On distributive semimodules

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DOI: http://dx.doi.org/10.31642/JoKMC/2018/100209

Received Apr. 14, 2023. Accepted for publication May. 30, 2023

Abstract—*This work considers the construction of the concept of distributive property for semimodules. Some characterizations of this property, with some examples are given. Some conditions on semiring or semimodules (like subtractive, semisubtractive, cancellative, and k-cyclic) are required to obtain interesting results. The main results are: Any subsemimodule and factor semimodule of a distributive semimodule is distributive. Moreover, weakly distributive, is also, introduced and investigated. It is found that the semimodule is distributive if and only if each subsemimodule is weak distributive. Taking advantage of the supplemented concept to find some properties of distributive semimodule. Finally, the summand sum and summand intersection properties for distributive semimodules with some conditions are valid.*

Keywords—distributive semimodule, subtractive, semisubtractive, k-cyclic, weak distributive.

I. INTRODUCTION

Throughout this paper, R represents a semiring and all semimodules are left R-semimodules. The semimodule $\mathcal{D}$ is called distributive if $\mathcal{H}_1 \cap (\mathcal{H}_2 + \mathcal{H}_3) = (\mathcal{H}_1 \cap \mathcal{H}_2) + (\mathcal{H}_1 \cap \mathcal{H}_3)$ for all subsemimodules $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ of $\mathcal{D}$ [1]. A semiring is called arithmetical semiring if it is considered a distributive semimodule over itself. This concept has not been studied independently for semimodule but has been occasionally referred to in some research, e.g. [1], [2]. This concept for modules was studied in many papers, for example See [3], [4], [5], and [6]. In this work, we study the structure and properties of distributive semimodules. In section 2, some definitions and facts that are important in our study will be given. In section 3, some properties, characterizations of distributive semimodule, and some examples of arithmetical semiring and distributive semimodule are given. In section 4, the concept of weak distributive semimodule and a relationship between distributive semimodule and weak distributive semimodule are studied. Moreover, a link between the supplement semimodules and distributive semimodules, to get more results are investigated. Also, by adding conditions, such as cancellative, and semisubtractive, results about summand and direct summand are obtained.

II. PRELIMINARIES

In this section, we will present some definitions and remarks that are needed in the next section.

Definition 2.1: [7, p. 149] Let R be a semiring. A left R-semimodule $\mathcal{D}$ is a commutative monoid $(\mathcal{D}, +, 0)$ for which we have a function $R \times \mathcal{D}: \rightarrow \mathcal{D}$, define by $(v, f) \mapsto vf$ such that $\forall\, v, \acute{v} \in R$, and $f, \acute{f} \in \mathcal{D}$, then

- $v\,(f + \acute{f}) = v\,f + v\,\acute{f}$
- $(v + \acute{v})f = v\,f + \acute{v}f$
- $(v\,\acute{v})f = v\,(\acute{v}\,f)$
- $0_R\, f = 0_{\mathcal{D}} = v\, 0_{\mathcal{D}}$

Definition 2.2: [7, p. 150] Let $\mathcal{D}$ be R-semimodule and $\emptyset \neq \mathcal{H} \subseteq \mathcal{D}$, if $\mathcal{H}$ closed under addition and scalar multiplication, then $\mathcal{H}$ is called a subsemimodule of $\mathcal{D}$. The set of all subsemimodules of $\mathcal{D}$ is denoted by $L(\mathcal{D})$.

Definition 2.3: [7, p. 49] A semiring R is called yoked if each $x, y \in R$, there exists $z \in R$ where $x + z = y$ or $y + z = x$.

Definition 2.4: [8] Let R be a semiring. The intersection of all maximal ideals of R is called the Jacobson radical of R and is denoted by J(R).

Definition 2.5 [9] Let R be a semiring. The R-semimodule $\mathcal{D}$ is called semisubtractive if each $\sigma, y \in \mathcal{D}$, there exists $g \in \mathcal{D}$ where $\sigma + g = y$ or $y + g = \sigma$.

Definition 2.6: [7, p. 49] Let R be a semiring. The R-semimodule $\mathcal{D}$ is called cancellative if each $f, y,$ and $z \in \mathcal{D}$, $f + y = f + z$, then $y = z$.

Definition 2.7: [10] Let R be a semiring, and $\mathcal{L}$ an ideal of R. If $s, s + n \in \mathcal{L}$ implies $n \in \mathcal{L}$, for all $s, n \in$ R, then the ideal $\mathcal{L}$ is called subtractive. The semiring is called subtractive if each ideals of R is subtractive.

Definition 2.8: [11] An R-subsemimodule $\mathcal{K}$ of R-semimodule $\mathcal{D}$ is subtractive if $s, s + n \in \mathcal{K}$ implies $n \in \mathcal{K}$, for all $s, n \in \mathcal{D}$. The semimodule $\mathcal{D}$ is called subtractive if any $\mathcal{K} \in L(\mathcal{D})$ is subtractive.

Definition 2.9: [12] An R-semimodule $\mathcal{D}$ is called *k-cyclic* if any cyclic subsemimodule of $\mathcal{D}$ is subtractive.

Definition 2.10: [7, p. 164] Let $\mathcal{D}$ be an R-semimodule, $\mathcal{F} \in L(\mathcal{D})$, then $\mathcal{F}$ induces an R-congruence relation $\approx_{\mathcal{F}}$ on $\mathcal{D}$ called the Bourne relation, defined by setting $a \approx_{\mathcal{F}} b$ if and only if there exist elements i and j of $\mathcal{F}$ such that $a + i = a + j$. If $a \in \mathcal{D}$ then we write ${}^{a}/_{\mathcal{F}}$ instead of ${}^{a}/_{\approx_{\mathcal{F}}}$. The factor semimodule ${}^{\mathcal{D}}/_{\approx_{\mathcal{F}}}$ is denoted by ${}^{\mathcal{D}}/_{\mathcal{F}}$.

Definition 2.11: [11] An R-subsemimodule $\mathcal{O}_1$ of an R-semimodule $\mathcal{D}$ is called small (superfluous) if for any $\mathcal{O}_2 \in L(\mathcal{D})$, with $\mathcal{O}_1 + \mathcal{O}_2 = \mathcal{D}$ then $\mathcal{O}_2 = \mathcal{D}$.

Definition 2.12: [13] An R-subsemimodule $\mathcal{M}$ of an R-semimodule $\mathcal{D}$ is called large if $\mathcal{M} \cap \mathcal{G} = 0$, implies $\mathcal{G} = 0$, for all $\mathcal{G} \in L(\mathcal{D})$. Expressed by $\mathcal{M} <^{e} \mathcal{D}$.

Definition 2.13: [9] An R-subsemimodule $\mathcal{M}$ of an R-semimodule $\mathcal{D}$ is called the relative complement of $\mathcal{F} \in L(\mathcal{D})$, if $\mathcal{M} \cap \mathcal{F} = 0$ and $\mathcal{M}$ is a maximal with this property.

Definition2.14:[7, p. 156]Let $\mathcal{M}$ and $\mathcal{D}$ be R-semimodules. The map $\eta: \mathcal{M} \to \mathcal{D}$ is called a homomorphism if $\forall\, e, n \in \mathcal{M}, s \in$ R

1- $\eta(e + n) = \eta(e) + \eta(n)$.

2-$\eta(s\, n) = s\eta(n)$.

Definition 2.15: [14] For a homomorphism of R-semimodules $\eta: \mathcal{B} \to \mathcal{D}$ we define :

1-$\eta(\mathcal{B}) = \{\, \eta(e) \mid e \in \mathcal{B}\}$.

2-ker(η) = $\{\, e \in \mathcal{B} \mid \eta(e) = 0\}$

3- $Im(\eta) = \{k \in \mathcal{D} | k + \eta(e) = \eta(\acute{e})$ for some $e, \acute{e} \in \mathcal{B}\}$.

4- η is $i-$regular if $\eta(\mathcal{B}) = Im(\mathcal{B})$.

5- η is $k-$regular, if $\eta(e) = \eta(\acute{e})$ implies $e + h = \acute{e} + \acute{h}$ for some $h, \acute{h} \in \ker(\eta)$.

6- $Hom\,(\mathcal{B}\,, \mathcal{D}) = \{\eta: \mathcal{B} \to \mathcal{D} \mid \eta$ is a homomorphism$\}$.

7- $End\,(\mathcal{B}) = \{\eta: \mathcal{B} \to \mathcal{B} \mid \eta$ is a homomorphis$\}$.

8- η is a monomorphism, if for any R - semimodules $\mathcal{H}$ and $\mathcal{B}$, a homomorphisms $\delta, \rho \in Hom(\mathcal{H}, \mathcal{B})$ with $\eta\delta = \eta\rho$, then $\delta = \rho$. (η is injective if and only if it is a monomorphism. [7, p. 169])

9- η is an epimorphism, if for any R - semimodules $\mathcal{H}$ and a homomorphism $\delta, \rho \in om(\mathcal{D}, \mathcal{H})$ with $\delta\eta = \rho\eta$, then $\delta = \rho$.

(η is surjective if and only if it is an epimorphism and $\eta(\mathcal{B})$ is subtractive [7, p. 169])

10- η is an isomorphism if η is monomorphism and epimorphism.

Definition 2.16: [7, p. 150] Let $\mathcal{D}$ be an R-semimodule, and $\mathrm{h}, \mathrm{k} \in \mathcal{D}$ we define :

1- $\mathrm{Rh} = \{\mathrm{rh} : \mathrm{r} \in \mathrm{R}\}$.

2- $(\mathrm{Rh}:\mathrm{k}) = \{\mathrm{r} \in \mathrm{R} : \mathrm{rk} \in \mathrm{Rh}\}$.

3- $(0:h) = l(h) = \{\mathrm{r} \in \mathrm{R} : \mathrm{rh} = 0\}$.

Lemma 2.17: [7, p. 156] Let $\mathcal{D}$ be an R-semimodule, $\mathcal{H}_1, \mathcal{H}_2$, and $\mathcal{H}_3 \in L(\mathcal{D})$ satisfying the conditions that $\mathcal{H}_1 \subseteq \mathcal{H}_2$ and $\mathcal{H}_2$ is subtractive, then $\mathcal{H}_2 \cap (\mathcal{H}_1 + \mathcal{H}_3) = \mathcal{H}_1 + (\mathcal{H}_2 \cap \mathcal{H}_3)$.

Lemma 2.18: [11] Let R be a semiring. Then J(R) is small in R.

Corollary 2.19: For a semiring R, every ideal subset of J(R) is small in R.

Proof: By Lemma 2.18

III. Some Characterizations and Properties of Distributive Semimodules

Consider an R-semimodule $\mathcal{D}$ and subsemimodules $\mathcal{A}, \mathcal{F}$, and $\mathcal{C}$ of $\mathcal{D}$. We note that $\mathcal{W} \cap (\mathcal{F} + \mathcal{C}) = (\mathcal{W} \cap \mathcal{F}) + (\mathcal{W} \cap \mathcal{C})$ is equivalent to $\mathcal{W} + (\mathcal{F} \cap \mathcal{C}) = (\mathcal{W} + \mathcal{F}) \cap (\mathcal{W} + \mathcal{C})$[1]. To see why, assume $\mathcal{W} \cap (\mathcal{F} + \mathcal{C}) = (\mathcal{W} \cap \mathcal{F}) + (\mathcal{W} \cap \mathcal{C})$. Then by hypothesis, we have $(\mathcal{W} + \mathcal{F}) \cap (\mathcal{W} + \mathcal{C}) = [(\mathcal{W} + \mathcal{F}) \cap \mathcal{W}] + [(\mathcal{W} + \mathcal{F}) \cap \mathcal{C}] = \mathcal{W} + (\mathcal{W} \cap \mathcal{C}) + (\mathcal{F} \cap \mathcal{C}) = \mathcal{W} + (\mathcal{F} \cap \mathcal{C})$. Conversely, if we have $\mathcal{W} + (\mathcal{F} \cap \mathcal{C}) = (\mathcal{W} + \mathcal{F}) \cap (\mathcal{W} + \mathcal{C})$, then we have $(\mathcal{W} \cap \mathcal{F}) + (\mathcal{W} \cap \mathcal{C}) = [(\mathcal{W} \cap \mathcal{F}) + \mathcal{W}] \cap [(\mathcal{W} \cap \mathcal{F}) + \mathcal{C}] = \mathcal{W} \cap [(\mathcal{W} \cap \mathcal{F}) + \mathcal{C}] = \mathcal{W} \cap [(\mathcal{C} + \mathcal{F}) \cap (\mathcal{C} + \mathcal{W})] = \mathcal{W} \cap (\mathcal{F} + \mathcal{C})$. Therefore, proving either one of the equivalencies above suffices to show that the semimodule is distributive.

Lemma 3.1: Let $\mathcal{D}$ be an R-semimodule, then

i) $l(h + k) \subseteq (\mathrm{Rh}:\mathrm{k}) + (\mathrm{Rk}:\mathrm{h})$, for all $h,\ k \in \mathcal{D}$, if $\mathcal{D}$ is a $k-$cyclic.

ii) $l(h) \subseteq (\mathrm{Rk}:\mathrm{h})$, for all $h,\ k \in \mathcal{D}$.

Proof:

i) Assume $r \in l(h + k)$, then $r(h + k) = 0 = rh + rk$, thus $rh, 0 \in \mathrm{R}h$. By subtractive of Rh, then $rk \in \mathrm{R}h$ so $\mathrm{r} \in (\mathrm{Rh}:\mathrm{k})$, thus $\mathrm{r} \in (\mathrm{Rh}:\mathrm{k}) + (\mathrm{Rk}:\mathrm{h})$.

ii) Assume $r \in l(h)$, then $rh = 0$, so $rh \in \mathrm{R}k$ and hence $r \in (\mathrm{R}k: h)$.

Lemma 3.2: Let R be a yoked cancellative, and $\hat{\mathrm{I}}$ a subtractive ideal of R. If $l(c) \subseteq \hat{\mathrm{I}}$, $c \in$ R, and $\mathrm{R}c = \hat{\mathrm{I}}c$, then $\mathrm{R} = \hat{\mathrm{I}}$.

Proof:

If $\mathrm{R} \neq \hat{\mathrm{I}}$, then there exists $a \in$ R and $a \notin \hat{\mathrm{I}}$, but $ac = sc$ for some $s \in \hat{\mathrm{I}}$. Since $a, s \in$ R there exists $\mathrm{h} \in \mathrm{R} \ni a + \mathrm{h} = \mathrm{s}$ or $\mathrm{s} + \mathrm{h} = a$. If $a + \mathrm{h} = \mathrm{s}$ then $ac + \mathrm{hc} = \mathrm{sc}$, hence $\mathrm{hc} = 0$, so $\mathrm{h} \in l(c) \subseteq \hat{\mathrm{I}}$, by subtractive $a \in \hat{\mathrm{I}}$, similarly if $\mathrm{s} + \mathrm{h} = a$.

This is a contradiction, therefore $\mathrm{R} = \hat{\mathrm{I}}$.

Lemma 3.3: Let R be a yoked, subtractive , cancellative, and $\mathcal{D}$ a $k-$cyclic R -semimodule. If $\mathrm{R}h + \mathrm{R}k = \mathrm{R}(h+k) + (\mathrm{R}h \cap \mathrm{R}k)$ for all $h,\ k \in \mathcal{D}$, then $\mathrm{R} = (\mathrm{Rh}:\mathrm{k}) + (\mathrm{Rk}:\mathrm{h})$.

Proof:

Since $\mathrm{R}h = \mathrm{R}h \cap (\mathrm{R}h + \mathrm{R}k)$, and by hypothesis so $\mathrm{R}h = \mathrm{R}h \cap [\mathrm{R}(h+k) + (\mathrm{R}h \cap \mathrm{R}k)]$. By Lemma 2.17 we get $\mathrm{R}h = [\ \mathrm{R}h \cap \mathrm{R}(h+k)] + (\mathrm{R}h \cap \mathrm{R}k)$. Since $[\mathrm{R}h \cap \mathrm{R}(h+k)] + (\mathrm{R}h \cap \mathrm{R}k) = [(\mathrm{R}h:(h+k))(h+k)] + (\mathrm{R}h \cap \mathrm{R}k) = (\mathrm{R}h:k)(h+k)] + (\mathrm{R}h \cap \mathrm{R}k)$ and $\mathrm{R}h \cap \mathrm{R}k = (\mathrm{R}h:k)k = (\mathrm{R}k:h)h$, then $[(\mathrm{R}h:k)(h+k)] + (\mathrm{R}h \cap \mathrm{R}k) = [(\mathrm{R}h:k) + (\mathrm{R}k:h)]h$. By Lemma 3.1 and Lemma 3.2 , then $\mathrm{R} = (\mathrm{Rh}:\mathrm{k}) + (\mathrm{Rk}:\mathrm{h})$.

In the following theorem we give a characterization of distributive R– semimodule.

Theorem 3.4: Let R be a semiring. Then $\mathcal{D}$ is a distributive R– semimodule, if and only if $\mathrm{R}(h+k) = (\mathrm{R}h \cap \mathrm{R}(h+k)) + (\mathrm{R}k \cap \mathrm{R}(h+k))$, for all $h,\ k \in \mathcal{D}$.

Proof:

Suppose that $\mathcal{D}$ is a distributive R – semimodule, and h, $k \in U$, then

$\mathrm{R}(h+k) = (\mathrm{R}h + \mathrm{R}k) \cap \mathrm{R}(h+k) = (\mathrm{R}h \cap \mathrm{R}(h+k)) + (\mathrm{R}k \cap \mathrm{R}(h+k))$. Conversely, let $\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3 \in L(\mathcal{D})$. It's clear $(\mathcal{W}_3 \cap \mathcal{W}_1) + (\mathcal{W}_3 \cap \mathcal{W}_2) \subseteq \mathcal{W}_3 \cap (\mathcal{W}_1 + \mathcal{W}_2)$, so we need to show the other direction $\mathcal{W}_3 \cap (\mathcal{W}_1 + \mathcal{W}_2) \subseteq (\mathcal{W}_3 \cap \mathcal{W}_1) + (\mathcal{W}_3 \cap \mathcal{W}_2)$.

Assume that $z = h + k \in \mathcal{W}_3 \cap (\mathcal{W}_1 + \mathcal{W}_2)$ where $z \in \mathcal{W}_3, h \in \mathcal{W}_1$, and $k \in \mathcal{W}_2$, then $\mathrm{R}z = \mathrm{R}(h+k)$ by hypothesis $\mathrm{R}z = (\mathrm{R}h \cap \mathrm{R}z) + (\mathrm{R}k \cap \mathrm{R}z)$, then $z \in (\mathcal{W}_3 \cap \mathcal{W}_1) + (\mathcal{W}_3 \cap \mathcal{W}_2)$.

Example 3.5: Let $(\mathrm{R}, +, .)$ be a semiring, where $\mathrm{R} = \{0,1,2\}$, and for all $h,\ c \in \mathrm{R},\ h + c = \max\{h, c\},\ h.k = \min\{h, c\}$.

If $h = 0,\ c = 1$, so $\mathrm{R}0 = \{0\}$, $\mathrm{R}1 = \{0,1\}$, $\mathrm{R}(0+1) = \mathrm{R}1$, then $\mathrm{R}(h+c) = (\mathrm{R}h \cap \mathrm{R}(h+c)) + (\mathrm{R}c \cap \mathrm{R}(h+c)) \rightarrow \mathrm{R}1 = \mathrm{R}0 \cap \mathrm{R}1 + \mathrm{R}1 \cap \mathrm{R}1$. If $h = 0, c = 2$, so $\mathrm{R}0 = \{0\}, \mathrm{R}2 = \mathrm{R}, \mathrm{R}(0+2) = \mathrm{R}$, then $\mathrm{R} = \mathrm{R}0 \cap \mathrm{R} + \mathrm{R}2 \cap \mathrm{R}$. If $h = 1, c = 2$, so $\mathrm{R}1 = \{0,1\}, \mathrm{R}2 = \mathrm{R}, \mathrm{R}(1+2) = \mathrm{R}2$, then $\mathrm{R} = \mathrm{R}2 = \mathrm{R}1 \cap \mathrm{R} + \mathrm{R} \cap \mathrm{R}$.By Theorem 3.4, R is a distributive R – semimodule**.**

Proposition 3.6: [12] Let $\mathcal{D}$ be an R - semimodule, $\mathcal{Q}$ is a distributive an R–semimodule. If $\mu, \omega \in Hom\ (\mathcal{D}, \mathcal{Q})$ are $i-regular$. then,

i) $\mathcal{D} = \omega^{-1}\mu(\mathcal{D}) + \mu^{-1}\omega(\mathcal{D})$.

ii) $\mathcal{F} = \mathcal{F} \cap \omega^{-1}(\mu(\mathcal{F})) + \mathcal{F} \cap \mu^{-1}(\omega(\mathcal{F}))$, for any $\mathcal{F} \in L(\mathcal{D})$.

Proposition 3.7: If $\mathcal{D}$ is a $k-$cyclic distributive R– semimodule, then $\mathrm{R} = (\mathrm{Rh}:\mathrm{k}) + (\mathrm{Rk}:\mathrm{h})$, for all $h,\ k \in \mathcal{D}$.

Proof:

Let $\xi, \pounds \in \mathrm{Hom}(\mathrm{R}, \mathcal{D}) \ni \xi(1) = \mathrm{h}$, and $\pounds(1) = k$.

$\pounds^{-1}\xi(\mathrm{R}) = \{r \in \mathrm{R}: \pounds(r) \in \mathrm{R}h\} = \{r \in \mathrm{R}: rk \in \mathrm{R}h\} = (\mathrm{R}h:k)$, similarly $\xi^{-1}\pounds(\mathrm{R}) = (\mathrm{R}k:h)$.

Since $\mathcal{D}$ is a $k-$cyclic, then $\xi, \pounds$ are $i-$regular. By Proposition 3.6 $\mathrm{R} = \pounds^{-1}\xi(\mathrm{R}) + \xi^{-1}\pounds(\mathrm{R})$, implies that $\mathrm{R} = (\mathrm{R}h:k) + (\mathrm{R}k:h)$.

Theorem 3.8: Let R be a yoked, subtractive, cancellative, if $\mathcal{D}$ is a subtractive R-semimodule, then the following are equivalent:

i) $\mathrm{R} = (\mathrm{R}m:\mathrm{k}) + (\mathrm{Rk}:m)$, for all $m,\ k \in \mathcal{D}$.

ii) $\mathrm{R}(m+k) = (\mathrm{R}m \cap \mathrm{R}(m+k)) + (\mathrm{R}k \cap \mathrm{R}(m+k))$, for all $h,\ k \in \mathcal{D}$.

iii) $\mathcal{D}$ is a distributive R – semimodule.

Proof: i ↔ii

$\mathrm{R}(m+k) = (\mathrm{R}m \cap \mathrm{R}(m+k)) + (\mathrm{R}k \cap \mathrm{R}(m+k))$

$\leftrightarrow \mathrm{R}(m+k) = (\mathrm{R}m:(m+k))(m+k) + (\mathrm{R}k:(m+k))(m+k) \leftrightarrow \mathrm{R}(m+k) = [(\mathrm{R}m:(m+k)) + (\mathrm{R}k:(m+k))](m+k) \leftrightarrow \mathrm{R} = [(\mathrm{R}m:(m+k)) + (\mathrm{R}k:(m+k))]$ by Lemma 3.1 and Lemma3.2.

ii→ iii by Theorem 3.4 and iii→ i by Proposition 3.7 .

Corollary 3.9: Let R be a yoked, subtractive, cancellative, if $\mathcal{D}$ is a subtractive R -semimodule. Then

i)If $\mathrm{R}h + \mathrm{R}k = \mathrm{R}(h+k) + (\mathrm{R}h \cap \mathrm{R}k)$for all $h,\ k \in \mathcal{D}$, then $\mathcal{D}$ is distributive R – semimodule.

ii) If $\mathcal{D}$ is a distributive R – semimodule, and $\mathrm{R}h \cap \mathrm{R}k = 0$ where $h,\ k \in \mathcal{D}$, then $l(h) + l(k) = \mathrm{R}$.

iii) If $\mathcal{D}$ is a distributive R – semimodule, and $l(h) \subseteq J(\mathrm{R}), h \in \mathcal{D}$, then Rh is large in $\mathcal{D}$.

Proof:

i) It is direct by using Lemma 3.3 and Theorem 3.8.

ii) If $\mathcal{D}$ is distributive R – semimodule, then by Theorem 2.5, $\mathrm{R} = (\mathrm{Rh}:\mathrm{k}) + (\mathrm{Rk}:\mathrm{h})$. Since $(\mathrm{Rh}:\mathrm{k}) = \{r \in \mathrm{R} : \mathrm{rk} \in \mathrm{Rh}\} = \{r \in \mathrm{R} : \mathrm{rk} = 0\} = l(k)$, because $\mathrm{R}h \cap \mathrm{R}k = 0$, similarly $(\mathrm{Rk}:\mathrm{h}) = l(h)$.

iii) Assume that $l(h) \subseteq J(\mathrm{R})$, and $l(h) \cap U = 0$, if $k \in U$, then $\mathrm{R}h \cap \mathrm{R}k = 0$. By (ii) then $l(h) + l(k) = \mathrm{R}$, but $l(h)$ is small in R by Corollary 1.22, so $l(k) = \mathrm{R}$ and $k = 0$, hence $U = 0$, therefore, $\mathrm{Rh} <^e \mathcal{D}$.

Proposition 3.10: If $\mathcal{D}$ is a distributive R– semimodule, and $\mathcal{B} \in L(\mathcal{D})$. Then $\mathcal{B}$ has a unique relative complement.

Proof:

Assume that $\mathcal{B}_1, \mathcal{B}_2 \in L(\mathcal{D})$ are relative complements of $\mathcal{B}$, then $\mathcal{B} \cap \mathcal{B}_1 = 0 \wedge \mathcal{B} \cap \mathcal{B}_2 = 0$. So, $(\mathcal{B} \cap \mathcal{B}_1) + (\mathcal{B} \cap \mathcal{B}_2) = 0 = \mathcal{B} \cap (\mathcal{B}_1 + \mathcal{B}_2)$, but $\mathcal{B}_1 \subseteq (\mathcal{B}_1 + \mathcal{B}_2)$, $\mathcal{B}_2 \subseteq (\mathcal{B}_1 + \mathcal{B}_2)$, but $\mathcal{B}_1, \mathcal{B}_2$ are relative complements, then $\mathcal{B}_1 = (\mathcal{B}_1 + \mathcal{B}_2) = \mathcal{B}_2$.

Proposition 3.11: If $\mathcal{D}$ is a distributive R– semimodule, and $\mathcal{O} \in L(\mathcal{D})$, then

i) $\mathcal{O}$ is distributive.

ii) $\mathcal{D}/_{\mathcal{O}}$ distributive if $\mathcal{O}$ is subtractive.

Proof:

i) Suppose that $\mathcal{O}$ is a subsemimodule of $\mathcal{D}$, and $\mathcal{W},\mathcal{H},\mathcal{P} \in L(\mathcal{O}) \rightarrow \mathcal{W},\mathcal{H},\mathcal{P} \in L(\mathcal{D})$, but $\mathcal{D}$ is a distributive, then $\mathcal{W} \cap (\mathcal{H}+\mathcal{P}) = (\mathcal{W}\cap\mathcal{H}) + (\mathcal{W}\cap\mathcal{P})$.

ii) Let $\mathcal{D}/_{\mathcal{O}}$ the factor R– semimodule and $\mathcal{A}_1/_{\mathcal{O}}, \mathcal{A}_2/_{\mathcal{O}}, \mathcal{A}_3/_{\mathcal{O}} \in L(\mathcal{D}/_{\mathcal{O}})$, where $\mathcal{O} \subseteq \mathcal{A}_{i\in\{1,2,3\}} \in L(\mathcal{D})$, then

$(\mathcal{A}_1/_{\mathcal{O}}) \cap (\mathcal{A}_2/_{\mathcal{O}} + \mathcal{A}_3/_{\mathcal{O}}) = \left(\mathcal{A}_1/_{\mathcal{O}}\right) \cap \left(\mathcal{A}_2+\mathcal{A}_3/_{\mathcal{O}}\right) = \mathcal{A}_1 \cap (\mathcal{A}_2+\mathcal{A}_3)/_{\mathcal{O}} = (\mathcal{A}_1 \cap \mathcal{A}_2 + \mathcal{A}_1 \cap \mathcal{A}_3)/_{\mathcal{O}} =$

$(\mathcal{A}_1 \cap \mathcal{A}_2)/_{\mathcal{O}} + (\mathcal{A}_1 \cap \mathcal{A}_3)/_{\mathcal{O}} = \left(\mathcal{A}_1/_{\mathcal{O}} \cap \mathcal{A}_2/_{\mathcal{O}}\right) + \left(\mathcal{A}_1/_{\mathcal{O}} \cap \mathcal{A}_3/_{\mathcal{O}}\right)$.

Proposition 3.12: If $\mathcal{D}$ is a uniserial R– semimodule, then $\mathcal{D}$ is distributive.

Proof:

Let $\mathcal{J}_1,\mathcal{J}_2,\mathcal{J}_3 \in L(U)$, since $\mathcal{D}$ is a uniserial , if

$\mathcal{J}_1 \subseteq \mathcal{J}_2 \subseteq \mathcal{J}_3 \rightarrow \mathcal{J}_1 \cap (\mathcal{J}_2+\mathcal{J}_3) = \mathcal{J}_1 = \mathcal{J}_1+\mathcal{J}_1 = (\mathcal{J}_1\cap\mathcal{J}_2)+(\mathcal{J}_1\cap\mathcal{J}_3)$, the rest of the possibilities are similar. Therefore, then $\mathcal{D}$ is distributive.

Proposition 3.13: If $\mathcal{D}$ is a simple R– semimodule, then $\mathcal{D}$ is distributive.

Proof: It's clear.

Example 3.14: Let Q be any nonempty set then $(P(Q),\cup,\cap)$ is arithmetical semiring. Since the first operation of semiring is union, and the intersection is distributive over the union in set theory, then $P(Q)$ is arithmetical.

Example 3.15: [13]Let $R = \{0, h, 1\}$ define addition and multiplication operations on R as follows.

$i)$ $0_R = 0, 1_R = 1$;

ii) $1+1 = 1+h = 1, h+h = h$;

iii) $0.0 = 0\,.h = 0, h\,.h = h$.

Then $(R,+,.)$ is a commutative semiring.

Let $\mathcal{D} = \{0,1,h,k\}$ with the same operations defined in R and

1) $0_{\mathcal{D}} = 0_R = 0$;

2) $k+k = k, k+1 = k+h = h$;

3) $0\,.k = h.k = 0, 1.k = k$

It is easy to see that $(\mathcal{D},+)$ is a distributive commutative R– semimodule, through proof

$R(c+v) = \left(Rc \cap R(c+v)\right) + (Rv \cap R(c+v))$, for all $c, v \in \mathcal{D}$.

Since $R0 = \{0\}$, $R1 = \{0,1,h\}$, $Rh = \{0,h\}$, $Rk = \{0,k\}$, then, if $c = 1, v = h$, then $R(1+h) = R1 = R1 + Rh = (R1 \cap R1) + (Rh \cap R1)$. If $c = 1, v = k$, then $R(1+k) = Rh = Rh + \{0\} = (R1 \cap Rh) + (Rk \cap Rh)$. If $c = h, v = k$, then $R(h+k) = Rh = Rh + \{0\} = (Rh \cap Rh) + (Rk \cap Rh)$.

Example 3.16: Let $R = \{0,1\}$ be a boolean semiring [7], recall that $(a) = \{an : n \in \mathbb{N}\}$. If $\mathcal{D} = \{(0),(2),(4),(8)\}$, where $(\mathcal{D},\cup)$ is a commutative monoid. Define multiplication operation as follows: $0.a = 0, 1.a = a$, for all $a \in \mathcal{D}$. Then $(\mathcal{D},\cup)$ is distributive R–semimodule because

$\mathcal{M}_1 \cap (\mathcal{M}_2+\mathcal{M}_3) = \mathcal{M}_1 \cap (\mathcal{M}_2 \cup \mathcal{M}_3) = (\mathcal{M}_1\cap\mathcal{M}_2) \cup (\mathcal{M}_1\cap\mathcal{M}_3) = (\mathcal{M}_1\cap\mathcal{M}_2) + (\mathcal{M}_1\cap\mathcal{M}_3)$, for all $\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3 \in L(\mathcal{D})$.

Example 3.17: Let $(\mathbb{N},+,.)$be the semiring, with usual operations, It is easy to see that $(Z_4,+_4)$ is a distributive $\mathbb{N}$– semimodule.

Example 3.18: The $(Z_2,+_2,._2)$ is yoked semiring, cancellative, and subtractive. Let $(X,+)$be commutative monoid, where $X = \{0,1,2,\ldots\ldots,k\}$, $k \in \mathbb{N}$, and $h+k = \max\{h,k\}$ for all $h,k \in X$. Define multiplication operation as follows: $0_{Z_2}.k = 0_X$ and $1_{Z_2}.k = k$, for all $k \in X$. Then the Z_2– semimodule X is subtractive and satisfies Theorem 3.8.

Definition 3.19: [7] If $\mathcal{D}_1$, and $\mathcal{D}_2$ are R– semimodules, the set $\mathcal{D}_1 \coprod \mathcal{D}_2 = \{(e,y) : e \in \mathcal{D}_1, y \in \mathcal{D}_2\}$ is an R– semimodule with two operations of addition and scalar multiplication on $\mathcal{D}_1 \coprod \mathcal{D}_2$ setting $(e_1,y_1)+(e_2,y_2) = (e_1+e_2, y_1+y_2)$, and for all $r \in R$, $r(e,y) = (re, ry)$.

Remark 3.20: If $\mathcal{D} \neq \{0\}$ is any R – semimodule, then $\mathcal{B} = \mathcal{D} \coprod \mathcal{D}$ is not a distributive R – semimodule.

Proof:

Let $w \neq 0 \in D$ imples that $(w,w) \in B$, hence $R(w,w) \in L(B)$, consider $\mathcal{D}\coprod\{0\}$, and $\{0\}\coprod\mathcal{D} \in L(\mathcal{B})$,

$R(w,w) \cap (\mathcal{D}\coprod\{0\} + \{0\}\coprod\mathcal{D}) = R(w,w) \cap \mathcal{B} = R(w,w)$,

$R(w,w) \cap D\coprod\{0\} + R(w,w) \cap \{0\}\coprod D = \{(0,0)\} + \{(0,0)\} = \{(0,0)\}$. Then $\mathcal{B}$ is not a distributive R – semimodule.

Proposition 3.21: [12] Let y be a k –regular homomorphism from a subtractive R– semimodule $\mathcal{D}$ to R-semimodule $\mathcal{A}$.

i) If $\mathcal{A}$ is distributive, then $y^{-1}(\mathcal{N}_1+\mathcal{N}_2) = y^{-1}(\mathcal{N}_1) + y^{-1}(\mathcal{N}_2)$, for any $\mathcal{N}_1,\mathcal{N}_2 \in L(\mathcal{A})$.

ii) If $\mathcal{D}$ is distributive, then $y(\mathcal{N}_1 \cap \mathcal{N}_2) = y(\mathcal{N}_1) \cap y(\mathcal{N}_2)$,for any $\mathcal{N}_1,\mathcal{N}_2 \in L(\mathcal{D})$.

Lemma 3.22: [14] Let $\mathcal{D}$ and $\mathcal{A}$ be R – semimodules, and $\eta \in Hom(\mathcal{D},\mathcal{A})$. The following statements are equivalent:

i) ɋ is $k-$regular and $Ker(ɋ)=0$.

ii) ɋ is a monomorphism

Proposition 3.23: Let $\mathcal{D}$ be a subtractive R–semimodule, $\mathcal{M}$ any R-semimodule, and $ɋ \in Hom\ (\mathcal{D},\mathcal{M})$, then

i) If $\mathcal{D}$ is a distributive semimodule and ɋ is $k-$regular, then $ɋ(\mathcal{D})$ is a distributive semimodule.

ii) If $\mathcal{M}$ is a distributive semimodule, and ɋ is a monomorphism, then $\mathcal{D}$ is distributive.

Proof:

i) Let $\mathcal{H}=ɋ^{-1}(\mathcal{A}_1), \mathcal{K}=ɋ^{-1}(\mathcal{A}_2)$, and $\mathcal{S}=ɋ^{-1}(\mathcal{A}_3)\in L(\mathcal{D}) \rightarrow ɋ(\mathcal{H})=\mathcal{A}_1, ɋ(\mathcal{K})=\mathcal{A}_2$, and $ɋ(\mathcal{S})=\mathcal{A}_3 \in L(ɋ(\mathcal{D}))$. Then

$\mathcal{A}_1\cap(\mathcal{A}_2+\mathcal{A}_3)=ɋ(\mathcal{H})\cap(ɋ(\mathcal{K})+ ɋ(\mathcal{S}))=ɋ(\mathcal{H})\cap ɋ(\mathcal{K}+\mathcal{S})$, since $ɋ\in Hom\ (\mathcal{D},\mathcal{M})$

$=ɋ(\mathcal{H}\cap(\mathcal{K}+\mathcal{S}))$ by Proposition 2.18.

$=ɋ\big((\mathcal{H}\cap\mathcal{K})+(\mathcal{H}\cap\mathcal{S})\big)=ɋ(\mathcal{H}\cap\mathcal{K})+ɋ(\mathcal{H}\cap\mathcal{S})$since $\mathcal{D}$ is a distributive

$=ɋ(\mathcal{H})\cap ɋ(\mathcal{K})+ɋ(\mathcal{H})\cap ɋ(\mathcal{S})=\mathcal{A}_1\cap\mathcal{A}_2+\mathcal{A}_1\cap\mathcal{A}_3$ by Proposition 2.18

ii) Let $\mathcal{Q},\mathcal{P}$, and $\mathcal{K}\in L(\mathcal{D})$, put $\mathcal{G}_1=ɋ(\mathcal{Q}), \mathcal{G}_2=ɋ(\mathcal{P})$ and $\mathcal{G}_3=ɋ(\mathcal{K})\in L(\mathcal{M})$, by monomorphism then $ɋ^{-1}(\mathcal{G}_1)=\mathcal{Q}, ɋ^{-1}(\mathcal{G}_2)=\mathcal{P}$, and $ɋ^{-1}(\mathcal{G}_3)=\mathcal{K}$, and ɋ is $k-$regular by Lemma 2.19 then

$\mathcal{Q}\cap(\mathcal{P}+\mathcal{K})=ɋ^{-1}(\mathcal{G}_1)\cap\big(ɋ^{-1}(\mathcal{G}_2)+ɋ^{-1}(\mathcal{G}_3)\big)=ɋ^{-1}(\mathcal{G}_1)\cap ɋ^{-1}(\mathcal{G}_2+\mathcal{G}_3)$ by Proposition 2.18

$=ɋ^{-1}(\mathcal{G}_1\cap(\mathcal{G}_2+\mathcal{G}_3))=ɋ^{-1}(\mathcal{G}_1\cap\mathcal{G}_2+\mathcal{G}_1\cap\mathcal{G}_3)$ since $\mathcal{M}$ is a distributive

$=ɋ^{-1}(\mathcal{G}_1\cap\mathcal{G}_2)+ɋ^{-1}(\mathcal{G}_1\cap\mathcal{G}_3)$ by Proposition 2.18

$=ɋ^{-1}(\mathcal{G}_1)\cap ɋ^{-1}(\mathcal{G}_2)+ɋ^{-1}(\mathcal{G}_1)\cap ɋ^{-1}(\mathcal{G}_3)=\mathcal{Q}\cap\mathcal{P}+\mathcal{Q}\cap\mathcal{K}$.

Corollary 3.24: Let $\mathcal{D}$ be subtractive R – semimodule, $\mathcal{M}$ any R–semimodule, and $£\in Hom\ (\mathcal{D},\mathcal{M})$, if £ isomorphism, then $\mathcal{D}$ is a distributive semimodule if and only if $\mathcal{M}$ is a distributive semimodule.

Proof: It's clear by using Proposition 3.23.

Proposition 3.25: Let $\mathcal{D}$ be R -semimodule. If for any two elements $x,y\in\mathcal{D}$ there are $t,v,r,s\in$ R such that $v+t=1$, $tx=ry$, and $vy=sx$, then $\mathcal{D}$ is a distributive semimodule.

Proof.

Assume that $\mathcal{A},\mathcal{B},\mathcal{C}\in L(\mathcal{D})$ and $u\in\mathcal{A}\cap(\mathcal{B}+\mathcal{C})$. We need to show that $u\in(\mathcal{A}\cap\mathcal{B}+\mathcal{A}\cap\mathcal{C})$. Let $u=x+y$, where $x\in\mathcal{B}$, $y\in\mathcal{C}$. By hypothesis, there are $t,v\in$ R such that $v+t=1$, $tx=ry$ so $tx\in$ Ry, $vy=sx$ so $vy\in$ Rx, implies that $vu=vx+vy\in$ R$u\cap$ R$x\subseteq\mathcal{A}\cap\mathcal{B}$ and $tu=tx+ty\in$ R$u\cap$ R$y\subseteq\mathcal{A}\cap\mathcal{C}$, and $vu+tu=u\in\mathcal{A}\cap\mathcal{B}+\mathcal{A}\cap\mathcal{C}$.

Proposition 3.26: Let $\{\mathcal{P}\}_{i\in I}$ be a family of subsemimodules of a distributive semimodule $\mathcal{D}$, and let $\mathcal{P}=\sum_{i\in I}\mathcal{P}_i$. Then

i. $\mathcal{B}\cap\sum_{i\in I}\mathcal{P}_i=\sum_{i\in I}(\mathcal{B}\cap\mathcal{P}_i)$ for any subsemimodule $\mathcal{B}$ of $\mathcal{D}$. However, the dual statement $\mathcal{B}+(\cap\mathcal{P}_i)=\cap_{i\in I}(\mathcal{B}+\mathcal{P}_i)$ does not hold in general.

ii. If $\mathcal{B}$ is a subsemimodule of $\mathcal{D}$ such that $\mathcal{B}\cap\sum_{i\in I}\mathcal{P}_i\neq 0$, then $\mathcal{B}\cap\mathcal{P}_i\neq 0$ for some $i\in I$.

iii. If $\{\mathcal{B}\}_{i\in I}$ is a family of subsemimodules of $\mathcal{D}$ such that $\mathcal{B}_i\leq^e\mathcal{P}_i$, then $\sum_{i\in I}\mathcal{B}_i\leq^e\sum_{i\in I}\mathcal{P}_i$.

iv. If $\mathcal{B}\leq^e\mathcal{P}_i$, then $\mathcal{B}\leq^e\sum_{i\in I}\mathcal{P}_i$

Proof.

(i) Let $x\in\mathcal{B}\cap\sum_{i\in I}\mathcal{P}_i$ **,** then there are finite set $K\subseteq I$, such that $x=\sum_{i\in K}g_i\in\mathcal{B}\cap\sum_{i\in K}\mathcal{P}_i$,

since $\mathcal{D}$ is distributive then $\mathcal{B}\cap\sum_{i\in K}\mathcal{P}_i=\sum_{i\in K}(\mathcal{B}\cap\mathcal{P}_i)\subseteq\sum_{i\in I}(\mathcal{B}\cap\mathcal{P}_i)$. In the other direction by this example, let $\mathbb{Z}$-semimodule $\mathbb{Z}$ and $\mathcal{B}=(3)$ and $\mathcal{P}_i=(2^i)$ for all natural number i. Clearly, $\mathbb{Z}$ is a distributive semimodule. However, $\mathcal{B}+(\cap\mathcal{P}_i)=(3)$ whereas $\cap_{i\in I}(\mathcal{B}+\mathcal{P}_i)=\mathbb{Z}$.

(ii) follow from (i).

(iii) Let $\mathcal{C}$ be a nonzero subsemimodule of $\sum_{i\in I}\mathcal{P}_i$, then $\mathcal{C}\cap\sum_{i\in I}\mathcal{P}_i\neq 0$ by (ii) $\mathcal{C}\cap\mathcal{P}_i\neq 0$ for some $j\in I$, in addition, since $\mathcal{B}_j$ is an essential $\mathcal{P}_i$ then $(\mathcal{C}\cap\mathcal{P}_i)\cap\mathcal{B}_j\neq 0$ hence $\mathcal{C}\cap\mathcal{B}_j\neq 0$ implies that $(\sum_{i\in I}\mathcal{C})\cap\mathcal{B}_i\neq 0$ (since $\mathcal{C}\cap\mathcal{B}_j$ subsemimodule of $(\sum_{i\in I}\mathcal{C})\cap\mathcal{B}_i$) by distributive we get that $\mathcal{C}\cap\sum_{i\in I}\mathcal{B}_i\neq 0$ and $\sum_{i\in I}\mathcal{B}_i$ is an essential of $\sum_{i\in I}\mathcal{P}_i$.

(iv) Follows from (iii).

Definition 3.27: [9] A subsemimodule $\mathcal{K}$ of semimodule $\mathcal{D}$ is called closed if $\mathcal{K}$ has no proper essential extension in $\mathcal{D}$.

Definition 3.28: [9]A subsemimodule $\mathcal{K}$ of a semimodule $\mathcal{D}$ is said to be a closure of a subsemimodule $\mathcal{N}$ in $\mathcal{D}$, if $\mathcal{K}$ is closed and $\mathcal{N}$ essential in $\mathcal{K}$.

Proposition 3.29. Let $\mathcal{D}$ be a distributive semimodule, and $\mathcal{H}\in L(\mathcal{D})$, then

i. $\mathcal{H}$ has a unique complement $\mathcal{B}$ in $\mathcal{D}$, and $\mathcal{B}$ coincides with the sum of all subsemimodules of $\mathcal{D}$ which have zero intersection with $\mathcal{H}$.

ii. $\mathcal{H}$ has a unique closure $\mathcal{A}$ in $\mathcal{D}$, and $\mathcal{A}$ coincides with the sum of all essential extensions of $\mathcal{H}$ in $\mathcal{D}$.

Proof.

(i) Let $\Omega=\{B_i\in L(\mathcal{D}): B_i\cap\mathcal{H}=0\}$ and let $\mathcal{B}=\sum_{B_i\in\Omega}B_i$, by Proposition (3.26)(ii) then $\sum_{B_i\in\Omega}B_i\cap\mathcal{H}=0$, hence $\mathcal{B}$ largest subsemimodule of $\mathcal{D}$ which has zero intersection with $\mathcal{H}$ and $\mathcal{B}$ is a unique complement (see [12], Proposition 2.10).

(ii) Indeed, taking $\mathcal{A}=\sum_{i\in I}\mathcal{A}_i$, where $\mathcal{H}\leq^e\mathcal{A}_i$. By Proposition (3.26)(v), $\mathcal{H}\leq^e\mathcal{A}$. If we take $\mathcal{A}\leq^e\mathcal{A}'$, then

$\mathcal{H} \leq^e \mathcal{A}'$, by the hypothesis $\mathcal{A}' = \mathcal{A}$ hence $\mathcal{A}$ is closed in $\mathcal{D}$. Therefore, $\mathcal{A}$ is a unique closure of $\mathcal{H}$ in $\mathcal{D}$.

Proposition 3.30. Let $\mathcal{D}$ be a distributive semimodule, and $\mathcal{T}, \mathcal{G} \in L(\mathcal{D})$, then

i. If $\mathcal{G} \cap \mathcal{T} \leq^e \mathcal{T}$, then $\mathcal{G} \leq^e \mathcal{G} + \mathcal{T}$.

ii. If $\mathcal{G}$ is closed in $\mathcal{D}$ and $\mathcal{G} \cap \mathcal{T} \leq^e \mathcal{T}$, then $\mathcal{T} \subseteq \mathcal{G}$.

iii. If $\mathcal{G}$ is closed in $\mathcal{D}$ and $\mathcal{T}$ is uniform, then either $\mathcal{T} \subseteq \mathcal{G}$ or $\mathcal{G} \cap \mathcal{T} = 0$.

iv. The intersection of any two distinct closed and uniform subsemimodules of $\mathcal{D}$ is equal to zero.

Proof.

(i) If we assume $\mathcal{G} \cap \mathcal{T} \leq^e \mathcal{T}$, and it's clear $\mathcal{G} \leq^e \mathcal{G}$. By Proposition (3.26)(iv) we get $\mathcal{G} + \mathcal{G} \cap \mathcal{T} = \mathcal{G} \leq^e \mathcal{G} + \mathcal{T}$.

(ii) Since $\mathcal{G}$ is closed and $\mathcal{G} \cap \mathcal{T} \leq^e \mathcal{T}$, then by (i) $\mathcal{G} \leq^e \mathcal{G} + \mathcal{T} \leq \mathcal{D}$, implies that $\mathcal{G} = \mathcal{G} + \mathcal{T}$, hence $\mathcal{T} \subseteq \mathcal{G}$.

(iii) If $\mathcal{G} \cap \mathcal{T} \neq 0$, then $\mathcal{G} \cap \mathcal{T} \leq^e \mathcal{T}$ since $\mathcal{T}$ is uniform. By (ii) then $\mathcal{T} \subseteq \mathcal{G}$.

(iv) Follows from (iii)

IV. WEAK DISTRIBUTIVE SEMIMODULE

Let $\mathcal{D}$ be R– semimodule, and $\mathcal{A} \in L(\mathcal{D})$. We shall call $\mathcal{A}$ a weak distributive subsemimodule of $\mathcal{D}$ if $\mathcal{A} = (\mathcal{A} \cap \mathcal{H}) + (\mathcal{A} \cap \mathcal{K})$, for all $\mathcal{H}, \mathcal{K} \in L(\mathcal{D})$ such that $\mathcal{H} + \mathcal{K} = \mathcal{D}$. While $\mathcal{D}$ is a weak distributive if each $\mathcal{A} \in L(\mathcal{D})$ is a weak distributive subsemimodule.

Example 4.1: Let $(\mathrm{R}, +, .)$be the semiring with identity, then the R– semimodule R is weak distributive.

Proof:

If $\mathcal{K} \in L(\mathrm{R})$, and $\mathrm{R} = \mathcal{A} + \mathcal{B}$, for some $\mathcal{A}, \mathcal{B} \in L(\mathrm{R})$, we need prove $\mathcal{K} \subseteq \mathcal{K} \cap \mathcal{A} + \mathcal{K} \cap \mathcal{B}$. Since $1 \in \mathrm{R}$ then $1 = s + p$, for some $s \in \mathcal{A}, p \in \mathcal{B}$ we have each $k \in \mathcal{K}$ implies that $k.1 = k(s + p) = ks + kp$, where $ks \in \mathcal{K} \cap \mathcal{A}$, and $kp \in \mathcal{K} \cap \mathcal{B}$, hence $k \in \mathcal{K} \cap \mathcal{A} + \mathcal{K} \cap \mathcal{B}$.

Proposition 4.2: Let $\mathcal{D}$ be R– semimodule, $\mathcal{D}$ is a distributive if and only if each $\mathcal{F} \in L(\mathcal{D})$ is weak distributive R– semimodule.

Proof:

Let $\mathcal{F} \in L(\mathcal{D})$ and $\mathcal{Q}, \mathcal{C} \in L(\mathcal{D})$, such that $\mathcal{Q} + \mathcal{C} = \mathcal{D}$, if $\mathcal{D}$ is a distributive, than

$\mathcal{F} \cap (\mathcal{Q} + \mathcal{C}) = \mathcal{F} \cap \mathcal{Q} + \mathcal{F} \cap \mathcal{C}$, hence $\mathcal{F} \cap \mathcal{D} = \mathcal{F} = \mathcal{F} \cap \mathcal{Q} + \mathcal{F} \cap \mathcal{C}$ hence $\mathcal{F}$ is weak distributive.

In the other direction, let $\mathcal{G}, \mathcal{T}, \mathcal{K} \in L(\mathcal{D})$, then

$\mathcal{G} \cap (\mathcal{T} + \mathcal{K}) = \mathcal{G} \cap (\mathcal{T} + \mathcal{K}) \cap (\mathcal{T} + \mathcal{K}) = [\mathcal{G} \cap (\mathcal{T} + \mathcal{K})] \cap \mathcal{T} + [\mathcal{G} \cap (\mathcal{T} + K)] \cap \mathcal{K}$

$= [(\mathcal{G} \cap \mathcal{T}) \cap (\mathcal{T} + K)] + [(\mathcal{G} \cap K) \cap (\mathcal{T} + K)] = (\mathcal{G} \cap \mathcal{T}) + (\mathcal{G} \cap K)$, since $\mathcal{G} \cap (\mathcal{T} + K) \in L(\mathcal{T} + K)$, and $\mathcal{T} + \mathcal{K}$ is weak distributive .

Definition 4.3: [15] Let $\mathcal{D}$ be R– semimodule, and $\mathcal{J}, \mathcal{S} \in L(\mathcal{D})$. The subsemimodule $\mathcal{J}$ is called supplement of $\mathcal{S}$ in $\mathcal{D}$ if $\mathcal{J} + \mathcal{S} = \mathcal{D}$, and $\mathcal{J}$ is minimal with the property. That is, if $\mathcal{F} + \mathcal{S} = \mathcal{D}$ and $\mathcal{F} \subseteq \mathcal{J}$, then $\mathcal{F} = \mathcal{J}$. A semimodule $\mathcal{D}$ is called a supplemented if each $\mathcal{S} \in L(\mathcal{D})$ has a supplement in $\mathcal{D}$.

The following Lemma mentioned in [15], will be proved under different conditions.

Lemma 4.4: Let $\mathcal{D}$ be a distributive R– semimodule, and $\mathcal{T}, \mathcal{L} \in L(\mathcal{D})$. Then $\mathcal{T}$ is a supplement of $\mathcal{L}$ if and only if $\mathcal{D} = \mathcal{L} + \mathcal{T}$ and $\mathcal{T} \cap \mathcal{L}$ is small in $\mathcal{T}$.

Proof:

Assume that $\mathcal{T}$ is a supplement of $\mathcal{L}$ and $\mathcal{X} \in L(\mathcal{T})$, where $\mathcal{T} \cap \mathcal{L} + \mathcal{X} = \mathcal{T}$. Since $\mathcal{D} = \mathcal{L} + \mathcal{T}$ by definition supplement then $\mathcal{D} = \mathcal{L} + (\mathcal{T} \cap \mathcal{L} + \mathcal{X}) = \mathcal{L} + \mathcal{X}$ and $\mathcal{X} = \mathcal{T}$ by the minimality of $\mathcal{T}$.

On the other hand, let $\mathcal{D} = \mathcal{L} + \mathcal{T}$ and $\mathcal{T} \cap \mathcal{L}$ small in $\mathcal{T}$. To prove that $\mathcal{T}$ is a supplement of $\mathcal{L}$,

If $\mathcal{C} \subseteq \mathcal{T}$ and $\mathcal{D} = \mathcal{L} + \mathcal{C}$, then $\mathcal{T} = \mathcal{T} \cap \mathcal{D} = \mathcal{T} \cap (\mathcal{L} + \mathcal{C}) = \mathcal{T} \cap \mathcal{L} + \mathcal{T} \cap \mathcal{C} = \mathcal{T} \cap \mathcal{L} + \mathcal{C}$ and $\mathcal{C} = \mathcal{T}$ since $\mathcal{T} \cap \mathcal{L}$ small in $\mathcal{T}$, hence $\mathcal{T}$ is a supplement of $\mathcal{L}$.

Proposition 4.5: Let $\mathcal{D}$ be an R– semimodule, $\mathcal{Q}, \mathcal{F}$, and $\mathcal{A} \in L(\mathcal{D})$. Then

i)If $\mathcal{D} = \mathcal{Q} + \mathcal{F} = \mathcal{Q} + \mathcal{A}$ and $\mathcal{F}$ is a weak distributive subsemimodule, then $\mathcal{D} = \mathcal{Q} + (\mathcal{F} \cap \mathcal{A})$.

ii)If $\mathcal{Q}$ is a supplement of $\mathcal{F}$ and $\mathcal{A}$, then $\mathcal{Q}$ is a supplement of $\mathcal{F} \cap \mathcal{A}$.

iii) If $\mathcal{D} = \mathcal{A} + \mathcal{B}$ and $\mathcal{B}$ is a weak distributive subsemimodule, then every supplement of $\mathcal{A}$ is contained in$\mathcal{B}$.

Proof:

i) Since $\mathcal{D} = \mathcal{Q} + \mathcal{A}$, we have $\mathcal{F} = \mathcal{F} \cap \mathcal{D} = \mathcal{F} \cap (\mathcal{Q} + \mathcal{A}) = (\mathcal{F} \cap \mathcal{Q}) + (\mathcal{F} \cap \mathcal{A})$. but $\mathcal{D} = \mathcal{Q} + \mathcal{F}$ then $\mathcal{D} = \mathcal{Q} + (\mathcal{F} \cap \mathcal{Q}) + (\mathcal{F} \cap \mathcal{A}) = \mathcal{Q} + (\mathcal{F} \cap \mathcal{A})$.

ii)By hypothesis $\mathcal{D} = \mathcal{Q} + \mathcal{F} = \mathcal{Q} + \mathcal{A}$, by Lemma4.4 $\mathcal{Q} \cap \mathcal{F}$ and $\mathcal{Q} \cap \mathcal{A}$ are small in $\mathcal{Q}$, since $\mathcal{Q} \cap \mathcal{F} \cap \mathcal{A} \subseteq \mathcal{Q} \cap \mathcal{F}$, then $\mathcal{Q} \cap \mathcal{F} \cap \mathcal{A}$ is small in $\mathcal{Q}$. By Lemma4.4 $\mathcal{Q}$ is a supplement of $\mathcal{F} \cap \mathcal{A}$.

iii) Let $\mathcal{D} = \mathcal{A} + \mathcal{B}$, and $\mathcal{O}$ is a supplement of $\mathcal{A}$ in $\mathcal{D}$, then $\mathcal{D} = \mathcal{A} + \mathcal{O}$. Hence $\mathcal{B} = \mathcal{B} \cap \mathcal{A} + \mathcal{B} \cap \mathcal{O}$ and so

$\mathcal{D} = \mathcal{A} + \mathcal{B} \cap \mathcal{A} + \mathcal{B} \cap \mathcal{O} = \mathcal{A} + \mathcal{B} \cap \mathcal{O}$ this implies$\mathcal{B} \cap \mathcal{O} = \mathcal{O}$ and $\mathcal{O} \subseteq \mathcal{B}$.

Definition 4.6: [15] A semimodule $\mathcal{D}$ is called an amply supplement if wherever $\mathcal{F} + \mathcal{H} = \mathcal{D}$, then $\mathcal{F}$ has a supplement in $\mathcal{D}$ contained in $\mathcal{H}$.

Theorem 4.7: If $\mathcal{D}$ is a distributive supplement R– semimodule. Then $\mathcal{D}$ is an amply supplement, and every subsemimodule has a unique supplement.

Proof:

Let $\mathcal{D} = \mathcal{M} + \mathcal{S}$, so $\mathcal{M}$ has a supplement in $\mathcal{D}$, say $\mathcal{F}$. By Proposition4.5 (iii) the subsemimodule $\mathcal{F}$ is contained in $\mathcal{S}$. Then $\mathcal{D}$ is an amply supplement.

Assume that $\mathcal{X}$is also a supplement of $\mathcal{M}$ in $\mathcal{D}$, then $\mathcal{D} = \mathcal{M} + \mathcal{F} \cap \mathcal{X}$ by proposition4.5 (i), by the minimality $\mathcal{X} = \mathcal{F}$.

Definition 4.8: [7] Let $\mathcal{D}$ be a R– semimodule, and $\mathcal{L}, \mathcal{L}_* \in L(\mathcal{D})$. $\mathcal{D}$ is called a direct sum of $\mathcal{L}$and $\mathcal{L}_*$, if each $u \in \mathcal{D}$ can be represented uniquely as $u = k + h$, where $k \in \mathcal{L}$ and $h \in \mathcal{L}_*$, then we say that $\mathcal{L}$ (similarly $\mathcal{L}_*$) is a direct summand of $\mathcal{D}$, and denoted by $\mathcal{D} = \mathcal{L} \oplus \mathcal{L}_*$.

Lemma 4.9: [16] Let $\mathcal{D}$ be a cancellative, and semisubtractive R– semimodule. If $\mathcal{K}_1, \mathcal{K}_2 \in L(\mathcal{D})$ are subtractive, $\mathcal{K}_1 + \mathcal{K}_2 = \mathcal{D}$ and $\mathcal{K}_1 \cap \mathcal{K}_2 = 0$, then $\mathcal{K}_1 \oplus \mathcal{K}_2 = \mathcal{D}$.

Proposition 4.10: Let $\mathcal{D}$ be a cancellative, semisubtractive, and distributive R– semimodule, if $\mathcal{A}_1$ and $\mathcal{A}_2$ are direct summands of $\mathcal{D}$, then $\mathcal{A}_1 \cap \mathcal{A}_2$ and $\mathcal{A}_1 + \mathcal{A}_2$ are direct summands of $\mathcal{D}$.

Proof:

By hypothesis $\mathcal{D} = \mathcal{A}_1 \oplus \mathcal{H}_1 = \mathcal{A}_2 \oplus \mathcal{H}_2$ for some $\mathcal{H}_1, \mathcal{H}_2 \in L(\mathcal{D})$.Hence $\mathcal{A}_1 = \mathcal{A}_1 \cap \mathcal{D} = \mathcal{A}_1 \cap (\mathcal{A}_2 \oplus \mathcal{H}_2) = \mathcal{A}_1 \cap \mathcal{A}_2 + \mathcal{A}_1 \cap \mathcal{H}_2$, since $\mathcal{D} = \mathcal{A}_1 \oplus \mathcal{H}_1$, then $\mathcal{D} = (\mathcal{A}_1 \cap \mathcal{A}_2) + (\mathcal{A}_1 \cap \mathcal{H}_2 + \mathcal{H}_1)$, since $\mathcal{A}_1 \cap \mathcal{A}_2$ and $\mathcal{A}_1 \cap \mathcal{H}_2 + \mathcal{H}_1$ are subtractive (see Lemma 2 in [17]) and $(\mathcal{A}_1 \cap \mathcal{A}_2) \cap (\mathcal{A}_1 \cap \mathcal{H}_2 + \mathcal{H}_1) = 0$, hence $\mathcal{A}_1 \cap \mathcal{A}_2$ is direct summand by Lemma 3.10.

Now to prove that $\mathcal{A}_1 + \mathcal{A}_2$ is a direct summand of $\mathcal{D}$, at first $\mathcal{A}_1 + \mathcal{A}_2 = \mathcal{A}_1 \cap \mathcal{D} + \mathcal{A}_2 \cap \mathcal{D} = \mathcal{A}_1 \cap \mathcal{A}_2 + \mathcal{H}_2 + \mathcal{A}_2 \cap \mathcal{A}_1 + \mathcal{H}_1 = \mathcal{A}_1 \cap \mathcal{A}_2 + \mathcal{A}_1 \cap \mathcal{H}_2 + \mathcal{A}_2 \cap \mathcal{A}_1 + \mathcal{A}_2 \cap \mathcal{H}_1 = \mathcal{A}_1 \cap \mathcal{A}_2 + \mathcal{A}_2 \cap \mathcal{H}_1 + \mathcal{A}_1 \cap \mathcal{H}_2 = \mathcal{A}_2 \cap (\mathcal{A}_1 + \mathcal{H}_1) + \mathcal{A}_1 \cap \mathcal{H}_2 = \mathcal{A}_2 + \mathcal{A}_1 \cap \mathcal{H}_2$. Now we get $\mathcal{D} = \mathcal{A}_2 + \mathcal{H}_2 \cap \mathcal{A}_1 + \mathcal{H}_1 = \mathcal{A}_2 + \mathcal{H}_2 \cap \mathcal{A}_1 + \mathcal{H}_2 \cap \mathcal{H}_1 = (\mathcal{A}_1 + \mathcal{A}_2) + (\mathcal{H}_2 \cap \mathcal{H}_1)$, since $(\mathcal{A}_1 + \mathcal{A}_2), (\mathcal{H}_2 \cap \mathcal{H}_1)$are subtractive (see Lemma 2 in [17]), and $(\mathcal{A}_1 + \mathcal{A}_2) \cap (\mathcal{H}_2 \cap \mathcal{H}_1) = 0$, then $\mathcal{A}_1 + \mathcal{A}_2$ is direct summand of $\mathcal{D}$ by Lemma 3.10.

Proposition 4.11: If $\mathcal{D} = \mathcal{T}_1 \oplus \mathcal{T}_2$ is a semimodule, and $\mathcal{T}_3 \in L(\mathcal{D})$ is subtractive and weak distributive, then $\mathcal{D}/_{\mathcal{T}_3} = (\mathcal{T}_1 + \mathcal{T}_3)/_{\mathcal{T}_3} \oplus (\mathcal{T}_2 + \mathcal{T}_3)/_{\mathcal{T}_3}$.

Proof:

Assume that $\mathcal{D} = \mathcal{T}_1 \oplus \mathcal{T}_2$, then it's clear that $\mathcal{D}/_{\mathcal{T}_3} = (\mathcal{T}_1 + \mathcal{T}_3)/_{\mathcal{T}_3} + (\mathcal{T}_2 + \mathcal{T}_3)/_{\mathcal{T}_3}$. Now, we need prove the unique representation of the elements of $\mathcal{D}/_{\mathcal{T}_3}$. Since $\mathcal{T}_3$ is weak distributive, then $\mathcal{T}_3 = (\mathcal{T}_1 + \mathcal{T}_3) \cap (\mathcal{T}_2 + \mathcal{T}_3)$, hence $\forall k \in \mathcal{T}_3, k = k_1 + k_2$ where $k_1 \in \mathcal{T}_1 \cap \mathcal{T}_3, k_2 \in \mathcal{T}_2 \cap \mathcal{T}_3$. Assume that $u + \mathcal{T}_3 \in \mathcal{D}/_{\mathcal{T}_3}$, assume $u + \mathcal{T}_3 = (h + t) + \mathcal{T}_3 = (h' + t') + \mathcal{T}_3$ ... $(*)$ Where $h, h' \in \mathcal{T}_1 + \mathcal{T}_3$ and $t, t' \in \mathcal{T}_2 + \mathcal{T}_3$ it can be assumed that $h, h' \in \mathcal{T}_1$ and $t, t' \in \mathcal{T}_2$, then $(h + t) + k = (h' + t') + k'$ for some $k, k' \in \mathcal{T}_3$ by $(*)$ $k = k_1 + k_2$ and $k' = k_1' + k_2'$ such that $k_1, k_1' \in \mathcal{T}_1 \cap \mathcal{T}_3$ and $k_2, k_2' \in \mathcal{T}_2 \cap \mathcal{T}_3$, then $h + k_1 = h' + k_1', t + k_2 = t' + k_2'$ by unique representation of the elements of $\mathcal{T}_1 \oplus \mathcal{T}_2$, it follows $h + k_1 = h' + k_1', t + k_2 = t' + k_2'$ where $k_1, k_1', k_2, k_2' \in \mathcal{T}_3$, then $h + \mathcal{T}_3 = h' + \mathcal{T}_3$ and $t + \mathcal{T}_3 = t' + \mathcal{T}_3$, therefore $\mathcal{D}/_{\mathcal{T}_3} = (\mathcal{T}_1 + \mathcal{T}_3)/_{\mathcal{T}_3} \oplus (\mathcal{T}_2 + \mathcal{T}_3)/_{\mathcal{T}_3}$.

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