary 2.22.** Every nonzero module satisfies ACC on proper (non-$e$-small) submodules is $e$-GH.

**Proof** We may assume that $M\neq 0$ has ACC on proper (non-$e$-small) submodules and that $F$ is the property of being nonzero. By (Proposition 2.20), $M$ must be $e$-GH. □

We will generalize the notion of $e$-GH modules as follows.

**Definition 2.23.** Let $M_1$ and $M_2$ be two $R$-modules. $M_1$ is called $e$-GH relative to $M_2$, if for each epimorphism $g: M_1 \to M_2$, we have $ker(g) \subseteq M_2$.

**Remark 2.24.** Given the definition above, we see that an $R$-module $M$ is $e$-GH if and only if $M$ is $e$-GH relative to $M$.

The features of the $e$-GH module relative to a module are described below.

**Proposition 2.25.** Let $M_1$ and $M_2$ be $R$-modules. Then the following conditions are equivalent:

1. $M_1$ is $e$-GH relative to $M_2$.
2. For each $L \subseteq M_1$, $L$ is $e$-GH relative to $M_2$.
3. For each $L \subseteq M_1$, $M_1/L$ is $e$-GH relative to $M_2$.

**Proof** (1) $\Rightarrow$ (2) Assume $M_1 = L \oplus K$, where $K \subseteq M_1$ and $g: L \to M_2$ is a surjective map. Then $g: M_1 \to M_2$ is a surjective and hence $ker(g \pi) \subseteq M_1$, from (1). It is obvious that $ker(g \pi) = \pi^{-1}(ker(g)) = ker(g \pi K)$. Thus, $ker(g \pi) \subseteq L \oplus K$ and hence $ker(g) \subseteq L$, by ([10], Proposition 2.5(3)).

(2) $\Rightarrow$ (1) By taking $L = M_1$.

(1) $\Rightarrow$ (3) Let $L \subseteq M_1$ and $g: M_1/L \to M_2$ be surjective. Hence $g: M_1/L \to M_2$ is a surjective map. Thus, $M_1 \to M_1/L$ is the natural map. By (1), $ker(g \pi) \subseteq M_1$. As $ker(g \pi) = \pi^{-1}(ker(g))$, hence $\pi^{-1}(ker(g)) \subseteq M_1$. Therefore $M_1/L$ is $e$-GH relative to $M_2$.

(3) $\Rightarrow$ (1) By taking $L = 0$. □

**Proposition 2.26.** If $M$ is a module with the property that for any $g \in End(M)$, there exists an $n \in \mathbb{Z}^+$ such that $ker(g^n) \cap Im(g^n) \subseteq M$, then $M$ is $e$-GH.

**Proof** Let $g \in End(M)$ be a surjective. By assumption, there is an integer $n \geq 1$ such that $ker(g^n) \cap Im(g^n) \subseteq M$. It follows that $g^n \in End(M)$ is a surjective, i.e., $Im(g^n) = M$. Thus, $ker(g^n) \cap Im(g^n) = ker(g^n) \cap M = ker(g^n) \subseteq M$. It is easy to see that $ker \leq ker(g^n)$, therefore $ker(g^n) \subseteq M$ by ([10], Proposition 2.5(a)). Hence $M$ is $e$-GH. □

**Proposition 2.27.** Let $M$ be an $R$-module. If for any $R$-epimorphism $\phi: M \to M$, there exist $n \geq 1$ such that $ker(\phi^n) = ker(\phi^{n+1})$ for all $i \in \mathbb{Z}^+$, then $M$ is $e$-GH.

**Proof** Let $\phi \in End(M)$ be any surjective. We claim that $ker(\phi^n) \cap Im(\phi^n) = 0$. Let $y \in ker(\phi^n) \cap Im(\phi^n)$. Thus $\phi^n(y) = 0$ and $y = \phi^n(x)$ for some $x \in M$. Hence $\phi^{2n}(x) = \phi^n(y) = 0$ and hence $x \in ker(g^{2n})$. But from our assumption we have that $ker(g^n) = ker(\phi^{n+1})$, and so $x \in ker(g^n)$ therefore $0 = \phi^n(x) = y$. Hence $ker(\phi^n) \cap Im(\phi^n) = 0$. As $\phi$ is a surjective, so $Im(\phi^n) = M$, thus $ker(\phi^n) = 0$. But $ker(\phi^n) \subseteq ker(g^n)$, then $ker(\phi^n) \subseteq M$. Therefore $M$ is $e$-GH. □

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