Decomposition Matrix for the projective Characters $S_{28}, p = 11$

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Abstract— *In this work, we compute the decomposition matrix to the spin characteristics of S_28 for a given field of characteristic p=11, the connections between the irreducible modular spin characters and the irreducible spin characters of S_28. The technique employed in this study is (r,r ̅)-inducing, which creates projective characters for S_28 by using the projective characters of S_27. Maple was used to seeing all available columns and then choosing the potentially appropriate columns of them. In order to explore irreducible modular spin characteristics, general correlations and theorems will be discovered as a result of this research.*

Keywords—Irreducible modular spin characters, Representation group, Decomposition matrix to the spin characters.

I. Introduction

The spin representations of S_n are representations without Z in their kernel, as shown in [11], and are created by a central subgroup Z={-1,1} in the representation group ̅S_n of the symmetric group S_n. The various components of the n-partitions serve as labels for the spin characters of the spin representations of S_n, and they are indicated by ⟨α⟩. In fact, there is one irreducible spin character represented by 〖⟨α⟩〗^* that is self-associate if α=(α_1,α_2,…α_m) and n-m is even. If n-m is odd, there are two associate spin characters denoted by ⟨α⟩ and ⟨α⟩' see [11,6]. The relationships between the irreducible spin characters and the irreducible modular spin characters of S_n serve as the decomposition matrix for spin characters. The number of rows corresponds to the number of projective characters, and the number of columns to the number of (p,α)-regular classes [9]. Any spin characters of S_n can be written as a liner combination using the irreducible spin character's non-negative coefficients [5]. The method (r,r ̅)-inducing (restricting) is used to distribute the spin characters into p-blocks see [4,10]. Numerous individuals have made contributions to this area of study and do research on this subject [13,14,1,12,3,2]. In this study, we identify the spin character decomposition matrix for the S_28 over fields of characteristic p=11. Let's establish certain notations and terminologies before we state any results. The terms "p.s." (p.i.s.) and "m.s." (i.m.s.) stand for "principal spin character" (indecomposable) and "modular spin character" (irreducible), respectively. "d_i" stands for "p.i.s." of S_n, "D_i" stands for "p.i.s." of S_(n-1), and ⟨ ⟩^no is the number of i.m.s.

II. Preliminaries

For the study, some important conclusions were needed.

Theorem 2.1 [11] Degree of the spin character $\langle \alpha_1, \dots, \alpha_{\mathrm{m}} \rangle = 2^{[(n-m)/2]} \frac{n!}{\prod_{i=1}^{m} \alpha_i!} \prod_{1 \le i < j \le m} \frac{(\alpha_i - \alpha_j)}{(\alpha_i + \alpha_j)}$

Theorem 2.2 [7] Let B be a p-block G of defect one $|\mathrm{G}| = \mathrm{m}_0 p^{\mathrm{a}}$ and let b be the number of p-conjugate classes to the irreducible ordinary character χ of G then:

1. There exists a positive integer number N such that the irreducible ordinary characters lying in the block B can be partitioned into two disjoint classes:
$$B_1 = \{\chi \in B | b, \quad deg\chi \equiv N mod p^a\},$$
$$B_2 = \{\chi \in B | b, \quad deg\chi \equiv -N mod p^a\}.$$
2. Each coefficient of the decomposition matrix of the block B is 1 or 0.

Theorem 2.3 [8] Let G be a group of order $m_0 p^a$, where $(p, m_0) = 0$. If c is a principal character of sub group H of G, then deg c $\equiv 0 \bmod p^a$

Theorem 2.4 [6] Let p be odd and n be even. If $p \nmid n$, then $\langle n\rangle = \varphi\langle n\rangle$ and $\langle n\rangle' = \varphi\langle n\rangle'$ are distinct irreducible modular spin characters of degree $2^{(n-2)/2}$.

III. DECOMPOSITION MATRIX FOR THE SPIN CHARACTERS OF S_{28}

Decomposition matrix for $\bar{S}_{28}$ of degree (332,276), and it is decomposed in to blocks of character it consists of 52 blocks which are B_1,B_2,B_3,B_4 of defect two, B_5, $B_6, \cdots, B_{18}$ of defect one the others block of defect zero.

case 1. The decomposition matrix for the blocks B_1 of type associative as shown in the **Tables 1**.

Table1. Block B_1

spin characters	Decomposition matrix																																						
⟨28⟩	1																																							
⟨28⟩′		1																																						
⟨22,6⟩*	1	1	1	1																																					
⟨21,6,1⟩			1		1																																				
⟨21,6,1⟩′				1		1																																			
⟨20,6,2⟩					1		1																																		
⟨20,6,2⟩′						1		1																																	
⟨19,6,3⟩							1		1																																
⟨19,6,3⟩′								1		1																															
⟨18,6,4⟩									1		1																														
⟨18,6,4⟩′										1		1																													
⟨17,11⟩*	1	1	1	1										1	1																									
⟨17,10,1⟩	1	1	1		1								1		1																								
⟨17,10,1⟩′	1	1		1		1								1		1																							
⟨17,9,2⟩					1		1								1		1																						
⟨17,9,2⟩′						1		1								1		1																					
⟨17,8,3⟩							1		1								1		1																				
⟨17,8,3⟩′								1		1								1		1																			
⟨17,7,4⟩									1		1								1		1																		
⟨17,7,4⟩′										1		1								1		1																	
⟨17,6,5⟩											1										1																		
⟨17,6,5⟩′												1										1																	
⟨15,7,6⟩																			1		1		1																
⟨15,7,6⟩′																				1		1		1															
⟨14,8,6⟩																	1		1				1		1														
⟨14,8,6⟩′																		1		1				1		1													
⟨13,9,6⟩															1		1								1		1												
⟨13,9,6⟩′																1		1								1		1											
⟨12,10,6⟩	2												1	1	1												1		1	1									
⟨12,10,6⟩′		2											1	1		1												1	1	1									
⟨11,10,6,1⟩*													1	1													1	1	1	1	1	1							
⟨11,9,6,2⟩*																									1	1	1	1	1	1	1	1	1	1					
⟨11,8,6,3⟩*																							1	1	1	1							1	1	1	1			
⟨11,7,6,4⟩*																							1	1											1	1			
⟨10,9,6,2,1⟩																													1				1				1			
⟨10,9,6,2,1⟩′																														1				1				1		
⟨10,8,6,3,1⟩																															1		1		1		1		1	
⟨10,8,6,3,1⟩′																																1		1		1		1		1
⟨10,7,6,4,1⟩																																			1				1	
⟨10,7,6,4,1⟩′																																				1				1
⟨9,8,6,3,2⟩																															1								1	
⟨9,8,6,3,2⟩′																																1								1
⟨9,7,6,4,2⟩																																					1		1	
⟨9,7,6,4,2⟩′																																						1		1
⟨8,7,6,4,3⟩																																					1			
⟨8,7,6,4,3⟩′																																						1		
	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_{10}	d_{11}	d_{12}	d_{13}	d_{14}	d_{15}	d_{16}	d_{17}	d_{18}	d_{19}	d_{20}	d_{21}	d_{22}	d_{23}	d_{24}	d_{25}	d_{26}	d_{27}	d_{28}	d_{29}	d_{30}	d_{31}	d_{32}	d_{33}	d_{34}	d_{35}	d_{36}	d_{37}	d_{38}	d_{39}	d_{40}

Proof. Using (6,6)-inducing of p.i.s. method on $D_1$for S_{27} to S_{28} we have

$$D_1\uparrow^{(6,6)} S_{28} = (\langle 27\rangle^* + \langle 22,5\rangle + \langle 22,5\rangle' + \langle 16,11\rangle + \langle 16,11\rangle' + 2\langle 16,10,1\rangle^* + 2\langle 12,10,5\rangle^*)\uparrow^{(6,6)} S_{28}$$
$$= \langle 28\rangle + \langle 28\rangle' + 2\langle 22,6\rangle^* + 2\langle 17,11\rangle^* + 2\langle 17,10,1\rangle + 2\langle 17,10,1\rangle' + 2\langle 12,10,6\rangle + 2\langle 12,10,6\rangle' = c_1$$

Similarly, using $(\mathrm{r},\bar{\mathrm{r}})$ −inducing of p.i.s. for S_{27} to S_{28} gives $D_{111}\uparrow^{(0,1)} S_{27} = d_3$,$D_{112}\uparrow^{(0,1)} S_{27} = d_4$, $D_3\uparrow^{(6,6)} S_{27} = c_2$, $D_4\uparrow^{(6,6)} S_{27} = c_3$, $D_5\uparrow^{(6,6)} S_{27} = c_4$, $D_6\uparrow^{(6,6)} S_{27} = c_5$, $D_{113}\uparrow^{(0,1)} S_{27} = d_{13}$,$D_{114}\uparrow^{(0,1)} S_{27} = d_{14}$,$D_8\uparrow^{(6,6)} S_{27} = c_6$, $D_9\uparrow^{(6,6)} S_{27} = c_7$, $D_{10}\uparrow^{(6,6)} S_{27} = c_8$, $D_{11}\uparrow^{(6,6)} S_{27} = c_9$, $D_{12}\uparrow^{(6,6)} S_{27} = c_{10}$, $D_{13}\uparrow^{(6,6)} S_{27} = c_{11}$,$D_{14}\uparrow^{(6,6)} S_{27} = c_{12}$, $D_{115}\uparrow^{(0,1)} S_{27} = d_{29}$,$D_{116}\uparrow^{(0,1)} S_{27} = d_{30}$,$D_{16}\uparrow^{(6,6)} S_{27} = c_{13}$, $D_{117}\uparrow^{(0,1)} S_{27} = d_{33}$, $D_{118}\uparrow^{(0,1)} S_{27} = d_{34}$, $D_{119}\uparrow^{(0,1)} S_{27} = d_{35}$, $D_{120}\uparrow^{(0,1)} S_{27} = d_{36}$, $D_{19}\uparrow^{(6,6)} S_{27} = c_{14}$, $D_{20}\uparrow^{(6,6)} S_{27} = c_{15}$,
respectively. Now that $\langle 28\rangle$ and $\langle 28\rangle'$ are distinct irreducible modular spin characters (**Theorem 2.4**), c_1 must divide into d_1 , d_2 .Since $\langle 20,6,2\rangle \neq \langle 20,6,2\rangle'$ are distinct irreducible modular spin characters so c_2 or c_3 is split. Suppose c_2 is split to d_5, d_6, but $\langle 19,6,3\rangle \neq \langle 19,6,3\rangle'$ then c_3 split to d_7, d_8. If c_3 split to d_7, d_8, and frome $(11,\alpha)$-regular classes

$$\langle 20,6,2\rangle - \langle 19,6,3\rangle + \langle 18,6,4\rangle \neq \langle 20,6,2\rangle' - \langle 19,6,3\rangle' + \langle 18,6,4\rangle' \quad (1)$$

then c_2 also split to d_5, d_6. So in both cases we get c_2 and c_3 are splits. A split is made in c_4 or c_5 because of $\langle 18,6,4\rangle \neq \langle 18,6,4\rangle'$. If c_4 is divided into d_9 and d_{10}, However, $\langle 17,6,5\rangle \neq \langle 17,6,5\rangle'$ split c_5 to d_{11}, d_{12} after that. If c_5 split, and frome$(11,\alpha)$-regular classes

$$\langle 17,7,4\rangle - \langle 17,6,5\rangle \neq \langle 17,7,4\rangle' - \langle 17,6,5\rangle' \quad (2)$$

then c_4 split , so in both cases we get c_4 and c_5 are splits. Since $\langle 17,9,2\rangle \neq \langle 17,9,2\rangle'$ so c_6 or c_7 is split. Suppose c_6 is split to d_{15}, d_{16}, but $\langle 14,8,6\rangle \neq \langle 14,8,6\rangle'$ then c_7 split to d_{17}, d_{18}. If c_7 split to d_{17}, d_{18} , and frome $(11,\alpha)$-regular classes

$$\langle 17,9,2\rangle - \langle 17,8,3\rangle + \langle 17,7,4\rangle \neq \langle 17,9,2\rangle' - \langle 17,8,3\rangle' + \langle 17,7,4\rangle' \quad (3)$$

then c_6 split, so in both cases we get c_6 and c_7 are splits. Since $\langle 17,7,4\rangle \neq \langle 17,4,4\rangle'$it follows that either c_8 or c_9 is split.

Suppose c_8 is split to d_{19}, d_{20}, but $\langle 17,6,5\rangle \neq \langle 17,6,5\rangle'$ then c_9 split to d_{21}, d_{22}.If c_9 split, and frome $(11,\alpha)$-regular classes

$$\langle 17,7,4\rangle - \langle 17,6,5\rangle \neq \langle 17,7,4\rangle' - \langle 17,6,5\rangle' \quad (4)$$

then c_8 split, so in both cases we get c_8 and c_9 are splits. c_{10} or $c_{11}$is split because of $\langle 14,8,6\rangle \neq \langle 14,8,6\rangle'$. Suppose c_{10} is divided into d_{23} and d_{24}. However, $\langle 13,9,6\rangle \neq \langle 13,9,6\rangle'$ then split c_{11} to d_{25}, d_{26}. If c_{11} split, and frome $(11,\alpha)$-regular classes

$$\langle 14,8,6\rangle - \langle 13,9,6\rangle + \langle 12,10,6\rangle \neq \langle 14,8,6\rangle' - \langle 13,9,6\rangle' + \langle 12,10,6\rangle' \quad (5)$$

then c_{10} split, so in both cases we get c_{10} and c_{11} are splits. Since $\langle 13,9,6\rangle \neq \langle 13,9,6\rangle'$ so $c_{12} = d_{27} + d_{28}$or there are two columns φ_1,φ_2 such that

$$\varphi_1 = a_1\langle 13,9,6\rangle + a_2\langle 12,10,6\rangle + a_3\langle 11,10,6,1\rangle^* + a_4\langle 11,9,6,2\rangle^* + a_5\langle 11,8,6,3\rangle^* + a_6\langle 11,7,6,4\rangle^* + a_7\langle 10,9,6,2,1\rangle + a_8\langle 10,8,6,3,1\rangle + a_9\langle 10,7,6,4,1\rangle + a_{10}\langle 9,8,6,3,2\rangle + a_{11}\langle 9,7,6,4,2\rangle + a_{12}\langle 8,7,6,4,3\rangle$$

$$\varphi_2 = a_1\langle 13,9,6\rangle' + a_2\langle 12,10,6\rangle + a_3\langle 11,10,6,1\rangle^* + a_4\langle 11,9,6,2\rangle^* + a_5\langle 11,8,6,3\rangle^* + a_6\langle 11,7,6,4\rangle^* + a_7\langle 10,9,6,2,1\rangle' + a_8\langle 10,8,6,3,1\rangle' + a_9\langle 10,7,6,4,1\rangle' + a_{10}\langle 9,8,6,3,2\rangle' + a_{11}\langle 9,7,6,4,2\rangle' + a_{12}\langle 8,7,6,4,3\rangle'$$

to find are since $\langle 13,9,6\rangle \downarrow S_{27} = \langle 12,9,6\rangle^{*2} + \langle 13,8,6\rangle^{*2} + \langle 13,9,5\rangle^{*4} = 8$ of i.m.s., then we have $a_1 \in \{0,1,2,3,4\}$. In the same way we got $a_2, a_6 \in \{0,1,\ldots,6\}$, $a_3 \in \{0,1,\ldots,8\}$, a_4, $a_5 \in \{0,1,\ldots,12\}$, a_7, $a_9 \in \{0,1,\ldots,4\}$, $a_8 \in \{0,1,\ldots,7\}$, $a_{10}, a_{11} \in \{0,1,2,3\}$, $a_{12} \in \{0,1,\}$. let $a_1 = 1$ (if $a_1 = 0$ then c_{12} is split). since $\langle 13,9,6\rangle \downarrow S_{27} \cap \langle 11,7,6,4\rangle^* \downarrow S_{27}$ has no i.m.s so $a_6 = 0$, the same way we get a_7, ..., a_{12} are equal to zero, , we must discuss all probabilities such that the degree $\varphi_1, \varphi_2 \equiv 0\ mod 11^2$, that's difficult, so we do the algorithm by maple program in appendix to help us and we find it are equal to **352** probabilities, therefore, we aim to limit the number of possibilities, since inducing m.s. is m.s. we have:

$$(\langle 14,7,6\rangle^* + \langle 10,7,6,3,1\rangle^* - \langle 11,7,6,3\rangle)\uparrow^{(4,8)} S_{28} = \langle 15,7,6\rangle + \langle 15,7,6\rangle' + \langle 14,8,6\rangle + \langle 14,8,6\rangle' + \langle 10,8,6,3,1\rangle + \langle 10,7,6,4,1\rangle' - \langle 11,10,6,1\rangle^* - \langle 11,7,6,4\rangle^*\ hence\ a_5 = 0 \quad (6)$$

Similarly in the following equations

$$(\langle 11,8,6,2\rangle - \langle 13,8,6\rangle^* + \langle 17,8,2\rangle^*)\uparrow^{(3,9)}\ hence\ a_4 \geq a_1 \quad (7)$$

$$(\langle 13,8,6\rangle^* - \langle 11,8,6,2\rangle - \langle 10,8,6,2,1\rangle^*)\uparrow^{(3,9)} S_{28}\ hence\ a_1 \geq a_4 \therefore a_1 = a_4 \quad (8)$$

then we get degrees $\varphi_1, \varphi_2 \equiv 0\ mod\ 11^2$ only when $\varphi_1 + \varphi_2 = m(d_{27} + d_{28})$, m ∈ {1,2,3,4} in order for "φ_1, φ_2 is split the c_{12}". Since $\langle 9,8,6,3,2\rangle \neq \langle 9,8,6,3,2\rangle'$ so c_{13} or c_{15} is split. Suppose c_{13} is split to d_{31}, d_{32}, but $\langle 9,7,6,4,2\rangle \neq \langle 9,7,6,4,2\rangle'$ then c_{15} split to d_{39}, d_{40}. If c_{15} split and frome $(11,\alpha)$-regular classes.

$$\langle 9,8,6,3,2\rangle - \langle 9,7,6,4,2\rangle + \langle 8,7,6,4,3\rangle \neq \langle 9,8,6,3,2\rangle' - \langle 9,7,6,4,2\rangle' + \langle 8,7,6,4,3\rangle' \quad (9)$$

then c_{13} split, so in both cases we get c_{13} and c_{15} are splits. Finly, since $\langle 8,7,6,4,3\rangle \neq \langle 8,7,6,4,3\rangle'$ so c_{14} divided or there are two columns φ_1,φ_2. Suppose $a_{12} = 1$, but block $\boldsymbol{B_1}$ is associate so there must be another column associate with c_{14} then $\langle 8,7,6,4,3\rangle$ has 3 i.m.s. , but $\langle 8,7,5,4,3\rangle \downarrow S_{27}$ has only two of i.m.s. so this contradicts the hypothesis , then $c_{14} = d_{39} + d_{40}$. Using the information above, we divide c_1 into c_{15} and then turn to **Table 1**.

Case 2. Table 2 shows the decomposition matrices for the spin block $\boldsymbol{B_2}$of the double type.

Table 2. Block B_2

Spin character	**Decomposition matrix**																			
$\langle 27,1\rangle^*$	1																			
$\langle 23,5\rangle^*$	1	1																		
$\langle 22,5,1\rangle$		1	1																	
$\langle 20,5,2,1\rangle^*$			1	1																
$\langle 19,5,3,1\rangle^*$				1	1															
$\langle 18,5,4,1\rangle^*$					1	1														
$\langle 16,12\rangle^*$		1					1													
$\langle 16,11,1\rangle$	1	1	1				1	1												
$\langle 16,9,2,1\rangle^*$			1	1				1	1											
$\langle 16,8,3,1\rangle^*$				1	1				1	1										
$\langle 16,7,4,1\rangle^*$					1	1				1	1									
$\langle 16,6,5,1\rangle^*$						1					1									
$\langle 15,7,5,1\rangle^*$										1	1	1								
$\langle 14,8,5,1\rangle^*$									1	1		1	1							
$\langle 13,9,5,1\rangle^*$								1	1				1	1						
$\langle 12,11,5\rangle$	1						1	1							1					
$\langle 12,10,5,1\rangle^*$							2	1						1	2	2				
$\langle 12,9,5,2\rangle^*$													1	1	1	2	1			
$\langle 12,8,5,3\rangle^*$												1	1				1	1		
$\langle 12,7,5,4\rangle^*$												1						1		
$\langle 11,9,5,2,1\rangle$															1	1	1		1	
$\langle 11,8,5,3,1\rangle$																1	1	1	1	1
$\langle 11,7,5,4,1\rangle$																		1		1
$\langle 9,8,5,3,2,1\rangle^*$																1				1
$\langle 9,7,5,4,2,1\rangle^*$																			1	1
$\langle 8,7,5,4,3,1\rangle^*$																			1	
	d_{41}	d_{42}	d_{43}	d_{44}	d_{45}	d_{46}	d_{47}	d_{48}	d_{49}	d_{50}	d_{51}	d_{52}	d_{53}	d_{54}	d_{55}	d_{56}	d_{57}	d_{58}	d_{59}	d_{60}

Proof. Using $(\mathrm{r},\overline{\mathrm{r}})$- inducing of p.i.s. D_{21}, D_2, D_3, …, D_{11}, D_{185}, D_{45}, D_{47}, D_{49}, D_{16}, D_{162}, D_{55}, D_{19}, D_{20}, for S_{27} to S_{28},and on $(11,\alpha)$-regular classes:

1. $\langle 22,5,1\rangle = \langle 22,5,1\rangle'$
2. $\langle 16,11,1\rangle = \langle 16,11,1\rangle'$
3. $\langle 12,11,5\rangle = \langle 12,11,5\rangle'$
4. $\langle 11,9,5,2,1\rangle = \langle 11,9,5,2,1\rangle'$
5. $\langle 11,8,5,3,1\rangle = \langle 11,8,5,3,1\rangle'$
6. $\langle 11,7,5,4,1\rangle = \langle 11,7,5,4,1\rangle'$
7. $\langle 9,7,5,4,2,1\rangle^* = \langle 8,7,5,4,3,1\rangle^* + \langle 11,7,5,4,1\rangle - \langle 12,7,5,4\rangle^* + \langle 15,7,5,1\rangle^* - \langle 16,7,4,1\rangle^* + \langle 18,5,4,1\rangle^*$
8. $\langle 11,7,5,4,1\rangle = \langle 11,8,5,3,1\rangle - \langle 11,9,5,2,1\rangle + \langle 12,11,5\rangle - \langle 16,11,1\rangle + \langle 23,5\rangle^* - \langle 27,1\rangle^*$
9. $\langle 12,8,5,3\rangle^* = \langle 12,7,5,4\rangle^* + \langle 12,9,5,2\rangle^* - \langle 12,10,5,1\rangle^* + \langle 12,11,5\rangle + \langle 16,12\rangle^* - \langle 23,5\rangle^*$
10. $\langle 16,7,4,1\rangle^* = \langle 16,6,5,1\rangle^* + \langle 16,8,3,1\rangle^* - \langle 16,9,2,1\rangle^* + \langle 16,11,1\rangle - \langle 16,12\rangle^* + \langle 27,1\rangle^*$
11. $\langle 12,10,5,1\rangle = 2\langle 9,8,5,3,2,1\rangle^* - 2\langle 9,7,5,4,2,1\rangle^* + 2\langle 8,7,5,4,3,1\rangle^* + 2\langle 12,11,5\rangle - 2\langle 27,1\rangle^* - \langle 13,9,5,1\rangle^* + \langle 14,8,5,1\rangle^* - \langle 15,7,5,1\rangle^* + \langle 16,6,5,1\rangle^* - \langle 18,5,4,1\rangle^* + \langle 19,5,3,1\rangle^* - \langle 20,5,2,1\rangle^* + \langle 22,5,1\rangle^* - \langle 23,5\rangle^* - \langle 27,1\rangle^*$.
12. $\langle 9,8,5,3,2,1\rangle^* = \langle 11,8,5,3,1\rangle - \langle 8,7,5,4,3,1\rangle^* - \langle 12,8,5,3\rangle^* + \langle 15,7,5,1\rangle^* - \langle 16,7,4,1\rangle^* + \langle 18,5,4,1\rangle^*$

Then the matrix contains at most 32 columns since the number of the i.m.s. is equal or less than the number of the spin characters, but **Table 2** contains at most 20 columns since there are 12 equations corresponding the spin characters of S_{28} in $\boldsymbol{B_2}$.Then we get **Table 2**.

Case 3. The decomposition matrix for the spin block $\boldsymbol{B_3}$ of type double is a**Table 3.**

Table 3. Block B_3

Spin character	Decomposition matrix																			
$\langle 26,2\rangle^*$	1																			
$\langle 24,4\rangle^*$	1	1																		
$\langle 22,4,2\rangle$		1	1																	
$\langle 21,4,2,1\rangle^*$			1	1																
$\langle 19,4,3,2\rangle^*$				1	1															
$\langle 17,5,4,2\rangle^*$					1	1														
$\langle 16,6,4,2\rangle^*$					1	1	1													
$\langle 15,13\rangle^*$		1						1												
$\langle 15,11,2\rangle$	1	1	1					1	1											
$\langle 15,10,2,1\rangle^*$			1	1					1	1										
$\langle 15,8,3,2\rangle^*$				1	1					1	1									
$\langle 15,7,4,2\rangle^*$					1		1				1	1								
$\langle 15,6,5,2\rangle^*$							1					1								
$\langle 14,8,4,2\rangle^*$										1	1	1	1							
$\langle 13,11,4\rangle$	1							1	1					1						
$\langle 13,10,4,1\rangle^*$								2	1	1				1	1					
$\langle 13,9,4,2\rangle^*$										1			1		1	1				
$\langle 13,8,4,3\rangle^*$												1	1			1	1			
$\langle 13,6,5,4\rangle^*$												1					1			
$\langle 12,10,4,2\rangle^*$								2						2	1	1		2		
$\langle 11,10,4,2,1\rangle$														1		1		1	1	
$\langle 11,8,4,3,2\rangle$																1	1	1	1	1
$\langle 11,6,5,4,2\rangle$																	1			1
$\langle 10,8,4,3,2,1\rangle^*$																		1		1
$\langle 10,6,5,4,2,1\rangle^*$																			1	1
$\langle 8,6,5,4,3,2\rangle^*$																			1	
	d_{61}	d_{62}	d_{63}	d_{64}	d_{65}	d_{66}	d_{67}	d_{68}	d_{69}	d_{70}	d_{71}	d_{72}	d_{73}	d_{74}	d_{75}	d_{76}	d_{77}	d_{78}	d_{79}	d_{80}

Proof. Using $(\mathrm{r},\overline{\mathrm{r}})$-inducing of p.i.s. D_{21}, D_{23}, D_{25}, D_{27}, D_{29}, D_{31}, D_{33}, D_{35}, D_{79}, D_{38}, D_{41}, D_{43}, D_{218}, D_{87}, D_{89}, D_{53}, D_{55}, D_{124}, D_{57}, $D_{59}$of S_{27} to S_{28}, and on $(11,\alpha)$-regular classes:

1. $\langle 22,4,2\rangle = \langle 22,4,2\rangle'$
2. $\langle 15,11,2\rangle = \langle 15,11,2\rangle'$
3. $\langle 13,11,4\rangle = \langle 13,11,4\rangle'$
4. $\langle 11,10,4,2,1\rangle = \langle 11,10,4,2,1\rangle'$
5. $\langle 11,8,4,3,2\rangle = \langle 11,8,4,3,2\rangle'$
6. $\langle 11,6,5,4,2\rangle = \langle 11,6,5,4,2\rangle'$
7. $\langle 15,6,5,2\rangle^* = \langle 15,7,4,2\rangle^* - \langle 15,8,3,2\rangle^* + \langle 15,10,2,1\rangle^* - \langle 15,11,2\rangle + \langle 15,13\rangle^* - \langle 26,2\rangle^*$
8. $\langle 13,6,5,4\rangle^* = \langle 13,8,4,3\rangle^* - \langle 13,9,4,2\rangle^* + \langle 13,10,4,1\rangle^* - \langle 13,11,4\rangle - \langle 15,13\rangle^* + \langle 26,2\rangle^*$
9. $\langle 11,6,5,4,2\rangle = \langle 11,8,4,3,2\rangle - \langle 11,10,4,2,1\rangle + \langle 13,11,4\rangle - \langle 15,11,2\rangle + \langle 22,4,2\rangle$
10. $\langle 10,6,5,4,2,1\rangle^* = \langle 11,6,5,4,2\rangle + \langle 8,6,5,4,3,2\rangle^* - \langle 13,6,5,4\rangle^* + \langle 15,6,5,2\rangle^* - \langle 16,6,4,2\rangle^* + \langle 17,5,4,2\rangle^*$
11. $2\langle 8,6,5,4,3,2\rangle^* = 2\langle 11,10,4,2,1\rangle - \langle 12,10,4,2\rangle^* - \langle 13,9,4,2\rangle^* + \langle 13,10,4,1\rangle^* - \langle 13,11,4\rangle + \langle 14,8,4,2\rangle^* - \langle 15,7,4,2\rangle^* - \langle 15,10,2,1\rangle^* + \langle 15,11,2\rangle + \langle 15,13\rangle^* + \langle 16,6,4,2\rangle^* - \langle 17,5,4,2\rangle^* + \langle 19,4,3,2\rangle^* - \langle 24,4\rangle^* + 2\langle 26,2\rangle^*$
12. $\langle 11,8,4,3,2\rangle = \langle 10,8,4,3,2,1\rangle^* + \langle 8,6,5,4,3,2\rangle^* - \langle 13,8,4,3\rangle^* + \langle 14,8,4,2\rangle^* - \langle 15,8,3,2\rangle^* + \langle 19,4,3,2\rangle^*$

Then the matrix contains at most 20 columns since there are 12 equations corresponding the spin characters of S_{28} in $\boldsymbol{B_3}$. Then we get **Table 3**.

Case 4. The block $\boldsymbol{B_4}$ of type associate's decomposition matrix, as given in **Tables 4.**

Table 4. Block B_4

Spin character	d_{81}	d_{82}	d_{83}	d_{84}	d_{85}	d_{86}	d_{87}	d_{88}	d_{89}	d_{90}	d_{91}	d_{92}	d_{93}	d_{94}	d_{95}	d_{96}	d_{97}	d_{98}	d_{99}	d_{100}
⟨25,2,1⟩	1																			
⟨25,2,1⟩′		1																		
⟨24,3,1⟩	1		1																	
24,3,1		1		1																
⟨23,3,2⟩			1		1															
⟨23,3,2⟩′				1		1														
⟨22,3,2,1⟩*					1	1	1	1												
⟨18,4,3,2,1⟩							1		1											
⟨18,4,3,2,1⟩′								1		1										
⟨17,5,3,2,1⟩									1		1									
⟨17,5,3,2,1⟩′										1		1								
⟨16,6,3,2,1⟩									1		1		1							
⟨16,6,3,2,1⟩′										1		1		1						
⟨15,7,3,2,1⟩							1		1				1		1					
⟨15,7,3,2,1⟩′								1		1				1		1				
⟨14,13,1⟩			1														1			
⟨14,13,1⟩′				1														1		
⟨14,12,2⟩	1		1		1												1		1	
⟨14,12,2⟩′		1		1		1												1		1
⟨14,11,2,1⟩*					1	1	1	1											1	1
⟨14,8,3,2,1⟩							1								1					
⟨14,8,3,2,1⟩′								1								1				
⟨14,7,4,2,1⟩													1		1					
⟨14,7,4,2,1⟩′														1		1				
⟨14,6,5,2,1⟩													1							
⟨14,6,5,2,1⟩′														1						
⟨13,12,3⟩	1																		1	
⟨13,12,3⟩′		1																		1
⟨13,11,3,1⟩*																	1	1	1	1
⟨13,9,3,2,1⟩																				
⟨13,9,3,2,1⟩′																				
⟨13,7,4,3,1⟩																				
⟨13,7,4,3,1⟩′																				
⟨13,6,5,3,1⟩																				
⟨13,6,5,3,1⟩′																				
⟨12,11,3,2⟩*																	1	1		
⟨12,10,3,2,1⟩																				
⟨12,10,3,2,1⟩′																				
⟨12,7,4,3,2⟩																				
⟨12,7,4,3,2⟩′																				
⟨12,6,5,3,2⟩																				
⟨12,6,5,3,2⟩′																				
⟨11,7,4,3,2,1⟩*																				
⟨11,6,5,3,2,1⟩*																				
⟨7,6,5,4,3,2,1⟩																				
⟨7,6,5,4,3,2,1⟩′																				

Spin character	d_{101}	d_{102}	d_{103}	d_{104}	d_{105}	d_{106}	d_{107}	d_{108}	d_{109}	d_{110}	d_{111}	d_{112}	d_{113}	d_{114}	d_{115}	d_{116}	d_{117}	d_{118}	d_{119}	d_{120}
⟨25,2,1⟩																				
⟨25,2,1⟩′																				
⟨24,3,1⟩																				
24,3,1																				
⟨23,3,2⟩																				
⟨23,3,2⟩′																				
⟨22,3,2,1⟩*																				
⟨18,4,3,2,1⟩																				
⟨18,4,3,2,1⟩′																				
⟨17,5,3,2,1⟩																				
⟨17,5,3,2,1⟩′																				
⟨16,6,3,2,1⟩																				
⟨16,6,3,2,1⟩′																				
⟨15,7,3,2,1⟩																				
⟨15,7,3,2,1⟩′																				
⟨14,13,1⟩																				
⟨14,13,1⟩′																				
⟨14,12,2⟩																				
⟨14,12,2⟩′																				
⟨14,11,2,1⟩*	1	1																		
⟨14,8,3,2,1⟩	1		1																	
⟨14,8,3,2,1⟩′		1		1																
⟨14,7,4,2,1⟩			1		1															
⟨14,7,4,2,1⟩′				1		1														
⟨14,6,5,2,1⟩					1															
⟨14,6,5,2,1⟩′						1														
⟨13,12,3⟩							1													
⟨13,12,3⟩′								1												
⟨13,11,3,1⟩*	1	1					1	1	1	1										
⟨13,9,3,2,1⟩	1		1						1		1									
⟨13,9,3,2,1⟩′		1		1						1		1								
⟨13,7,4,3,1⟩			1		1						1		1							
⟨13,7,4,3,1⟩′				1		1						1		1						
⟨13,6,5,3,1⟩					1								1							
⟨13,6,5,3,1⟩′						1								1						
⟨12,11,3,2⟩*							1	1	1	1					1	1				
⟨12,10,3,2,1⟩							2		1		1				2		1	1		
⟨12,10,3,2,1⟩′								2		1		1				2	1	1		
⟨12,7,4,3,2⟩											1		1		1		1	1	1	
⟨12,7,4,3,2⟩′												1		1		1	1	1		1
⟨12,6,5,3,2⟩													1						1	
⟨12,6,5,3,2⟩′														1						1
⟨11,7,4,3,2,1⟩*															1	1	1	1	1	1
⟨11,6,5,3,2,1⟩*																	1	1	1	1
⟨7,6,5,4,3,2,1⟩																	1			
⟨7,6,5,4,3,2,1⟩′																		1		

Proof. The following values are obtained by using $(r,\bar{r})$-inducing of p.i.s. for S_{27} to S_{28} D_{101}, D_{106}, D_{65}, D_{67}, D_{68}, ..., D_{76}, D_{107}, D_{79}, D_{81}, D_{82}, ..., D_{87}, D_{89}, D_{90}, D_{109}, D_{110}, D_{95}, D_{97}, D_{98}, D_{99} respectively. Since $\langle 25,2,1\rangle \neq \langle 25,2,1\rangle'$ so c_1 divided or there are two columns φ_1,φ_2 also since $\langle 25,2,1\rangle \downarrow S_{27} = \langle 24,2,1\rangle^{*1} + \langle 25,2\rangle^{1} = 2$ of i.m.s., then we have $a_1 \in \{0,1\}$. In the same way we got $a_{26} = 0$, $a_5 \in \{0,1\}$, a_4, a_6, $a_9 \in \{0,1,2\}$, a_2, a_3, a_7, a_8, a_{14}, $a_{25} \in \{0,1,2,3\}$, a_{12}, a_{17}, $a_{19} \in \{0,1,\dots,4\}$, a_{13}, a_{23}, $a_{24} \in \{0,1,\dots,5\}$, a_{18}, $a_{20} \in \{0,1,\dots,6\}$, a_{11}, $a_{15} \in \{0,1,\dots,7\}$, $a_{22} \in \{0,1,\dots,8\}$, a_{10}, $a_{16} \in \{0,1,\dots,9\}$, $a_{21} \in \{0,1,\dots,10\}$. let $a_1 = 1$(if $a_1 = 0$ then c_1 is split), and since block B_4 is associate so there must be another column associate with c_1 then $\langle 25,2,1\rangle$ has 3 i.m.s but $\langle 25,2,1\rangle \downarrow S_{27}$ has only two of i.m.s. so that $a_1 = 0$, then $c_1 = d_{81} + d_{82}$. Since $\langle 23,3,2\rangle \neq \langle 23,2,1\rangle'$ so $c_2 = d_{83} + d_{84}$, $c_3 = d_{85} + d_{86}$ or there are two columns φ_1,φ_2.

To find are let $a_3 \in \{1,2,3\}$, since $\langle 23,3,2\rangle \downarrow S_{27} \cap \langle 18,4,3,2\rangle \downarrow S_{27}$ has no i.m.s so $a_5 = 0$, the same way we get a_6, a_7, a_8, a_{12}, a_{13}, ..., a_{25} are equal to zero, we get

$\varphi_1 = a_2\langle 24,3,1\rangle + a_3\langle 23,3,2\rangle + a_4\langle 22,3,2,1\rangle^{*} + a_9\langle 14,13,1\rangle + a_{10}\langle 14,12,2\rangle + a_{11}\langle 14,11,2,1\rangle^{*}$

$\varphi_2 = a_2\langle 24,3,1\rangle' + a_3\langle 23,3,2\rangle' + a_4\langle 22,3,2,1\rangle^{*} + a_9\langle 14,13,1\rangle' + a_{10}\langle 14,12,2\rangle' + a_{11}\langle 14,11,2,1\rangle^{*}$

we must discuss all probabilities such that the degree $\varphi_1, \varphi_2 \equiv 0\, mod 11^2$, so we do the algorithm by maple program in appendix to help us and we find it are equal to **123** probabilities, therefore, since inducing m.s. is m.s. then:

$$(\langle 14,13\rangle - \langle 24,3\rangle + \langle 25,2\rangle) \uparrow^{(0,1)} S_{28} \quad hence\ a_9 \geq a_2 \qquad (10)$$

$$(\langle 24,3\rangle - \langle 14,13\rangle + \langle 12,10,3,2\rangle) \uparrow^{(0,1)} S_{28} \quad hence\ a_2 \geq a_9\ \therefore a_2 = a_9 \qquad (11)$$

$$(\langle 23,3,1\rangle^{*} - \langle 14,12,1\rangle^{*} + \langle 12,11,3,1\rangle) \uparrow^{(0,1)} S_{28}\ hence\ a_2 + a_3 \geq a_9 + a_{10} \qquad (12)$$

$$(\langle 14,12,1\rangle^* - \langle 23,3,1\rangle^*) \uparrow^{(0,1)} S_{28}\ hence \quad (13)$$
$$a_9 + a_{10} \geq a_2 + a_3 \ \therefore a_2 + a_3 = a_9 + a_{10}$$

We get degrees $\varphi_1, \varphi_2 \equiv 0 \ mod\ 11^2$ only when $\varphi_1 + \varphi_2 = m(d_{83} + d_{84}) + n(d_{85} + d_{86})$ such that $m = 0, n \in \{1,2,3\}$ or $m = 1, n \in \{0,1,2\}$ or $m = 2, n \in \{0,1\}$ then c_2 and c_3 splits. Since $\langle 14,13,1\rangle \neq \langle 14,13,1\rangle'$ so $c_4 = d_{97} + d_{98}$ or there are two columns φ_1, φ_2. Let $a_9 \in \{1,2\}$ (if $a_9 = 0$ then c_4 is split), since $\langle 14,13,1\rangle \downarrow S_{27} \cap \langle 14,8,3,2,1\rangle \downarrow S_{27}$ has no i.m.s so $a_{12} = 0$, the same way we get a_{13}, a_{14}, a_{17}, a_{18}, a_{19}, a_{22} ..., a_{26} are equal to zero, then we get

$$\varphi_1 = a_9\langle 14,13,1\rangle + a_{10}\langle 14,12,2\rangle + a_{11}\langle 14,11,2,1\rangle^* + a_{15}\langle 13,12,3\rangle + a_{16}\langle 13,11,3,1\rangle^* + a_{20}\langle 12,11,3,2\rangle^* + a_{21}\langle 12,10,3,2,1\rangle,$$

$\varphi_2 = a_9\langle 14,13,1\rangle' + a_{10}\langle 14,12,2\rangle' + a_{11}\langle 14,11,2,1\rangle^* + a_{15}\langle 13,12,3\rangle' + a_{16}\langle 13,11,3,1\rangle^* + a_{20}\langle 12,11,3,2\rangle^* + a_{21}\langle 12,10,3,2,1\rangle'$. so we do the algorithm by maple program in appendix to help us and we find it are equal to 8146 probabilities, therefore, since inducing m.s. is m.s. then:

$$(\langle 13,11,2,1\rangle + \langle 24,2,1\rangle^* - \langle 13,12,2\rangle) \uparrow^{(3,9)} S_{28}\ hence\ a_{11} + a_{16} \geq a_{10} + a_{15} \quad (14)$$

$$(\langle 13,12,2\rangle^* - \langle 13,11,2,1\rangle^* + \langle 13,8,3,2,1\rangle) \uparrow^{(3,9)} S_{28}\ hence\ a_{10} + a_{15} \geq a_{11} + a_{16} \ \therefore a_{10} + a_{15} = a_{11} + a_{16} \quad (15)$$

$$(\langle 12,11,3,1\rangle + \langle 23,3,1\rangle^* - \langle 14,12,1\rangle^*) \uparrow^{(2,10)} S_{28}\ hence\ a_{20} \geq a_{10} \quad (16)$$

$$(\langle 14,12,1\rangle^* - \langle 12,11,3,1\rangle + \langle 12,9,3,2,1\rangle^*) \uparrow^{(2,10)} S_{28}\ hence\ a_{10} \geq a_{20} \ \therefore a_{10} = a_{20} \quad (17)$$

$$(\langle 13,10,3,1\rangle^* - \langle 14,13\rangle + \langle 24,3\rangle) \uparrow^{(0,1)} S_{28}\ hence\ a_{16} \geq a_9 \quad (18)$$

$$(\langle 13,10,3,1\rangle + \langle 13,10,3,1\rangle' - \langle 13,11,3\rangle^* + \langle 25,2\rangle + \langle 25,2\rangle^*) \uparrow^{(0,1)} S_{28}\ hence\ a_{16} \geq a_{15} \quad (19)$$

$$(\langle 14,12,1\rangle^* - \langle 12,9,3,2,1\rangle^* + \langle 12,11,3,1\rangle) \uparrow^{(2,10)} S_{28} hence\ a_9 + a_{21} \geq a_{16} \quad (20)$$

$$(\langle 11,10,3,2,1\rangle^* + \langle 14,13\rangle + \langle 14,13\rangle' + \langle 13,9,3,2\rangle - \langle 12,10,3,2\rangle) \uparrow^{(0,1)} S_{28}\ hence\ a_9 \geq a_{20} \quad (21)$$

then we get degrees $\varphi_1, \varphi_2 \equiv 0 \ mod 11^2$ only when $\varphi_1 + \varphi_2 = m(d_{97} + d_{98}), m \in \{1,2\}$, so that $c_4 = d_{97} + d_{98}$. Since $\langle 14,12,2\rangle \neq \langle 14,12,2\rangle'$ so c_5 split or there are two columns φ_1, φ_2. Let $a_{10} \in \{1,2,\ldots,9\}$, since $\langle 14,12,2\rangle \downarrow S_{27} \cap \langle 14,8,3,2,1\rangle \downarrow S_{27}$ has no i.m.s then $a_{12} = 0$, in the same a_{13}, a_{14}, a_{17}, a_{18}, a_{19}, a_{22} ..., a_{25} are equal to zero, then we get

$\varphi_1 = a_{10}\langle 14,12,2\rangle + a_{11}\langle 14,11,2,1\rangle^* + a_{15}\langle 13,12,3\rangle + a_{16}\langle 13,11,3,1\rangle^* + a_{20}\langle 12,11,3,2\rangle^* + a_{21}\langle 12,10,3,2,1\rangle$,

$\varphi_2 = a_{10}\langle 14,12,2\rangle' + a_{11}\langle 14,11,2,1\rangle^* + a_{15}\langle 13,12,3\rangle' + a_{16}\langle 13,11,3,1\rangle^* + a_{20}\langle 12,11,3,2\rangle^* + a_{21}\langle 12,10,3,2,1\rangle'$. we must discuss all probabilities such that the degree $\varphi_1, \varphi_2 \equiv 0 \ mod 11^2$, by maple program in appendix to help us and we find it are equal to **3667** probabilities, therefore,

$$(\langle 11,10,3,2,1\rangle^* + \langle 14,13\rangle + \langle 14,13\rangle' + \langle 13,9,3,2\rangle - \langle 12,10,3,2\rangle) \uparrow^{(0,1)} S_{28}\ hence\ a_{21} \geq a_{20} + a_{21} \ \therefore a_{20} = 0 \quad (22)$$

and Eqs. (14-17) get degrees $\varphi_1, \varphi_2 \not\equiv 0 \ mod 11^2$ contradiction with Theorem 2.3. Then $c_5 = d_{99} + d_{100}$.Since $\langle 13,12,3\rangle \neq \langle 13,12,3\rangle'$ so $c_6 = d_{107} + d_{108}$ or there are two columns φ_1, φ_2. Let $a_{15} \in \{1,2,\ldots,7\}$ since $\langle 13,12,3\rangle \downarrow S_{27} \cap \langle 13,9,3,2,1\rangle \downarrow S_{27}$ has no i.m.s so $a_{17} = 0$, the same way we get a_{17}, a_{18}, a_{19}, a_{22} ..., a_{25} are equal to zero, then we get

$\varphi_1 = a_{15}\langle 13,12,3\rangle + a_{16}\langle 13,11,3,1\rangle^* + a_{20}\langle 12,11,3,2\rangle^* + a_{21}\langle 12,10,3,2,1\rangle$,

$\varphi_2 = a_{15}\langle 13,12,3\rangle' + a_{16}\langle 13,11,3,1\rangle^* + a_{20}\langle 12,11,3,2\rangle^* + a_{21}\langle 12,10,3,2,1\rangle'$.

by maple program in appendix we find it are equal to **44** probabilities, therefore,

$$(\langle 13,12,2\rangle^* - \langle 13,11,2,1\rangle + \langle 13,8,3,2,1\rangle^*) \uparrow^{(3,9)} S_{28}\ hence\ a_{15} \geq a_{16} \quad (23)$$

$$(\langle 13,11,2,1\rangle - \langle 13,12,2\rangle^* + \langle 24,2,1\rangle^*) \uparrow^{(3,9)} S_{28}\ hence\ a_{16} \geq a_{15} \ \therefore a_{15} = a_{16} \quad (24)$$

and Eq. (22) we get degrees $\varphi_1, \varphi_2 \not\equiv 0 \ mod 11^2$ contradiction with **(Theorem 2.3.)**, then $c_6 = d_{107} + d_{108}$. $\langle 13,9,3,2,1\rangle \neq \langle 13,9,3,2,1\rangle'$ so $c_7 = d_{111} + d_{112}$ or there are two columns φ_1, φ_2. Let $a_{17} \in \{1,2,3,4\}$. Since $\langle 13,9,3,2,1\rangle \downarrow S_{27} \cap \langle 13,6,5,3,1\rangle \downarrow S_{27}$ has no i.m.s so $a_{19} = 0$, the same way we get a_{23}, a_{25} are equal to zero, then we get

$\varphi_1 = a_{17}\langle 13,9,3,2,1\rangle + a_{18}\langle 13,7,4,3,1\rangle + a_{20}\langle 12,11,3,2\rangle^* + a_{21}\langle 12,10,3,2,1\rangle + a_{22}\langle 12,7,4,3,2\rangle + a_{24}\langle 11,7,4,3,2,1\rangle^*$

$\varphi_2 = a_{17}\langle 13,9,3,2,1\rangle' + a_{18}\langle 13,7,4,3,1\rangle' + a_{20}\langle 12,11,3,2\rangle^* + a_{21}\langle 12,10,3,2,1\rangle' + a_{22}\langle 12,7,4,3,2\rangle' + a_{24}\langle 11,7,4,3,2,1\rangle^*$,

by maple program in appendix we find it are equal to **967** probabilities, therefore,

$$(\langle 13,8,3,2,1\rangle^* - \langle 13,7,4,2,1\rangle^* + \langle 13,6,5,2,1\rangle^*) \uparrow^{(3,9)} S_{28}\ hence\ a_{17} \geq a_{18} \quad (25)$$

$$(\langle 13,7,4,2,1\rangle^* - \langle 13,8,3,2,1\rangle^* + \langle 13,11,2,1\rangle) \uparrow^{(3,9)} S_{28}\ hence\ a_{18} \geq a_{17}, \therefore a_{17} = a_{18} \quad (26)$$

$$(\langle 13,6,4,3,1\rangle^* + \langle 11,6,3,2,1\rangle - \langle 12,6,4,3,2\rangle^*) \uparrow^{(5,7)} S_{28}\ hence\ a_{18} + a_{24} \geq a_{22} \quad (27)$$

$$(\langle 12,6,4,3,2\rangle^* - \langle 13,6,4,3,1\rangle - \langle 11,6,4,3,2,1\rangle^* + \langle 14,6,4,2,1\rangle^*) \uparrow^{(5,7)} S_{28}\ hence\ a_{22} \geq a_{18} + a_{24}, \therefore a_{22} = a_{18} + a_{24} \quad (28)$$

$$(\langle 12,10,3,2\rangle - \langle 13,9,3,2\rangle + \langle 14,8,3,2\rangle) \uparrow^{(0,1)} S_{28}\ hence\ a_{20} + a_{21} \geq a_{17} \quad (29)$$

$$(\langle 13,9,3,2\rangle - \langle 12,10,3,2\rangle + \langle 13,11,3\rangle^*) \uparrow^{(0,1)} S_{28}\ hence\ a_{17} \geq a_{20} + a_{21}, \ \therefore a_{17} = a_{20} + a_{21} \quad (30)$$

We get degrees $\varphi_1, \varphi_2 \equiv 0 \ mod 11^2$ only when $\varphi_1 + \varphi_2 = m(d_{111} + d_{112}), m \in \{1,2,3,4\}$. So that c_7 is split. $\langle 13,7,4,3,1\rangle \neq \langle 13,7,4,3,1\rangle'$ so $c_8 = d_{113} + d_{114}$ or there are two columns.Let $a_{18} \in \{1,2,\ldots,6\}$.

$\varphi_1 = a_{18}\langle 13,7,4,3,1\rangle + a_{19}\langle 13,6,5,3,1\rangle + a_{20}\langle 12,11,3,2\rangle^* + a_{21}\langle 12,10,3,2,1\rangle + a_{22}\langle 12,7,4,3,2\rangle + a_{23}\langle 12,6,5,3,2\rangle + a_{24}\langle 11,7,4,3,2,1\rangle^*$,

$\varphi_2 = a_{17}\langle 13,9,3,2,1\rangle' + a_{18}\langle 13,7,4,3,1\rangle' + a_{19}\langle 13,6,5,3,1\rangle' + a_{20}\langle 12,11,3,2\rangle^* + a_{21}\langle 12,10,3,2,1\rangle' + a_{22}\langle 12,7,4,3,2\rangle' + a_{23}\langle 12,6,5,3,2\rangle' + a_{24}\langle 11,7,4,3,2,1\rangle^*$.

by maple program in appendix we find it are equal to **27794** probabilities, therefore,

$$(\langle 13,7,4,3\rangle - \langle 13,6,5,3\rangle) \uparrow^{(0,1)} S_{28}\ hence\ a_{18} \geq a_{19} \quad (31)$$

$$(\langle 13,6,5,3\rangle - \langle 13,7,4,3\rangle + \langle 13,9,3,2\rangle) \uparrow^{(0,1)} S_{28}\ hence\ a_{19} \geq a_{18}, \therefore a_{18} = a_{19} \quad (32)$$

$$(\langle 11,10,3,2,1\rangle^* - \langle 12,10,3,2\rangle + \langle 13,10,3,1\rangle) \uparrow^{(0,1)} S_{28} hence\ 0 \geq a_{22} \quad (33)$$

$$(\langle 12,9,3,2,1\rangle^* - \langle 12,7,4,3,1\rangle^* + \langle 12,6,5,3,1\rangle^*) \uparrow^{(2,10)} S_{28}\ hence\ a_{21} + a_{23} \geq a_{22} \quad (34)$$

$$(\langle 12,7,4,3,1\rangle^* - \langle 12,6,5,3,1\rangle^* - \langle 12,9,3,2,1\rangle^* + \langle 14,12,1\rangle) \uparrow^{(2,10)} S_{28}$$
$$hence a_{22} \geq a_{21} + a_{23}, \therefore a_{22} = a_{21} + a_{23} \quad (35)$$

$$(\langle 11,7,4,3,2\rangle^* - \langle 11,6,5,3,2\rangle^* - \langle 11,10,3,2,1\rangle^* + \langle\langle 13,11,3\rangle^*\rangle) \uparrow^{(0,1)} S_{28}$$
$$hence a_{22} + a_{24} \geq a_{23} + a_{25} + a_{21} \quad (36)$$

$$(\langle 11,6,5,3,2\rangle^* + \langle 11,10,3,2,1\rangle^* - \langle 11,7,4,3,2\rangle^*) \uparrow^{(0,1)} S_{28}\ hence a_{25} + a_{23} + a_{21} \geq a_{22} + a_{24},$$
$$\therefore a_{25} + a_{23} + a_{21} = a_{22} + a_{24} \quad (37)$$

$$(\langle 13,6,5,3\rangle + \langle 13,6,5,3\rangle' - \langle 11,6,5,3,2\rangle^* + \langle 10,6,5,3,2,1\rangle + \langle 10,6,5,3,2,1\rangle') \uparrow^{(0,1)} S_{28}$$
$$hence\ 2a_{19} + a_{25} \geq a_{23} \quad (38)$$

$$(\langle 11,6,5,3,2\rangle^* - \langle 13,6,5,3\rangle + \langle 13,6,5,3\rangle' - \langle 10,6,5,3,2,1\rangle - \langle 10,6,5,3,2,1\rangle' + \langle 7,6,5,4,3,2\rangle + \langle 7,6,5,4,3,2\rangle' + \langle 14,6,5,2\rangle + \langle 14,6,5,2\rangle') \uparrow^{(0,1)} S_{28}\ hence\ a_{23} \geq 2a_{19} + a_{25},$$
$$\therefore a_{23} = 2a_{19} + a_{19} \quad (39)$$

$\varphi_1 = a_{22}\langle 12,7,4,3,2\rangle + a_{23}\langle 12,6,5,3,2\rangle + a_{24}\langle 11,7,4,3,2,1\rangle^*$
then we get degree $\varphi_1, \varphi_2 \not\equiv 0\ mod 11^2$. So that c_8 is split. Since $\langle 12,10,3,2,1\rangle \neq \langle 12,10,3,2,1\rangle'$ so $c_9 = d_{115} + d_{116}$ or there are two columns φ_1, φ_2. Let $a_{21} \in \{1,2,\ldots,10\}$.
$+a_{25}\langle 11,6,5,3,2,1\rangle^*$
$\varphi_2 = a_{22}\langle 12,7,4,3,2\rangle' + a_{23}\langle 12,6,5,3,2\rangle' + a_{24}\langle 11,7,4,3,2,1\rangle^*$ $+a_{25}\langle 11,6,5,3,2,1\rangle^*$.
by maple program in appendix we find it are equal to **9** probabilities, therefore,

$$(\langle 12,7,4,3,1\rangle^* - \langle 12,6,5,3,1\rangle^*) \uparrow^{(2,10)} S_{28} hence\ a_{22} \geq a_{21} \quad (40)$$

Frome Eq.(34) we get degrees $\varphi_1, \varphi_2 \equiv 0\ mod 11^2$. only when $\varphi_1 + \varphi_2 = m(d_{119} + d_{120}), m \in \{1,2,3\}$. So that c_{10} is split.Using the information above, we divide c_1 into c_{10} and then turn to **Table 5**.

Case 5 Decomposition matrix for the blocks $\boldsymbol{B_5}$, B_7 of type duble and B_6 of type associte as shown in the **Tables 5**.

Table 5. Blocks B_5, B_6, B_7

Block	**Spin character**	**Decomposition matrix**																			
$\boldsymbol{B_5}$	$\langle 25,3\rangle^*$	1																			
	$\langle 14,11,3\rangle$	1	1																		
	$\langle 14,10,3,1\rangle^*$		1	1																	
	$\langle 14,9,3,2\rangle^*$			1	1																
	$\langle 14,7,4,3\rangle^*$				1	1															
	$\langle 14,6,5,3\rangle^*$					1															
$\boldsymbol{B_6}$	$\langle 23,4,1\rangle$						1														
	$\langle 23,4,1\rangle'$							1													
	$\langle 15,12,1\rangle$						1		1												
	$\langle 15,12,1\rangle'$							1		1											
	$\langle 12,11,4,1\rangle^*$								1	1	1	1									
	$\langle 12,9,4,2,1\rangle$										1		1								
	$\langle 12,9,4,2,1\rangle'$											1		1							
	$\langle 12,8,4,3,1\rangle$												1		1						
	$\langle 12,8,4,3,1\rangle'$													1		1					
	$\langle 12,6,5,4,1\rangle$														1						
	$\langle 12,6,5,4,1\rangle'$															1					
$\boldsymbol{B_7}$	$\langle 21,7\rangle^*$																1				
	$\langle 18,10\rangle^*$																1	1			
	$\langle 11,10,7\rangle$																	1	1		
	$\langle 10,9,7,2\rangle^*$																		1	1	
	$\langle 10,8,7,3\rangle^*$																			1	1
	$\langle 10,7,6,5\rangle^*$																				1
		d_{121}	d_{122}	d_{123}	d_{124}	d_{125}	d_{126}	d_{127}	d_{128}	d_{129}	d_{130}	d_{131}	d_{132}	d_{133}	d_{134}	d_{135}	d_{136}	d_{137}	d_{138}	d_{139}	d_{140}

$\varphi_1 = a_{20}\langle 12,11,3,2\rangle^* + a_{21}\langle 12,10,3,2,1\rangle + a_{22}\langle 12,7,4,3,2\rangle + a_{23}\langle 12,6,5,3,2\rangle + a_{24}\langle 11,7,4,3,2,1\rangle^* + a_{25}\langle 11,6,5,3,2,1\rangle^*$,
$\varphi_2 = a_{20}\langle 12,11,3,2\rangle^* + a_{21}\langle 12,10,3,2,1\rangle' + a_{22}\langle 12,7,4,3,2\rangle'$ $+a_{23}\langle 12,6,5,3,2\rangle' + a_{24}\langle 11,7,4,3,2,1\rangle^* + a_{25}\langle 11,6,5,3,2,1\rangle^*$.
by maple program in appendix we find it are equal to **722** probabilities, therefore,
Frome Eqs.(33-37) we get degree $\varphi_1, \varphi_2 \not\equiv 0\ mod 11^2$. So that c_9 is split.
$\langle 12,7,4,3,2\rangle \neq \langle 127,4,3,2\rangle'$ so c_{10} split or there are two columns φ_1, φ_2. Let $a_{22} \in \{1,2,\ldots,8\}$.

Proof. By using$(\mathrm{r},\bar{\mathrm{r}})$-inducing of p.i.s., D_{63}, D_{65}, D_{67}, D_{75}, D_{73}, D_{25}, D_{37}, D_{47}, D_{48}, D_{45}, D_{46}, D_{43}, D_{44}, D_{111}, D_{113}, D_{115}, D_{117}, D_{119} for S_{27} to S_{28} gives d_{121}, d_{122},…, d_{125}, c_1, c_2, d_{130}, d_{131}, ..., d_{140}, respectively. for S_{27} to S_{28}. To find $\boldsymbol{B_5}$, $\boldsymbol{B_7}$, since

- degree $\{\langle 14,11,3\rangle + \langle 14,11,3\rangle', \langle 14,9,3,2\rangle^*, \langle 14,6,5,3\rangle^*\} \equiv 99\ mod\ 11^2$,
- degree $\{\langle 25,3\rangle^*, \langle 14,10,3,1\rangle^*, \langle 14,7,4,3\rangle^*\} \equiv -99\ mod\ 11^2$,
- degree $\{\langle 21,7\rangle^*, \langle 11,10,7\rangle + \langle 11,10,7\rangle', \langle 10,8,7,3\rangle^*\} \equiv 99\ mod\ 11^2$,

- degree$\{\langle 18,10\rangle^*, \langle 10,9,7,2\rangle^*, \langle 10,7,6,5\rangle^*\} \equiv -99 \; mod \; 11^2$,

and on$(11,\alpha)$-regular classes

2.2) c_2 must split to $d_{128}+d_{129}$. Hence the decomposition matrix for this block is **Table 5**.

Case 6. Decomposition matrix for the blocks $\boldsymbol{B_8}$, $\boldsymbol{B_9}$ of type associte as shown in the **Tables 8.**

Table 6. Blocks B_8, B_9

Block	Spin character	Decomposition matrix																			
$\boldsymbol{B_8}$	$\langle 21,5,2\rangle$	1																			
	$\langle 21,5,2\rangle'$		1																		
	$\langle 16,10,2\rangle$	1		1																	
	$\langle 16,10,2\rangle'$		1		1																
	$\langle 13,10,5\rangle$			1		1															
	$\langle 13,10,5\rangle'$				1		1														
	$\langle 11,10,5,2\rangle^*$					1	1	1	1												
	$\langle 10,8,5,3,2\rangle$							1		1											
	$\langle 10,8,5,3,2\rangle'$								1		1										
	$\langle 10,7,5,4,2\rangle$									1											
	$\langle 10,7,5,4,2\rangle'$										1										
$\boldsymbol{B_9}$	$\langle 21,4,3\rangle$											1									
	$\langle 21,4,3\rangle'$												1								
	$\langle 15,10,3\rangle$											1		1							
	$\langle 15,10,3\rangle'$												1		1						
	$\langle 14,10,4\rangle$													1		1					
	$\langle 14,10,4\rangle'$														1		1				
	$\langle 11,10,4,3\rangle^*$															1	1	1	1		
	$\langle 10,9,4,3,2\rangle$																	1		1	
	$\langle 10,9,4,3,2\rangle'$																		1		1
	$\langle 10,6,5,4,3\rangle$																			1	
	$\langle 10,6,5,4,3\rangle'$																				1
		d_{141}	d_{142}	d_{143}	d_{144}	d_{145}	d_{146}	d_{147}	d_{148}	d_{149}	d_{150}	d_{151}	d_{152}	d_{153}	d_{154}	d_{155}	d_{156}	d_{157}	d_{158}	d_{159}	d_{160}

Proof. Using $(\mathrm{r},\bar{\mathrm{r}})$-inducing of p.i.s. D_2, D_9, D_{13}, D_{17}, D_{18}, $D_{121}, D_{122}, \cdots, D_{125}$ to S_{28} gives c_1, c_2, c_3,..., c_{10}, respectively. Since $\langle 21,5,2\rangle \neq \langle 21,5,2\rangle'$ and $\langle 21,5,2\rangle \downarrow S_{27} \cap \{\langle 13,10,5\rangle, \langle 11,10,5,2\rangle^*, \langle 10,8,5,3,2\rangle, \langle 10,7,5,4,2\rangle\} \downarrow S_{27}$ has no i.m.s.Then $c_1 = d_{141}+d_{142}$. Since $\langle 13,10,5\rangle \neq \langle 13,10,5\rangle'$ then c_2 split or there are two columns. Suppose there are columns:

$\phi_1 = a_1\langle 16,10,2\rangle + a_2\langle 13,10,5\rangle + a_3\langle 11,10,5,2\rangle^* + a_4\langle 10,8,5,3,2\rangle + a_5\langle 10,7,5,4,2\rangle$,

$\phi_2 = a_1\langle 16,10,2\rangle' + a_2\langle 13,10,5\rangle' + a_3\langle 11,10,5,2\rangle^* + a_4\langle 10,8,5,3,2\rangle' + a_5\langle 10,7,5,4,2\rangle'$

to describe it since $\langle 13,10,5\rangle \downarrow S_{27}$ has 12 of i.m.s and $\boldsymbol{B_8}$ of defect one then we have $a_1, a_2 \cdots a_5 \in \{0,1\}$. Suppose $a_2 = 1$, since $\langle 13,10,5\rangle \downarrow S_{27} \cap \{\langle 10,8,5,3,2\rangle, \langle 10,7,5,4,2\rangle\} \downarrow S_{27}$ has no i.m.s. so $a_4, a_5 = 0$, but degree $\phi_1, \phi_2 \equiv 0 \; mod \; 11^2$ only when

1. $\langle 14,11,3\rangle = \langle 14,11,3\rangle'$
2. $\langle 14,6,5,3\rangle^* = \langle 14,7,4,3\rangle^* - \langle 14,9,3,2\rangle^* + \langle 14,10,3,1\rangle^* - \langle 14,11,3\rangle + \langle 25,3\rangle^*$
3. $\langle 11,10,7\rangle = \langle 11,10,7\rangle'$
4. $\langle 10,7,6,5\rangle^* = \langle 10,8,7,3\rangle^* - \langle 10,9,7,2\rangle^* + \langle 11,10,7\rangle - \langle 18,10\rangle^* + \langle 21,7\rangle^*$

then ech blocks contains at 5. To find $\boldsymbol{B_6}$ since $\langle 23,4,1\rangle \neq \langle 23,4,1\rangle'$, and $\langle 23,4,1\rangle \downarrow S_{27} \cap \{\langle 12,11,4,1\rangle^*, \langle 12,9,4,2,1\rangle, \langle 12,8,4,3,1\rangle, \langle 12,6,5,4,2,1\rangle\} \downarrow S_{27}$ has no i.m.s then $c_1 = d_{126}+d_{127}$, also since $\boldsymbol{B_6}$ of defect one then from (**Theorem**

- $a_1 = a_2 = 1$ and $a_3 = 0 \Rightarrow c_2 = d_{143}+d_{144}$ or
- $a_1 = 0$ and $a_2 = a_3 = 1 \Rightarrow c_3 = d_{145}+d_{146}$

So $c_2 = d_{143}+d_{144}$. As since $\boldsymbol{B_8}$ of defect one then from (**Theorem 2.2**) c_3, c_4 must splits to $d_{145}+d_{146}$ and $d_{147}+d_{148}$, respectively. Since $\langle 10,7,5,4,2\rangle \neq \langle 10,7,5,4,2\rangle'$ then c_5 must split to $d_{149}+d_{150}$ then we get $\boldsymbol{B_8}$. To find the blocks $\boldsymbol{B_9}$ since $\langle 21,4,3\rangle \neq \langle 21,4,3\rangle'$,$\langle 21,4,3\rangle \downarrow S_{27} \cap \{\langle 14,10,4\rangle, \langle 11,10,4,3\rangle^*, \langle 10,9,4,3,2\rangle, \langle 10,6,5,4,3\rangle\} \downarrow S_{27}$ has no i.m.s then $c_6 = d_{161}+d_{162}$. Since $\langle 14,10,4\rangle \neq \langle 14,10,4\rangle'$ then c_7 divided or there are two columns. Suppose there columns:

$\phi_1 = a_1\langle 15,10,3\rangle + a_2\langle 14,10,4\rangle + a_3\langle 11,10,4,3\rangle^* + a_4\langle 10,9,4,3,2\rangle + a_5\langle 10,6,5,4,3\rangle$,

$\phi_2 = a_1\langle 15,10,3\rangle' + a_2\langle 14,10,4\rangle' + a_3\langle 11,10,4,3\rangle^* + a_4\langle 10,9,4,3,2\rangle' + a_5\langle 10,6,5,4,3\rangle'$,

to describe it since $\langle 14,10,4\rangle \downarrow S_{27}$ has 7 of i.m.s and B_9 of defect one then $a_1, a_2 \cdots a_5 \in \{0,1\}$. Suppose $a_1 = 1$ ($a_1 = 0 \Rightarrow c_7$ is split) since $\langle 14,10,4\rangle \downarrow S_{27} \cap \{\langle 10,9,4,3,2\rangle, \langle 10,6,5,4,3\rangle\} \downarrow S_{27}$ has no i.m.s. so $a_4, a_5 = 0$. But degree $\phi_1, \phi_2 \equiv 0 \; mod \; 11^2$ only when

- $a_1 = a_2 = 1$ and $a_3 = 0 \Rightarrow c_7 = d_{153}+d_{154}$ or
- $a_1 = 0$ and $a_2 = a_3 = 1 \Rightarrow c_8 = d_{155}+d_{156}$

Table 7. Blocks B_8, B_9

Block	Spin character	Decomposition matrix																								
$\boldsymbol{B_{10}}$	$\langle 20,8\rangle^*$	1																								
	$\langle 19,9\rangle^*$	1	1																							
	$\langle 11,9,8\rangle$		1	1																						
	$\langle 10,9,8,1\rangle^*$			1	1																					
	$\langle 9,8,7,4\rangle^*$				1	1																				
	$\langle 9,8,6,5\rangle^*$					1																				
$\boldsymbol{B_{11}}$	$\langle 20,7,1\rangle$						1																			
	$\langle 20,7,1\rangle'$							1																		
	$\langle 18,9,1\rangle$						1		1																	
	$\langle 18,9,1\rangle'$							1		1																
	$\langle 12,9,7\rangle$								1		1															
	$\langle 12,9,7\rangle'$									1		1														
	$\langle 11,9,7,1\rangle^*$										1	1	1	1												
	$\langle 9,8,7,3,1\rangle$												1		1											
	$\langle 9,8,7,3,1\rangle'$													1		1										
	$\langle 9,7,6,5,1\rangle$														1											
	$\langle 9,7,6,5,1\rangle'$															1										
$\boldsymbol{B_{12}}$	$\langle 20,5,3\rangle$																1									
	$\langle 20,5,3\rangle'$																	1								
	$\langle 16,9,3\rangle$																1		1							
	$\langle 16,9,3\rangle'$																	1		1						
	$\langle 14,9,4\rangle$																		1		1					
	$\langle 14,9,5\rangle'$																			1		1				
	$\langle 11,9,5,3\rangle^*$																				1	1	1	1		
	$\langle 10,9,5,3,1\rangle$																						1		1	
	$\langle 10,9,5,3,1\rangle'$																							1		1
	$\langle 9,7,5,4,3\rangle$																								1	
	$\langle 9,7,5,4,3\rangle'$																									1
		d_{161}	d_{162}	d_{163}	d_{164}	d_{165}	d_{166}	d_{167}	d_{168}	d_{169}	d_{170}	d_{171}	d_{172}	d_{173}	d_{174}	d_{175}	d_{176}	d_{177}	d_{178}	d_{179}	d_{180}	d_{181}	d_{182}	d_{183}	d_{184}	d_{185}

Proof. Since

- degree $\{\langle 19,9\rangle^*, \langle 10,9,8,1\rangle^*, \langle 9,8,6,5\rangle^*\} \equiv 110 \ mod\ 11^2$
- degree $\{\langle 20,8\rangle^*, \langle 11,9,8\rangle + \langle 11,9,8\rangle', \langle 9,8,7,4\rangle^*\} \equiv -110 \ mod\ 11^2$

By using (4,8)-inducing of p.i.s. D_{126}, D_{128}, D_{130}, D_{132}, D_{134} for S_{27} to S_{28}, and from $(11,\alpha)$-regular classes:

1. $\langle 11,9,8\rangle \neq \langle 11,9,8\rangle'$
2. $\langle 9,8,6,5\rangle^* = \langle 9,8,7,4\rangle^* - \langle 10,9,8,1\rangle^* + \langle 11,9,8\rangle - \langle 19,9\rangle^* + \langle 20,8\rangle^*$

then we get blocks $\boldsymbol{B_{10}}$. To find blocks $\boldsymbol{B_{11}}$, $\boldsymbol{B_{12}}$, using $(r,\bar{r})$-inducing of p.i.s. D_{126}, D_{127}, D_{130}, D_{132}, D_{133}, D_{134}, D_{135}, D_{137} , D_3, D_8, D_{12}, D_{232}, D_{233}, $D_{20}$to S_{28} gives d_{166}, d_{167}, c_1, c_2, d_{172}, d_{173}, d_{174}, d_{175}, c_3, c_4, c_5, d_{182}, d_{183}, $c_6$respectively. Since $\langle 12,9,7\rangle \neq \langle 12,9,7\rangle'$ $\langle 12,9,7\rangle \downarrow S_{27} \cap \{\langle 9,8,7,3,1\rangle, \langle 9,7,6,5,1\rangle\} \downarrow S_{27}$ has no i.m.s then c_1 divided or there are two columns. Suppose there are two columns:

So $c_7 = d_{153} + d_{154}$, also since $\boldsymbol{B_9}$ of defect one then c_8, c_9 must splits to d_{155}, d_{156} and d_{157}, d_{158}, respectively. finally since $\langle 10,6,5,4,3\rangle \neq \langle 10,6,5,4,3\rangle'$ then c_{10} must split to $d_{159} + d_{160}$. Hence the decomposition matrix for this block is **Table** 6.

Case 7. The decomposition matrix for the blocks $\boldsymbol{B_{10}}$ of type duble and $\boldsymbol{B_{11}}, \boldsymbol{B_{12}}$ of type associte as shown in the **Tables** 7.

$\phi_1 = a_1\langle 18,9,1\rangle + a_2\langle 12,9,7\rangle + a_3\langle 11,9,7,1\rangle^*$,$\phi_2 = a_1\langle 18,9,1\rangle' + a_2\langle 12,9,7\rangle' + a_3\langle 11,9,7,1\rangle^*$,

to describe columns since $\langle 12,9,7\rangle \downarrow S_{27}$ has 8 of i.m.s and since $\boldsymbol{B_{11}}$ of defect one then we have $a_1, a_2, a_3 \in \{0,1\}$. Suppose $a_2 = 1$, but degrees$\phi_1, \phi_2 \ \equiv 0 \ mod\ 11^2$ only when

- $a_1 = a_2 = 1$ and $a_3 = 0 \Rightarrow c_1 = d_{168} + d_{169}$ or
- $a_1 = 0$ and $a_2 = a_3 = 1 \Rightarrow c_2 = d_{170} + d_{171}$

So $c_1 = d_{168} + d_{169}$. As since $\boldsymbol{B_{11}}$ of defect one then c_2 must splits to d_{170}, d_{171} then we get block $\boldsymbol{B_{11}}$. Since $\langle 20,5,3\rangle \neq \langle 20,5,3\rangle'$ and $\langle 20,5,3\rangle \downarrow S_{27} \cap \{\langle 14,9,4\rangle, \langle 11,9,5,3\rangle^*, \langle 10,9,5,3,1\rangle, \langle 9,7,5,4,3\rangle\} \downarrow S_{27}$ has no i.m.s then $c_3 = d_{176} + d_{177}$. Since $\langle 14,9,4\rangle \neq \langle 14,9,5\rangle'$ then c_4 divided or there are two columns. Suppose there are two columns:

$\phi_1 = a_1\langle 16,9,3\rangle + a_2\langle 14,9,4\rangle + a_3\langle 11,9,5,3\rangle^* + a_4\langle 10,9,5,3,1\rangle + a_5\langle 9,7,5,4,3\rangle$,

$\phi_1 = a_1\langle 16,9,3\rangle' + a_2\langle 14,9,4\rangle' + a_3\langle 11,9,5,3\rangle^* + a_4\langle 10,9,5,3,1\rangle' + a_5\langle 9,7,5,4,3\rangle'$,

to describe it since $\langle 14,9,4\rangle \downarrow S_{27}$ has 8 of i.m.s and since $\boldsymbol{B_{12}}$ of defect one then we have $a_1, a_2 \cdots a_5 \in \{0,1\}$. Suppose $a_2 = 1$, since $\langle 14,9,4\rangle \downarrow S_{27} \cap \{\langle 10,9,5,3,1\rangle, \langle 9,7,5,4,3\rangle\} \downarrow S_{27}$ has no i.m.s. so $a_4, a_5 = 0$. But degree $\phi_1, \phi_2 \ \equiv 0 \ mod\ 11^2$ only when

- $a_1 = a_2 = 1$ and $a_3 = 0 \Rightarrow c_4 = d_{178} + d_{179}$ or
- $a_1 = 0$ and $a_2 = a_3 = 1 \Rightarrow c_5 = d_{180} + d_{181}$

So $c_4 = d_{178} + d_{179}$. As since $\boldsymbol{B_{12}}$ of defect one then c_5 must split to d_{180}, d_{181}. finally since $\langle 9,7,5,4,3\rangle \neq \langle 9,7,5,4,3\rangle'$ then c_6 must split to d_{184}, d_{185}. Hence the decomposition matrix for this block is **Table** 7.

Case 8. The decomposition matrix for the blocks $\boldsymbol{B_{13}}, \boldsymbol{B_{15}}$ of type duble and $\boldsymbol{B_{14}}$ of type associte as shown in the **Tables** 8.

Table 8. Blocks B_{13}, B_{14}, B_{15}

Block	Spin character	Decomposition matrix																			
B_{13}	$\langle 20,4,3,1\rangle^*$	1																			
	$\langle 15,9,3,1\rangle^*$	1	1																		
	$\langle 14,9,4,1\rangle^*$		1	1																	
	$\langle 12,9,4,3\rangle^*$			1	1																
	$\langle 11,9,4,3,1\rangle$				1	1															
	$\langle 9,6,5,4,3,1\rangle^*$					1															
B_{14}	$\langle 19,7,2\rangle$						1														
	$\langle 19,7,2\rangle'$							1													
	$\langle 18,8,2\rangle$						1		1												
	$\langle 18,8,2\rangle'$							1		1											
	$\langle 13,8,7\rangle$								1		1										
	$\langle 13,8,7\rangle'$									1		1									
	$\langle 11,8,7,2\rangle^*$										1	1	1	1							
	$\langle 10,8,7,2,1\rangle$												1		1						
	$\langle 10,8,7,2,1\rangle'$													1		1					
	$\langle 8,7,6,5,2\rangle$														1						
	$\langle 8,7,6,5,2\rangle'$															1					
B_{15}	$\langle 19,6,2,1\rangle^*$																1				
	$\langle 17,8,2,1\rangle^*$																1	1			
	$\langle 13,8,6,1\rangle^*$																	1	1		
	$\langle 12,8,6,2\rangle^*$																		1	1	
	$\langle 11,8,6,2,1\rangle$																			1	1
	$\langle 8,7,6,4,2,1\rangle^*$																				1
		d_{186}	d_{187}	d_{188}	d_{189}	d_{190}	d_{191}	d_{192}	d_{193}	d_{194}	d_{195}	d_{196}	d_{197}	d_{198}	d_{199}	d_{200}	d_{201}	d_{202}	d_{203}	d_{204}	d_{205}

Proof. To find $\boldsymbol{B_{13}}, \boldsymbol{B_{15}}$ Since

- degree $\{\langle 15,9,3,1\rangle^*, \langle 12,9,4,3\rangle^*, \langle 9,6,5,4,3,1\rangle^*\} \equiv 88 \; mod \; 11^2$
- degree $\{\langle 20,4,3,1\rangle^*, \langle 14,9,4,1\rangle^*, \langle 11,9,4,3,1\rangle + \langle 11,9,4,3,1\rangle'\} \equiv -88 \; mod \; 11^2$
- degree $\{\langle 19,6,2,1\rangle^*, \langle 13,8,6,1\rangle^*, \langle 11,8,6,2,1\rangle + \langle 11,8,6,2,1\rangle'\} \equiv 66 \; mod \; 11^2$
- degree$\{\langle 17,8,2,1\rangle^*, \langle 12,8,6,2\rangle^*, \langle 8,7,6,4,2,1\rangle^*\} \equiv -66 \; mod \; 11^2$

By using $(r,\bar{r})$-inducing of p.i.s.D_{29}, D_{37}, D_{43}, D_{49}, D_{59}, D_{151}, D_{152}, D_{160}, D_{162}, D_{155} for S_{27} to S_{28}, and since on $(11,\alpha)$-regular classes:

1. $\langle 11,9,4,3,1\rangle = \langle 11,9,4,3,1\rangle'$
2. $\langle 9,6,5,4,3,1\rangle^* = \langle 11,9,4,3,1\rangle - \langle 12,9,4,3\rangle^* + \langle 14,9,4,1\rangle^* - \langle 15,9,3,1\rangle^* + \langle 20,4,3,1\rangle^*$
3. $\langle 11,8,6,2,1\rangle \neq \langle 11,8,6,2,1\rangle'$
4. $\langle 8,7,6,4,2,1\rangle^* = \langle 11,8,6,2,1\rangle - \langle 12,8,6,2\rangle^* + \langle 13,8,6,1\rangle^* - \langle 17,8,2,1\rangle^* + \langle 19,6,2,1\rangle^*$.

then each block contains at most 5 columns. To find block $\boldsymbol{B_{14}}$ **u**sing $(r,\bar{r})$ $-$inducing of p.i.s. D_{146}, D_{147}, D_{148}, D_{150}, D_{234}, D_{235} of S_{27} to S_{28} gives c_1, c_2, c_3, d_{197}, d_{198}, c_4,respectively. Since $\langle 19,7,2\rangle \neq \langle 19,7,2\rangle'$ and $\langle 19,7,2\rangle \downarrow S_{27} \cap \{\langle 13,8,7\rangle, \langle 11,8,7,2\rangle^*, \langle 10,8,7,2,1\rangle, \langle 8,7,6,5,2\rangle\} \downarrow S_{27}$ has no i.m.s then $c_1 = d_{191} + d_{192}$. Since $\langle 13,8,7\rangle \neq \langle 13,8,7\rangle'$ then c_2 divided or there are two columns. Suppose there are two columns:

$\phi_1 = a_1\langle 18,8,2\rangle + a_2\langle 13,8,7\rangle + a_3\langle 11,8,7,2\rangle^* + a_4\langle 10,8,7,2,1\rangle + a_5\langle 8,7,6,5,2\rangle$,

$\phi_1 = a_1\langle 18,8,2\rangle' + a_2\langle 13,8,7\rangle' + a_3\langle 11,8,7,2\rangle^* + a_4\langle 10,8,7,2,1\rangle' + a_5\langle 8,7,6,5,2\rangle'$,

to describe it since $\langle 13,8,7\rangle \downarrow S_{27}$ has 4 of i.m.s and B_{12} of defect one then $a_1, a_2 \cdots a_5 \in \{0,1\}$. Suppose $a_2 = 1$,since $\langle 13,8,7\rangle \downarrow S_{27} \cap \{\langle 10,8,7,2,1\rangle, \langle 8,7,6,5,2\rangle\} \downarrow S_{27}$ has no i.m.s. so $a_4, a_5 = 0$. But degree $\phi_1, \phi_2 \equiv 0 \; mod \; 11^2$ only when

- $a_1 = a_2 = 1$ and $a_3 = 0 \Rightarrow c_2 = d_{193} + d_{194}$ or
- $a_1 = 0$ and $a_2 = a_3 = 1 \Rightarrow c_3 = d_{195} + d_{196}$

So $c_2 = d_{193} + d_{194}$. As since $\boldsymbol{B_{14}}$ of defect one then $c_3 = d_{195} + d_{196}$. finally since $\langle 8,7,6,5,2\rangle \neq \langle 8,7,6,5,2\rangle'$ then c_4 must split to $d_{199} + d_{200}$. Hence the decomposition matrix for this block is **Table** 8.

Case 9. The decomposition matrix for the blocks $\boldsymbol{B_{16}}$ of type associte and $\boldsymbol{B_{17}}, \boldsymbol{B_{18}}$ of type duble as shown in **Tables** 9.

Table 9. Blocks B_{16}, B_{17}, B_{18}

Block	Spin character	Decomposition matrix																			
$\boldsymbol{B_{16}}$	$\langle 19,5,4\rangle$	1																			
	$\langle 19,5,4\rangle'$		1																		
	$\langle 16,8,4\rangle$	1		1																	
	$\langle 16,8,4\rangle'$		1		1																
	$\langle 15,8,5\rangle$			1		1															
	$\langle 15,8,5\rangle'$				1		1														
	$\langle 11,8,5,4\rangle^*$					1	1	1	1												
	$\langle 10,8,5,4,1\rangle$							1		1											
	$\langle 10,8,5,4,1\rangle'$								1		1										
	$\langle 9,8,5,4,2\rangle$									1											
	$\langle 9,8,5,4,2\rangle'$										1										
$\boldsymbol{B_{17}}$	$\langle 18,6,3,1\rangle^*$											1									
	$\langle 17,7,3,1\rangle^*$											1	1								
	$\langle 14,7,6,1\rangle^*$												1	1							
	$\langle 12,7,6,3\rangle^*$													1	1						
	$\langle 11,7,6,3,1\rangle$														1	1					
	$\langle 9,7,6,3,2,1\rangle^*$															1					
$\boldsymbol{B_{18}}$	$\langle 18,5,3,2\rangle^*$																1				
	$\langle 16,7,3,2\rangle^*$																1	1			
	$\langle 14,7,5,2\rangle^*$																	1	1		
	$\langle 13,7,5,3\rangle^*$																		1	1	
	$\langle 11,7,5,3,2\rangle$																			1	1
	$\langle 10,7,5,3,2,1\rangle^*$																				1
		d_{206}	d_{207}	d_{208}	d_{209}	d_{210}	d_{211}	d_{212}	d_{213}	d_{214}	d_{215}	d_{216}	d_{217}	d_{218}	d_{219}	d_{220}	d_{221}	d_{222}	d_{223}	d_{224}	d_{225}

Proof. To find block $\boldsymbol{B_{16}}$ using $(r,\bar{r})$-inducing of p.i.s. D_6, D_{11}, D_{13}, D_{236}, D_{237}, D_{16} of S_{27} to S_{28} gives c_1, c_2, c_3, d_{212}, d_{213}, c_4, respectively. Since $\langle 19,5,4\rangle \neq \langle 19,5,4\rangle'$ and $\langle 19,5,4\rangle \downarrow S_{27} \cap \{\langle 15,8,5\rangle, \langle 11,8,5,4\rangle^*,$ $\langle 10,8,5,4,1\rangle, \langle 9,8,5,4,2\rangle\} \downarrow S_{27}$ has no i.m.s then $c_1 = d_{206} + d_{207}$. Since $\langle 15,8,5\rangle \neq \langle 15,8,5\rangle'$ then c_2 divided or there are two columns. Suppose there are two columns:

$\phi_1 = a_1\langle 16,8,4\rangle + a_2\langle 15,8,5\rangle + a_3\langle 11,8,5,4\rangle^* + a_4\langle 10,8,5,4,1\rangle + a_5\langle 9,8,5,4,2\rangle,$

$\phi_1 = a_1\langle 16,8,4\rangle' + a_2\langle 15,8,5\rangle' + a_3\langle 11,8,5,4\rangle^* + a_4\langle 10,8,5,4,1\rangle' + a_5\langle 9,8,5,4,2\rangle'.$

To describe columns since $\langle 15,8,5\rangle \downarrow S_{27}$ has 6 of i.m.s and since $\boldsymbol{B_{16}}$ of defect one then we have $a_1, a_2 \cdots a_5 \in \{0,1\}$. Suppose $a_2 = 1$, since $\langle 15,8,5\rangle \downarrow S_{27} \cap \{\langle 10,8,5,4,1\rangle, \langle 9,8,5,4,2\rangle\} \downarrow S_{27}$ has no i.m.s. so $a_4, a_5 = 0$. But degree $\phi_1, \phi_2 \equiv 0 \; mod \; 11^2$ only when

- $a_1 = a_2 = 1$ and $a_3 = 0 \Rightarrow c_2 = d_{208} + d_{209}$ or
- $a_1 = 0$ and $a_2 = a_3 = 1 \Rightarrow c_3 = d_{210} + d_{211}$

So $c_2 = d_{208} + d_{209}$, also since $\boldsymbol{B_{16}}$ of defect one then $c_3 = d_{210} + d_{211}$. Since $\langle 9,8,5,4,2\rangle \neq \langle 9,8,5,4,2\rangle'$ then c_4 must split to $d_{214} + d_{215}$. Hence block $\boldsymbol{B_{16}}$. To find blocks $\boldsymbol{B_{17}}$ and $\boldsymbol{B_{18}}$, since

- degree $\{\langle 17,7,3,1\rangle^*, \langle 12,7,6,3\rangle^*, \langle 9,7,6,3,2,1\rangle^*\} \equiv 88 \; mod \; 11^2$
- degree $\{\langle 18,6,3,1\rangle^*, \langle 14,7,6,1\rangle^*, \langle 11,7,6,3,1\rangle + \langle 11,7,6,3,1\rangle'\} \equiv -88 \; mod \; 11^2$
- degree $\{\langle 18,5,3,2\rangle^*, \langle 14,7,5,2\rangle^* \langle 11,7,5,3,2\rangle + \langle 11,7,5,3,2\rangle'\} \equiv 77 \; mod \; 11^2$
- degree $\{\langle\langle 16,7,3,2\rangle^*, \langle 13,7,5,3\rangle^*, \langle 10,7,5,3,2,1\rangle^*\} \equiv -77 \; mod \; 11^2$

By using $(r,\bar{r})$ −inducing of p.i.s.D_{166}, D_{167}, D_{175}, D_{177}, D_{170}, D_{71}, D_{75}, D_{83}, D_{91}, D_{95} for S_{27} to S_{28} , and since on $(11,\alpha)$-regular classes:

1. $\langle 11,7,6,3,1\rangle = \langle 11,7,6,3,1\rangle'$
2. $\langle 9,7,6,3,2,1\rangle^* = \langle 11,7,6,3,1\rangle - \langle 12,7,6,3\rangle^* + \langle 14,7,6,1\rangle^* - \langle 17,7,3,1\rangle^* \langle 18,6,3,1\rangle^*$
3. $\langle 11,7,5,3,2\rangle = \langle 11,7,5,3,2\rangle'$
4. $\langle 10,7,5,3,2,1\rangle = \langle 11,7,5,3,2\rangle - \langle 13,7,5,3\rangle^* + \langle 14,7,5,2\rangle^* - \langle 16,7,3,2\rangle^* + \langle 18,5,3,2\rangle^*.$

then each block contains at most 5 columns so we get **Table** 9.

4. CONCLUSIONS

In this research, the decomposition matrix of the S_{28} If the field characteristic is prime was found, which is equal to $\boldsymbol{B_1} \oplus \boldsymbol{B_2} \oplus, \ldots, \oplus \boldsymbol{B_{69}}$, the decomposition matrix for the spin characters of the symmetric group is connected between the irreducible spin characters and the irreducible modular spin characters. We had to conduct multiple studies in order to have enough data to discover new properties and theorems because there isn't a general formula for studying the topic, especially when we prove the field and the change of groups. This is what previous researchers did when they looked at the division matrix at the field where the characteristic is 0.We used maple programming to see all correct probabilities. then examine irreducible modular spin characters, and lastly classify groups.

Appendix(Maple Programming)

Let approximation matrix for the block to the spin characters of S_{28} and field has characteristic $p = 11$

Degree	Spin characters	Approximation matrix	
D_1	α	a_1	
D_1	α'		a_1
D_2	γ^*	a_2	a_2
.	.	.	.
.	.	.	.
D_i	δ	a_i	
A_i	δ'		a_i
		φ_1	φ_2

$a_i \in \mathbb{N}$, $\forall i$. To find discuss all probabilities such that the degree $Y_1, Y_2 \equiv 0 \; mod 7^3$.

```
>P:=p^3;
A_1 := D_1;
A_2 := D_2;
.
.
A_i := D_i;
S := 0;
j := 1;
for a_1 from 0 to n_1 do
for a_2 from 0 to n_2 do
.
.
for a_i from 0 to n_i do
S := A_1 * a_1 + A_2 * a_2 + ... + A_i * a_I;
G := modp(S, P);
if G = 0 then
print(j, 'a_1' = a_1, 'a_2' = a_2, ..., 'a_i' = a_i);
j := j + 1;
fi;
S := S;
od;
od;
.
.
od;
```

REFERENCES

[1] A. H. Jassim, “11-Brauer trees of S_{20}, “ Journal of Basrah Researches(Sciences).40, pp. 76-86, 2014.

[2] A. H. Jassim, “11- Modular Characters for $\bar{S}_{27}$, ” Journal of University of Babylon for Pure and Applied Sciences. 27, pp. 303–317,2019.

[3] A. H. Jassim, and M. M. Jawad, “Decomposition matrices for the spin characters of S_{26} Modulo 11, ” Basrah Journal of Science. 37, pp. 253–261, 2019.

[4] A. K. Yaseen, “Modular spin representations of the symmetric groups,” Ph.D. Dissertation, Aberystwyth University, Aberystwyth, Aberystwyth, United Kingdom, 1987.

[5] A. O. Morris, "The spin representation of the symmetric group, ” proc. London Math, Soc., 3(12), pp.55-76, 1962

[6] A. O. Morris, and A.K.Yaseen, “Decomposition matrices for spin characters of symmetric group,” Proc. of Royal society of Edinburgh. 108A,pp.145-164, 1988.

[7] B. M. Puttaswamaiah, andJ. D. Dixon, “Modular representation of finite groups, ” New York, Academic Press, 1977.

[8] G. James, “The modular characters of Mathieu groups, ” J. Algebra. 27, pp.57–111, 1973.

[9] J. F. Humphreys, “Blocks of the Projective representations of symmetric groups, ” J. London Math. Society. **33**, pp. 441-452, 1986.

[10] J. F. Humphreys, “Projective modular representations of finite groups, ” J. London Math, Soc., 16, pp.51-66, 1977.

[11] J. Schur, “Uber die Darstellung der symmetrischen und der alternierenden gruppe durch gebrochene lineare subtituttionen, ” j,Reine ang.Math. 139, pp.155-250, 1911.

[12] N. M. Yacoob, “Decomposition Matrices of the Spin Characters of the Symmetric Group S_{21}, S_{22} Modulo p = 11 and S_{22} modulo p = 7, ” M.Sc. Thesis Basrah University. 2014.

[13] S. A. Taban, and A. H. Jassim, “The Brauer trees of the symmetric groups S_{17} and S_{18} modulo $p = 11$, ” Basrah Journal of Science, 30A, pp. 149-165, 2012.

[14] S. A. Taban, and A. H. Jassim, “Irreducible modular spin characters of the symmetric group S_{19} modulo $p = 11$, ” Basrah Journal of Science 31A, pp. 59–74, 2013.