Modules in which every surjective endomorphism has an e-small kernel

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Abstract- *In this paper we introduce the notion of e-gH modules which is a proper generalization of Hopfian modules and defined as, a module M is called an e-gH if any surjective R-endomorphism g of M has an e-small kernel, a ring R is called an e-gH ring if R is an e-gH as R-module. We give some characterizations and properties of this modules.*

Keywords-component*; Hopfian module, generalized Hopfian module, e-small submodule, e-gH module.*

1. Introduction

Throughout this paper all modules are unitary right R-modules and R is an associative ring with identity. A nonzero submodule $L \leq M$ is said to be essential in M, denoted by $L \trianglelefteq M$, if $N \cap L \neq 0$ for every nonzero submodule N of M [7]. A submodule S of M is called small, denoted by $S \ll M$, if $S \neq M$ and for every submodule $L \leq M$ with the property $M = S + L$ implies $L = M$. A submodule $E \leq M$ is called e-small, denoted by $E \ll_e M$, if for every essential submodule S of M with the property $M = E + S$ implies $S = M$ [10]. In 1986, V. A. Hiremath introduced the concept of Hopfian module, defined as a module $M$is called Hopfian if for every surjective R-endomorphism of M is an isomorphism [8]. Gorbani and Haghany introduced generalized for Hopfian called generalized Hopfian (gH), as a module is called gH, if for every surjective R-endomorphism of M has an small kernel [6]. Now we are represent a new definition of a proper generalized of Hopfian, called, e-gH, defined as, a module M is called, e-gH, if for every surjective R-endomorphism f of M has an e-small kernel (i.e., $Kerf \ll_e M$). In this paper we show many properties and examples of e-gH modules. Also we generalize the notion of e-gH modules to concept of e-gH relative to a module.

2. e-gH and some basic propertes.

Definition 2.1. A nonzero R-module M is said to be an e-gH module if every surjective R-endomorphism g of M has an e-small kernel, i.e., $kerg \ll_e M$. Moreover, a ring R is called e-gH if, R_R is e-gH.

Remarks and Examples 2.2.
(1) Every gH module is an e-gH module.
Proof Since every small submodule is e-small, then the result is follows. □
(2) Every Hopfian module is an e-gH module.
Proof Since every Hopfian module is gH module, and so it is *an* e-gH module, by (1). □
(3) Every Noetherian module is an e-gH module.
Proof It follows directly by ([8], Proposition 6(i)) and (2). □
(4) The concept of e-gH modules is a proper generalization of Hopfian modules, as example: consider the 2-prüfer group $\mathbb{Z}_{2^\infty}$ as a $\mathbb{Z}$-module. From ([3], p.15) $\mathbb{Z}_{2^\infty}$ is a hollow $\mathbb{Z}$-module, and so it is gH as $\mathbb{Z}$-module. By (1), $\mathbb{Z}$-module $\mathbb{Z}_{2^\infty}$ is an e-gH module, but it is not Hopfian, see ([8], Remark 7).
(5) The two rings $\mathbb{Q}$ and $\mathbb{Z}$ are e-gH, because that the only rings homomorphism of them is identity map.
(6) The $\mathbb{Z}$-module $\mathbb{Z}$, and $\mathbb{Q}$-module (also $\mathbb{Z}$-module) $\mathbb{Q}$ are e-gH, in fact, they are Hopfian, see ([9], Examples 1.5(b)).

Theorem 2.3. The following are equivalent for an R-module M.
(1) M is an e-gH module.
(2) If $E \leq M$ and there is an epimorphism $g: M/E \to M$, then $E \ll_e M$.
(3) If $S \lhd M$ (i.e., S is a proper essential submodule of M) and if $f \in End(M)$ is surjective, then $f(S) \neq M$.

Proof (1) $\Longrightarrow$ (2) Assume $g: M/E \to M$ is an epimorphism. Then $g\pi \in End(M)$ is a surjective, where $\pi: M \to M/E$ is a natural map. By (1), we deduce that $kerg\pi \ll_e M$. If $e \in E = ker\pi$, then $\pi(e) = 0$ and so $g\pi(e) = f(0) = 0$, hence $e \in kerg\pi$. Therefore $E \leq kerg\pi$ and hence $E \ll_e M$.
(2) $\Longrightarrow$ (3) Let $S \lhd M$ and $f: M \to M$ an epimorphism. Suppose $f(S) = M$. By ([4], Lemma 3.1.8(2)), we have $M = f^{-1}(M) = f^{-1}(f(S) = S + kerf$. Moreover $\overline{f}: M/kerf \to M$ is an epimorphism, so by (2), $kerf \ll_e M$ and hence $S = M$ that is a contradiction. Hence $f(S) \neq M$.
(3) $\Longrightarrow$ (1) Let $f \in End(M)$ and f a surjective. To prove that $kerf \ll_e M$. Assume that $S \trianglelefteq M$ such that $kerf + S = M$. If $S \neq M$, so by (3), $f(S) \neq M$ and hence $f^{-1}(f(S)) \neq M$, (since if, $f^{-1}(f(S)) = M$, then $f(f^{-1}(f(S)) = f(M) = M$, and so by ([4], Lemma 3.1.8(3)), $f(S) = f(S) \cap f(M) = M$, therefore $kerf + S \neq M$ which is a contradiction. Then $S = M$ and hence $kerf \ll_e M$. So (1), holds. □

Corollary 2.4. Let M be a module. Then M is e-gH if and only if $g: M/N \to M$ is a non-epimorphism, for all N non e-small in M.

Proposition 2.5. Let M be an R-module. Then the following are equivalent.
(1) M is e-gH.
(2) For all epimorphism $\varphi \in End_R(M)$, if there exist $C \leq M$ with $\varphi(C) = \varphi(M)$, then C is closed in M.

Proof (1) $\Longrightarrow$ (2) Suppose that M is e-gH and $\varphi \in End_R(M)$ be an epimorphism. Assume $\varphi(C) = \varphi(M)$ for some $C \leq M$. Let S be any complement for C in M, then we have $C \oplus S \trianglelefteq M$. It is obvious that $C + S + ker\varphi = M$. Since $C + S \trianglelefteq M$ and $ker\varphi \ll_e M$, then $C + S = M$, and so $C \oplus S = M$. Therefore C is a direct summand, and hence C is closed in M.
(2) $\Longrightarrow$ (1) Let $\varphi \in End_R(M)$ be an epimorphism. Assume $ker\varphi + C = M$ where $C \trianglelefteq M$, then $\varphi(C) = \varphi(M)$. By (2), C is closed in M, and thus $C = M$. Hence $ker\varphi \ll_e M$ and M is e-gH. □

Proposition 2.6. Let M be an R-module such that for any $N \leq M$, $Z_2(M) \subseteq C$. Then the following are equivalent:
(1) M is e-gH.
(2) For any epimorphism $\varphi \in End_R(M)$, if there exist $C \leq M$ with $ker\varphi + C = M$, then M/C is nonsingular.
(3) For any epimorphism $\varphi \in End_R(M)$, if there exist $C \leq M$ with $ker\varphi + C = M$, then C is closed.

Proof (1) $\Longrightarrow$ (2) Let $\varphi \in End_R(M)$ be an epimorphism. If $ker\varphi + C = M$ for some $C \leq M$. Then $\varphi(C) = \varphi(M)$. From (Proposition 2.5), C is closed in M. By our assumption, $Z_2(M) \subseteq C$. So by ([2], Proposition 2.6), M/C nonsingular.
(2) $\Longrightarrow$ (3) Let $\varphi \in End_R(M)$ be an epimorphism. Assume that there exist $C \leq M$ with $ker\varphi + C = M$. Then $\varphi(C) = \varphi(M)$, by (2) M/C is nonsingular. Hence by ([2], Proposition 2.6), C is closed in M.
(3) $\Longrightarrow$ (1) Let $\varphi \in End_R(M)$ be an epimorphism. Assume that there exist $C \leq M$ with $\varphi(C) = \varphi(M)$, then $\varphi^{-1}(\varphi(C)) = \varphi^{-1}(\varphi(M))$ and so $ker\varphi + C = M$. By (3), C is closed in M. From (Proposition 2.5), M is e-gH. □

Proposition 2.7. Every direct summand of an e-gH module is e-gH.

Proof Let M be an e-gH module and $N \leq^{\oplus} M$. So, $M = N \oplus K$ for some $K \leq M$. Assume that $f \in End(N)$ and an epimorphism. Consider $I_K: K \to K$ is an identity map over K. Then $f \oplus I_K(M) = f \oplus I_K(N \oplus K) = f(N) \oplus I_K(K) = N \oplus K = M$, that means $f \oplus I_K$ is an epimorphism. But M is e-gH, then $ker(f \oplus I_K) \ll_e M$. It follow that $ker(f \oplus I_K) = kerf \oplus kerI_K = kerf \oplus 0 = kerf$, and then $kerf \ll_e M$. Since $kerf \leq N \leq^{\oplus} M$, then $kerf \ll_e N$, by ([5], Lemma 2.12 (1)). Therefore N is e-gH. □

Proposition 2.8. Let $M = M_1 \oplus M_2$ such that M_1 and M_2 be fully invariant under every surjection of M. Then M is e-gH if and only if M_i is e-gH for all $i = 1,2$.

Proof "If" part is follows directly by (Proposition 2.7).
"Only if" part. Let $f: M \to M$ be $an\ R-ep$imorphism. Then $f_i = f|_{M_i}: M_i \to M_i$ is an R-epimorphism for all $i = 1,2$, because M_1 and M_2 are fully invariant submodules. Since M_i is e-gH, for all $i = 1,2$, then $kerf_i \ll_e M_i$, so $kerf = ker(f_1 \oplus f_2) = ker\, f_1 \oplus ker\, f_1 \ll_e M_1 \oplus M_2 = M$, by ([10], Proposition 2.5(3)). Therefore $M = M_1 \oplus M_2$ is an e-gH module. □

Corollary 2.9. Let $M = \oplus_{i=1}^{n} M_i$ such that M_i be fully invariant under every surjection of M for all $i = 1,2,\dots,n$. Then M is e-gH if and only if M_i is e-gH for all $i = 1,2,\dots,n$.

Proposition 2.10. If $M = M_1 \oplus M_2$ with $r_R(M_1) \oplus r_R(M_2) = R$, then M is e-gH if and only if M_i is e-gH for all $i = 1,2$.

Proof "If" part is follows directly by (Proposition 2.7).
"Only if" part. Let $f: M \to M$ be an R-epimorphism. As $r_R(M_1) \oplus r_R(M_2) = R$ and $Imf \leq M_1 \oplus M_2$, then by ([1], Proposition 1.4.2) there exists $N \leq M_1$ and $K \leq M_2$ such that $Imf = N \oplus K$ implies $Imf|_{M_1} \oplus Imf|_{M_2} = N \oplus K$, thus $Imf|_{M_1} \leq M_1$ and $Imf|_{M_2} \leq M_2$. As M_i is e-gH and $f|_{M_i}$ is an R-epimorphism for all $i = 1,2$, then $kerf_i \ll_e M_i$. Then $kerf = ker(f_1 \oplus f_2) = ker\, f_1 \oplus ker\, f_1 \ll_e M_1 \oplus M_2 = M$, by ([10], Proposition 2.5(3)). Hence $M = M_1 \oplus M_2$ is an e-gH module. □

Lemma 2.11. Let $f: M \to \acute{M}$ be an R-isomorphism. If A is not e-small in $\acute{M}$ then $f^{-1}(A)$ is not e-small in M.

Proof If A is not e-small in $\acute{M}$, then there is a proper essential submodule E of $\acute{M}$ such that $A + E = \acute{M}$. Then $f^{-1}(A) + f^{-1}(E) = f^{-1}(A + E) = f^{-1}(\acute{M}) = M$ with $f^{-1}(E)$ is an essential in M. If $f^{-1}(E) = M$, then $E = f\left(f^{-1}(E)\right) = f(M) = \acute{M}$ that is a contradiction. Thus, $f^{-1}(E) \lhd M$ and henec $f^{-1}(A)$ is not e-small in M. □

Proposition 2.12. Let M be an R-module such that M/N be an e-gH R-module for any $0 \neq N \leq M$. Then M is e-gH.

Proof If false, then there is an R-epimorphism $f \in End(M)$ such that $kerf$ is not e-small, then $kerf \neq 0$. From 1st isomorphism theorem, there is an R-isomorphism $g: M/kerf \to M$. Let $\pi: M \to M/kerf$ be the natural R-epimorphism. It follows that $\pi g: M/kerf \to M/kerf$ is an R-epimorphism which $ker\, \pi g = g^{-1}(ker\pi) = g^{-1}(kerf)$ is not e-small in $M/kerf$, by (Lemma 2.11), a contradiction. Hence M is I e-gH R-module. □

Proposition 2.13. Let M be a nonsingular and e-gH R-module with $f \in End(M)$ is an epimorphism and $M/f(N)$ is singular for all and $N \leq M$. Then $f(L) \ll_e M$ if and only if $L \ll_e M$.

Proof If $f(L) \ll_e M$. Assume $L + K = M$ where $K \trianglelefteq M$. Then $f(L) + f(K) = M$. By hypothesis, $M/f(K)$ is singular and M a nonsingular R-module, ([7], Proposition 1.21) implies that $f(K) \trianglelefteq M$. As $f(L) \ll_e M$, then $f(K) = M$ and hence $K + kerf = M$. Since M is an e-gH R-module, then $kerf \ll_e M$ and so $K = M$. Therefore $L \ll_e M$. The converse is follows by ([10], Proposition 2.5(2)). □

Proposition 2.14. Let M be a module. Consider the following assertions:

(1) for any epimorphism $f \in End(M)$, if $N \ll_e M$ then $f^{-1}(N) \ll_e M$.

(2) M is e-gH.

Then (1) $\Longrightarrow$ (2). If M is a uniform module, then (2) $\Longrightarrow$ (1).

Proof (1) $\Longrightarrow$ (2) Suppose (1) hold. If $g \in End(M)$ is a surjective has not e-small kernel, then $g^{-1}(0)$ is not e-small in M, by assumption, (0) is not e-small in M, which is a contradiction. Hence $kerg \ll_e M$ and M is e-gH.

(2) $\Longrightarrow$ (1) Assume $f \in End(M)$ is a surjective and $N \ll_e M$. Let $f^{-1}(N) + K = M$ for some $K \trianglelefteq M$. Thus, $N + f(K) = M$. Since M is uniform, then $f(K) \trianglelefteq M$. As $N \ll_e M$ then $f(K) = M$. It follows that $K + kerf = M$. By (2), $kerf \ll_e M$, hence $K = M$. Therefore, $f^{-1}(N) \ll_e M$. □

Theorem 2.15. Let M be a quasi-projective uniform module. Then M is e-gH if and only if M/E is e-gH, for all $E \ll_e M$.

Proof The sufficiency is clear by taking $E = 0$. Assume M is e-gH, $E \ll_e M$ and $f: M/E \to M/E$ a surjective. Consider $\pi: M \to M/E$ is a natural map. Therefore $f\pi: M \to M/E$ is an homomorphism. As M is a quasi-projective module, there is a homomorphism $g: M \to M$ such that $\pi g = f\pi$. So $(Img + E)/E = \pi(Img) = f(\pi(M)) = f(M/E) = M/E$, hence $Img + E = M$. Since $E \ll_e M$ and Img is essential in M (as M is uniform), then $Img = M$, i.e., g is an epimorphism. Thus, $kerg \ll_e M$. As will as, $\pi g(E) = f\pi(E) = E$ imply that $g(E) + E = g(E) + ker\pi = \pi^{-1}\left(\pi g(E)\right) = \pi^{-1}(E) = E$, and then $g(E) \leq E$. So $E + kerg = g^{-1}(g(E)) \leq g^{-1}(E)$, i.e., $E \leq g^{-1}(E)$. As $f\pi = \pi g$, then $ker(f\pi) = ker(\pi g)$, so $\pi^{-1}(kerf) = g^{-1}(ker\pi) = g^{-1}(E)$. Thus $kerf = \pi(\pi^{-1}(kerf)) = \pi(g^{-1}(E)) = g^{-1}(E)/E$. From (Proposition 2.14), $g^{-1}(E) \ll_e M$ and hence $g^{-1}(E)/E = \pi(g^{-1}(E)) \ll_e M/E$, by ([10], Proposition 2.5(2)). So $kerf \ll_e M/E$ and M/E is e-gH. □

Proposition 2.16. Let M be a module such that N is a fully invariant submodule of M and M/N is Hopfian. If N is an e-gH module, then so is M.

Proof Consider a surjective $f \in End(M)$. Define $g \in End(M/N)$ by $g(m + N) = f(m) + N$ for all $m + N \in M/N$. Thus $Img = g(M/N) = f(M)/N = M/N$, i.e., g is an R-epimorphism, so g is an R-isomorphism (i.e., $kerg = N$), since M/N is Hopfian. We conclude that $kerf + N = \{m + N \mid m \in kerf\} = \{m + N \mid f(m) = 0\} = \{m{+}N \mid f(m) + N = N\} = \{m + N \mid g(m + N) = N\} = kerg = N$. So $kerf \subseteq N$. Put $h = f|_N$, then h is also an R-epimorphism, because N is fully invariant. Since N is e-gH, $kerh \ll_e N$. It follow that $kerh = kerf|_N = kerf \cap N = kerf$, then $kerf \ll_e N \leq M$, thus $kerf \ll_e M$. Hence M is an e-gH module. □

Proposition 2.17. Let M_1 and M_2 be an R-modules such that $M_1 \cong M_2$. Then M_1 is e-gH if and only if M_2 is e-gH.

Proof Assume that M_1 is an e-gH R-module such that $M_1 \cong M_2$. Then there is an R-isomorphism $g: M_1 \to M_2$. Notice that $g^{-1}: M_2 \to M_1$ is an R-isomorphism. Let $f \in End(M_2)$ be an R-epimorphism. Put $h = g^{-1} \circ f \circ g$. Then $h \in End(M_1)$ and $h(M_1) = g^{-1} \circ f\left(g(M_1)\right) = g^{-1} \circ f(M_2) = g^{-1}(M_2) = M_1$, i.e., h is a surjective. Therefore $kerh \ll_e M_1$, as M_1 is e-gH. By ([10], Proposition 2.5(2)), $g(kerh) \ll_e M_2$. So $g(kerh) = g(ker(g^{-1} \circ f \circ g) = g(g^{-1}(ker(g^{-1} \circ f) = f^{-1}(kerg^{-1}) = f^{-1}(0) = kerf$. Hence $kerf \ll_e M_2$ and M_2 is e-gH. □

Proposition 2.18. Let M be an R-module and let $K, S \leq M$ with $M = K + S$ and $M/(K \cap S)$ is e-gH. Then M/K and M/S are e-gH.

Proof If $M = K + S$, it follows that $M/(K \cap S) = (K/(K \cap S)) \oplus (S/(K \cap S))$, thus $K/(K \cap S)$ and $S/(K \cap S)$ are e-gH by (Proposition 2.7). As $K/(K \cap S) \cong M/S$ and $S/(K \cap S) \cong M/K$, so M/K and M/S are e-gH, by (Proposition 2.17). □

The next is a characterization of e-gH modules.

Theorem 2.19. Let M be a module. Then M is e-gH if and only if for any proper $N \leq M$ with $M/N \cong M$, N is an e-small submodule of M.

Proof Suppose that M is an e-gH module. Let N be a proper submodule of M such that $M/N \cong M$. Then there is an isomorphism $\psi: M/N \to M$. Now, consider the sequence $M \xrightarrow{\pi} M/N \xrightarrow{\psi} M$ where π is a canonical epimorphism map. Since M is an e-gH module and $\psi \circ \pi \in End(M)$ is a surjective, then $ker(\psi \circ \pi) \ll_e M$. But, we have that $ker(\psi \circ \pi) = \pi^{-1}(ker\psi) = \pi^{-1}(N) = ker\pi = N$, therefore $N \ll_e M$. Conversely, if $g: M \to M$ is an epimorphism, 1st isomorphism theorem implies that $M/kerg \cong M$ and $kerg \neq M$ (if $kerg =$

M then $g = 0$, a contradiction) so by assumption, $kerg \ll_e M$, and this ends the proof. □

Proposition 2.20. Let $\mathcal{F}$ be a property of modules preserved under isomorphism. Assume a module M has the property $\mathcal{F}$ and satisfies ACC on proper (non e-small) submodules N with M/N has the property $\mathcal{F}$, then M is e-gH.

Proof Assume that M is not e-gH, then by (Theorem 2.19), there exists a proper non e-small submodule L_1 of M such that $M/L_1 \cong M$. Therefore M/L_1 is not e-gH and has the property $\mathcal{F}$, by hypothesis. Again by (Theorem 2.19),, there is a proper non e-small submodule L_2/L_1 of M/L_1 with $M/L_2 \cong (M/L_1)/(L_2/L_1) \cong M/L_1$. Note that L_2 is a proper non e-small submodule of M. By repeating this argument, we get an ascending chain $L_1 \subseteq L_2 \subseteq L_3 \subseteq \cdots$ of proper (non e-small) submodules of M, which is a contradiction. Thus must be M is e-gH. □

Corollary 2.21. If M is a module has ACC on proper (non e-small) submodules N with M/N is not e-gH module, then M is e-gH.

Proof Let $\mathcal{F}$ be the property for not being e-gH, and assume M is not e-gH. From (Proposition 2.20), M must be e-gH. This contradiction proves that M is e-gH. □

Corollary 2.22. Every nonzero module satisfies ACC on proper (non e-small) submodules is e-gH.

Proof We may assume that $M \neq 0$ has ACC on proper (non e-small) submodules and that $\mathcal{F}$ is the property of being nonzero. By (Proposition 2.20), M must be e-gH. □

We will generalize the notion of e-gH modules as follows**.**

Definition 2.23. Let M_1 and M_2 be two R-modules. M_1 is called e-gH relative to M_2, if for each epimorphism $g: M_1 \to M_2$, we have $kerg \ll_e M_1$.

Remark 2.24. Given the definition above,, we see that an R-module M is e-gH if and only if M is e-gH relative to M.

The features of the e-gH module relative to a module are described below.

Proposition 2.25. Let M_1 and M_2 be R-modules. Then the following conditions are equivalent:

(1) M_1 is e-gH relative to M_2.

(2) For each $L \leq^{\oplus} M_1$, L is e-gH relative to M_2.

(3) For each $L \leq M_1$, M_1/L is e-gH relative to M_2.

Proof (1) ⟹ (2) Assume $M_1 = L \oplus K$, where $K \leq M_1$ and $g: L \to M_2$ a surjective. Let $\pi: M_1 \to L$ be a projection map. Then $g\pi: M_1 \to M_2$ is a surjective and hence $ker(g\pi) \ll_e M_1$, from (1). It is obvious that $ker(g\pi) = \pi^{-1}(kerg) = kerg \oplus K$. Thus, $kerg \oplus K \ll_e L \oplus K$ and hence $kerg \ll_e L$, by ([10], Proposition 2.5(3)).

(2) ⟹ (1) By taking $L = M_1$.

(1) ⟹ (3) Let $L \leq M_1$ and $g: M_1/L \to M_2$ a surjective. Hence $g\pi: M_1 \to M_2$ is a surjective, where $\pi: M_1 \to M_1/L$ is the natural map. By (1), $ker(g\pi) \ll_e M_1$. As $ker(g\pi) = \pi^{-1}(kerg)$, hence $\pi^{-1}(kerg) \ll_e M_1$, ([10], Proposition 2.5(1)) implies that $kerg = \pi(\pi^{-1}(kerg)) \ll_e M_1/L$. Therefore M_1/L is e-gH relative to M_2.

(3) ⟹ (1) By taking $L = 0$. □

Proposition 2.26. If M is a module with the property that for any $g \in End(M)$, there exists an $n \in \mathbb{Z}^+$ such that $kerg^n \cap Img^n \ll_e M$, then M is e-gH.

Proof Let $g \in End(M)$ is a surjective. By assumption, there is an integer $n \geq 1$ such that $kerg^n \cap Img^n \ll_e M$. It follows that $g^n \in End(M)$ is a surjective, i.e., $Img^n = M$. Thus, $kerg^n \cap Img^n = kerg^n \cap M = kerg^n \ll_e M$. It is easy to see that $kerg \leq kerg^n$, therefore $kerg \ll_e M$ by ([10], Proposition 2.5(a)). Hence M is e-gH. □

Proposition 2.27. Let M be an R-module. If for any R-epimorphism $\varphi: M \to M$, there exist $n \geq 1$ such that $ker\varphi^n = ker\varphi^{n+i}$ for all $i \in \mathbb{Z}^+$, then M is e-gH.

Proof Let $\varphi \in End(M)$ be any surjective. We claim that $ker\,\varphi^n \cap Im\varphi^n = 0$. Let $y \in ker\,\varphi^n \cap Im\varphi^n$. Thus $\varphi^n(y) = 0$ and $y = \varphi^n(x)$ for some $x \in M$. Hence $\varphi^{2n}(x) = \varphi^n(y) = 0$ and hence $x \in ker\varphi^{2n}$. But from our assumption we have that $ker\varphi^n = ker\varphi^{n+n} = ker\varphi^{2n}$, and so $x \in ker\varphi^n$ therefore $0 = \varphi^n(x) = y$. Hence $ker\,\varphi^n \cap Im\varphi^n = 0$. As φ is a surjective, so $Im\varphi^n = M$, thus $ker\varphi^n = 0$. But $ker\varphi \subseteq ker\varphi^n$, then $ker\varphi \ll_e M$. Therefore M is e-gH. □

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