# On The Translation Bifuzzy $\psi$ -ideal of $\psi$ –algebra

*Areej Tawfeeq Hameed*
*Department of Mathematics*
*Faculty of Education for Girls,*
*University of Kufa*
*Najaf, Iraq*

areej.tawfeeq@uokufa.edu.iq

Orcid.org/0000-0002-8749-1310

*Naba Hasoon Jabir AL-Saedi*
*Department of Mathematics*
*Faculty of Education for Girls,*
*University of Kufa*
*Najaf, Iraq*

nabaah.al-saedi@uokufa.edu.iq

Orcid.org/0000-0003-4992-1313

**DOI:** http://dx.doi.org/10.31642/JoKMC/2018/100223

**Received Apr. 19, 2023. Accepted for publication Jun.20, 2023**

***Abstract.*** *we introduce the notion of translation bifuzzy , bifuzzy extensions , translation bifuzzy of bifuzzy $\psi$ -subalgebra , bifuzzy $\psi$ -ideals on $\psi$ -algebra and investigate some of their properties, we discussion the R-intersection , the P-intresection, the R-union , the P-union of any sets of translation bifuzzy $\psi$-subalgebra , we will present some results on images and pre-images of translation bifuzzy $\psi$- subalgebras and translation $\psi$-ideals of $\psi$-algebra and we introduce the notions of Cartesian product of bifuzzy $\psi$ -subalgebras and bifuzzy ideals in a $\psi$ -algebra.*

***Keywords: $\psi$ -algebra , bifuzzy $\psi$ -subalgebra, bifuzzy $\psi$ -ideal, bifuzzy translation, bifuzzy extension, homomorphism of $\psi$ -algebra.***

**2010 Mathematics Subject Classification**: 06F35, 03G25, 08A72.

## 1. Introduction

Several authors have introduced of BCK-algebras as a generalization of the concept of set-theoretic difference and propositional calculus and studied some important properties. The concept of a fuzzy set, was introduced by L.A. Zadeh. A.T. Hameed and others, introduced KUS-ideals in KUS-algebras and introduced the notions fuzzy KUS-subalgebras, fuzzy KUS-ideals of KUS-algebras and investigated relations among them. J. Meng and Y. B. Jun applied the concept of fuzzy set to BCK/BCI-algebras and gave some of its properties. A.T. Hameed and others, discussed fuzzy α-translation, fuzzy extension of fuzzy (KUS/CI/QS)-subalgebras on (KUS/CI/QS)-algebra. They discussed fuzzy α-translation and fuzzy extension of fuzzy (KUS/CI/QS)-ideals in (KUS/CI/QS)-algebra. A.T.Hameed and N.H.Jaber, introduced $\psi$-subalgebras and $\psi$ -ideals on $\psi$-algebras and introduced the notions fuzzy $\psi$ -subalgebras, fuzzy $\psi$ -ideals of $\psi$-algebras and introduced the notions bifuzzy $\psi$ -subalgebras, bifuzzy $\psi$-ideals of $\psi$-algebras and investigated relations among them.

In this paper, we define a translation bifuzzy $\psi$ -subalgebras and a translation bifuzzy $\psi$ -ideals of $\psi$ -algebras and look for some of their properties accurately by using the concepts of bifuzzy $\psi$ -subalgebra and bifuzzy $\psi$ -ideal and look for some of their properties accurately by using the concepts of fuzzy $\psi$ -subalgebra and fuzzy $\psi$ -ideal.

## 2. Preliminaries:

We review some definitions and properties that will be useful in our results.

**Definition 2.1([3]).** Let $(X;+,-,0)$ be an algebra with two operations $(+)$ and $(-)$ and constant $(0)$. $X$ is called an ***ψ*-algebra** if it satisfies the following properties: for all $x,y,z \in X$,

$(\psi_1)$ $x - x = 0$,

$(\psi_2)$ $(0 - x) + x = 0$,

$(\psi_3)$ $(x - y) - z = x - (z + y)$,

$(\psi_4)$ $(y + x) - (x - z) = y + z$.

In $X$, we can define a binary relation ( $\leq$ ) by : $x \leq y$ if and only if $x + y = 0$ and $x - y = 0$ , $x, y \in X$.

**Lemma 2.2([3]).**

Let $(X;+,-,0)$ be a $\psi$-algebra. Then for any $x, y \in X$,

$$(L_1)\ x + y = x - (-y).$$

$$(L_2)\ x - y = x + (-y),$$

$$(L_3)\ x - y = -y + x.$$

**Proposition 2.3([3]).** Let $(X;+,-,0)$ be a $\psi$-algebra, then the following holds: for any $x, y, z \in X$,

$$(a_1)(x - y) - z = (x - z) - y,$$

$$(a_2)0 - (x - y) = (y - x),$$

$$(a_3)x - y \leq z \ \ imply\ \ x - z \leq y,$$

$$(a_4)x \leq y\ \ imply\ \ z + y \leq z + x,$$

$$(a_5)\ (x - y) - (z - y) \leq x - z \text{ and } (x - y) - (x - z) \leq z - y,$$

$$(a_6)\ x \leq y\ and\ \ y \leq z\ \ imply\ \ x \leq z.$$

**Definition 2.4([3]).** Let $(X;+,-,0)$ be a $\psi$-algebra and let $S$ be a nonempty set of $X$. $S$ is called **a *ψ*-subalgebra of *X*** if

$x + y \in S$ and $x - y \in S$, whenever $x, y \in S$.

**Definition 2.5 ([3]).** A nonempty subset $I$ of a $\psi$-algebra $(X;+,-,0)$ is called **a *ψ*-ideal of *X*** if it satisfies: for $x, y, z \in X$, (1) $0 \in I$,

(2) $(y + z) \in I$ and $(x - z) \in I$ imply $(y + x) \in I$.

**Proposition 2.6([3]).** Every $\psi$-ideal of $\psi$-algebra is a $\psi$-subalgebra of $X$ and the converse is not true.

**Lemma 2.7([3]).** An $\psi$-ideal $I$ of $\psi$-algebra $(X;+,-,0)$ has the following property:

1) For any $x \in X$, for all $y \in I, x \leq y \Rightarrow x \in I$.

2) If for any $x \in I \Rightarrow -x \in I$.

**Definition 2.8([7]).** Let $X$ be a nonempty set, a **fuzzy subset *μ* of *X*** is a mapping $\mu: X \to [0,1]$.

**Definition 2.9([7]).** Let $X$ be a nonempty set and $\mu$ be a fuzzy subset of $X$, for $t \in [0,1]$ , the set $\mu_t = \{x \in X | \mu(x) \geq t\}$ is called **a level subset of μ** .

**Definition 2.10 ([6]).**

1- A fuzzy subset $\mu$ of a set $X$ **has sup property** if for any subset $T$ of $X$ , there exist $t_0 \in T$ such that $\mu(t_0) = sup\,\{\mu(t) | \, t \in T\}$.

2- A fuzzy subset $\mu$ of a set $X$ **has inf property** if for any subset $T$ of $X$ , there exist $t_0 \in T$ such that $\mu(t_0) = inf\,\{\mu(t) | \, t \in T\}$.

**Definition 2.11 ([4] )**

Let $(X;+,-,0)$ be a $\psi$-algebra, fuzzy subset $\mu$ of $X$ is called a **fuzzy *ψ*-subalgebra of *X*** if it satisfies: for all $x, y \in X$ ,

$$(FS_1)\quad \mu(x + y) \geq min\{\mu(x), \mu(y)\},$$

$$(FS_2)\quad \mu(x - y) \geq min\{\mu(x), \mu(y)\}.$$

**Definition 2.12 ([4] )**

Let $(X;+,-,0)$ be a $\psi$-algebra, fuzzy subset $\mu$ of $X$ is called a **fuzzy *ψ*-ideal of *X*** if it satisfies: for all $x, y, z \in X$ ,

$$(FI_1)\quad \mu(0) \geq \mu(x),$$

$$(FI_2)\quad \mu(y + x) \geq min\{\mu(y + z), \mu(x - z)\}.$$

**Definition 2.13([2]).** A bifuzzy subset A in a nonempty set X is an object having the form $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$ where the functions $\mu_A: X \to [0,1]$ and $\nu_A: X \to [0,1]$ denote the fuzzy function and the degree of doubl fuzzy function respectively and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ , for all $x \in X$ .

**Remark 2.14([2]).** If a bifuzzy subset A in a nonempty set X , then $\mu_A(x) + \nu_A(x) = 1$ , i.e., when $\nu_A(x) = 1 - \mu_A(x) = \mu_A^c(x)$ for all $x \in X$ . Then $\mu_A$ is called a fuzzy subset and $\nu_A = \mu_A^c$ is the complement of $\mu_A$.

**Definition 2.15([5]).** Let $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$ be a bifuzzy subset of a $\psi$-algebra X. A is said to be an bifuzzy $\psi$-subalgebra of X if : for all x, y ∈X,

(IFS$_1$) $\mu_A(x+y) \geq \min\{\mu_A(x), \mu_A(y)\}, and$

$\mu_A(x-y) \geq min\{\mu_A(x), \mu_A(y)\}$.

(IFS$_2$) $\nu_A(x+y) \leq \max\{\nu_A(x), \nu_A(y)\}$ *and*

$\nu_A(x-y) \leq max\{\nu_A(x), \nu_A(y)\}$.

**Definition 2.16([5]).** Let $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$ be a bifuzzy subset of a $\psi$-algebra $(X; +, -, 0)$. A is said to be an bifuzzy $\psi$-ideal of X if : for all x, y, z ∈X,

$(IF\psi_1)$ $\mu_A(0) \geq \mu_A(x)$ and $\nu_A(0) \leq \nu_A(x)$,

$(IF\psi_2)$ $\mu_A(y+x) \geq min\{\mu_A(y+z), \mu_A(x-z)\}$ and

$\nu_A(y+x) \leq \max\{\nu_A(y+z), \nu_A(x-z)\}$.

**Proposition 2.17 ([3,5])**

1- Every $\psi -$ ideal of $\psi -$ algebra X is an $\psi$ -subalgebra of X **.**

2- Every fuzzy $\psi$ -ideal of $\psi$ -algebra X is a fuzzy $\psi$ -subalgebra of X **.**

3- Every bifuzzy $\psi$ -ideal of $\psi$ -algebra X is a bifuzzy $\psi$ -subalgebra of X **.**

**Definition 2.18 [4]:**

Let $f : (X; +, -, 0) \longrightarrow (Y; +', -', 0')$ be a mapping from an $\psi$ -algebra X into an $\psi$-algebra Y. If μ is a fuzzy subset of X, then

$$f(\mu)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \mu(x), & f^{-1}(y) = \{x \in X \mid f(x) = y\} \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

is said to be **the image of μ under *f*** and is denoted by $f(\mu)$.

**Definition 2.19 [4]:**

Let $f : (X; +, -, 0) \longrightarrow (Y; +', -', 0')$ be a mapping from an $\psi$ -algebra X into an $\psi$ -algebra Y. If β is a fuzzy subset of $\psi$ -algebra Y, then the fuzzy subset $\mu = \beta \circ f$ of X ( i. e, the fuzzy subset defined by $\mu(x) = \beta(f(x))$, for all x ∈ X) is called **the pre-image of β under *f* .**

**Proposition 2.20 [3]:**

Let $f : (X; +, -, 0) \longrightarrow (Y; +', -', 0')$ be a homomorphism from X into Y , then

1- If A is a $\psi$ -subalgebra of X, then the image $f$(A) is a $\psi$ -subalgebra of Y, where $f$ is onto.
2- If A is a $\psi$ -ideal of X, then the image $f$(A) is a $\psi$ -ideal of Y , where $f$ is onto.
3- If A is a $\psi$ -subalgebra of Y, then the inverse image $f^{-1}$(A) is a $\psi$ -subalgebra of X.
4- If A is a $\psi$ -ideal of Y, then the inverse image $f^{-1}$(A) is a $\psi$ -ideal of X.

**Proposition 2.21 [4]:**

Let $f : (X; +, -, 0) \longrightarrow (Y; +', -', 0')$ be a homomorphism from X into Y with sup-inf property, then

1- If $\mu$ is a fuzzy $\psi$ -subalgebra of X, then the image $f(\mu)$ is a fuzzy $\psi$ -subalgebra of Y, where $f$ is onto.
2- If $\mu$ is a fuzzy $\psi$ -ideal of X, then the image $f(\mu)$ is a fuzzy $\psi$ -ideal of Y , where $f$ is onto.
3- If $\mu$ is a fuzzy $\psi$ -subalgebra of Y, then the pre-image $f^{-1}(\mu)$ is a fuzzy $\psi$ -subalgebra of X.
4- If $\mu$ is a fuzzy $\psi$ -ideal of Y, then the pre-image $f^{-1}(\mu)$ is a fuzzy $\psi$ -ideal of X.

**Definition 2.22. [5]:**

Let $(X; +, -, 0)$ be an $\psi$-algebra, a fuzzy subset $\nu$ of $X$ is called **an anti-fuzzy *ψ*-subalgebra of *X*** if for all $x, y \in X$,

$\nu(x+y) \leq max\{\nu(x), \nu(y)\}$ and $\nu(x-y) \leq max\{\nu(x), \nu(y)\}$

**Definition 2.23. [5]:**

Let $(X; +, -, 0)$ be an $\psi$-algebra, a fuzzy subset $\nu$ of $X$ is called **an anti-fuzzy *ψ*-ideal of *X*** if it satisfies the following conditions, for all $x, y, z \in X$,

(DFI1) $\nu(0) \leq \nu(x)$ ,

(DFI2) $\nu(y+x) \leq max\{\nu(y+z), \nu(x-z)\}$ .

**Proposition 2.24 [5]:**

Let $f : (X; +, -, 0) \longrightarrow (Y; +', -', 0')$ be a homomorphism from X into Y with inf property, then

1- If $\nu$ is anti-fuzzy $\psi$ -subalgebra of X, then the image $f(\nu)$ is anti-fuzzy $\psi$ -subalgebra of Y, where $f$ is onto.
2- If $\nu$ is anti-fuzzy $\psi$ -ideal of X, then the image $f(\nu)$ is anti-fuzzy $\psi$ -ideal of Y , where $f$ is onto.

3- If $\nu$ is anti-fuzzy $\psi$ -subalgebra of Y, then the inverse image $f^{-1}(\nu)$ is anti-fuzzy $\psi$ -subalgebra of X.
4- If $\nu$ is anti-fuzzy $\psi$ -ideal of Y, then the inverse image $f^{-1}(\nu)$ is anti-fuzzy $\psi$ -ideal of X.

**Proposition 2.25 [5]:**

Let $f$ :( X;+, −,0)⟶ ( Y; +′, −′, 0′) be a homomorphism from X into Y with sup-inf property, then

1- If $A$ is a bifuzzy $\psi$ -subalgebra of X, then the image $f(A)$ is a bifuzzy $\psi$ -subalgebra of Y, where $f$ is onto.
2- If $A$ is a bifuzzy $\psi$ -ideal of X, then the image $f(A)$ is a bifuzzy $\psi$ -ideal of Y , where $f$ is onto.
3- If $A$ is a bifuzzy $\psi$ -subalgebra of Y, then the inverse image $f^{-1}(A)$ is a bifuzzy $\psi$ -subalgebra of X.
4- If $A$ is a bifuzzy $\psi$ -ideal of Y, then the inverse image $f^{-1}(A)$ is a bifuzzy $\psi$ -ideal of X.

**Definition 2.26[1,2]:**

Let X be a nonempty set and μ be a fuzzy subset of X and let α∈ [0,T]. A mapping $\mu_\alpha^T$ : X → [0,1] is called a **translation fuzzy subset** of μ if it satisfies:

$\mu_\alpha^T(x) = \mu(x) + \alpha$, for all x ∈ X, where$T = 1 - \sup\{\mu(x): x \in X\}$.

## 3. The Translations Bifuzzy $\psi$ - subalgebras of $\psi$ -algebra

In this section, we discuss translation on $\psi$-algebras and we get some of relations, theorems, propositions and give examples of α-translation of bifuzzy $\psi$-subalgebra. We show the notion of translation bifuzzy $\psi$-subalgebras of $\psi$-algebra and investigate some of their properties.

In follows, let $(X;+,-,0)$ denote an $\psi$-algebra, and for any fuzzy subset μ of X, we denote $T = 1 - sup\{\mu(x) \mid x \in X\}$ and $K = inf\{\nu(x) \mid x \in X\}$.

**Definition 3.1:**

Let $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$ be an bifuzzy subset of a $\psi$-algebra $(X;+,-,0)$, $\mu_A$ be a fuzzy subset of $X$ such that α ∈[0,T] and $\nu_A$ be a fuzzy subset of $X$ such that $\varepsilon$ ∈[0,K]. A mapping $(\mu_A)_\alpha^T$ :X→[0,1] and $(\nu_A)_\varepsilon^K$ :X→[0,1],

$A_{(\alpha,\varepsilon)}^{(T,K)} = \{(x, (\mu_A)_\alpha^T(x), (\nu_A)_\varepsilon^K(x)) \mid x \in X\}$ is called a **translation of A** if it satisfies: $\mu_\alpha^T(x) = \mu_A(x) + \alpha$ and $(\nu_A)_\varepsilon^K(x) = \nu_A(x) - \varepsilon$, for all x ∈X.

**Definition 3.2:**

Let $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$ be an bifuzzy subset of a $\psi$-algebra and α ∈ [0,T] , $\varepsilon$ ∈ [0,K], of $(X;+,-,0)$, then $A_{(\alpha,\varepsilon)}^{(T,K)} = \{(x, (\mu_A)_\alpha^T(x), (\nu_A)_\varepsilon^K(x)) \mid x \in X\}$ is called a **translation bifuzzy $\psi$ -subalgebra of X** , if for all $x$ , $y$ ∈X,

1- $(\mu_A)_\alpha^T(x+y) \geq \min\{(\mu_A)_\alpha^T(x), (\mu_A)_\alpha^T(y)\}$,
2- $(\mu_A)_\alpha^T(x-y) \geq \min\{(\mu_A)_\alpha^T(x), (\mu_A)_\alpha^T(y)\}$ and
3- $(\nu_A)_\varepsilon^K(x+y) \leq \max\{(\nu_A)_\varepsilon^K(x), (\nu_A)_\varepsilon^K(y)\}$,
4- $(\nu_A)_\varepsilon^K(x-y) \leq \max\{(\nu_A)_\varepsilon^K(x), (\nu_A)_\varepsilon^K(y)\}$.

i.e.,

1- $\mu_A(x+y) + \alpha \geq \min\{\mu_A(x) + \alpha, \mu_A(y) + \alpha\} = \min\{\mu_A(x), \mu_A(y)\} + \alpha$,
2- $\mu_A(x-y) + \alpha \geq \min\{\mu_A(x) + \alpha, \mu_A(y) + \alpha\} = \min\{\mu_A(x), \mu_A(y)\} + \alpha$,
3- $\nu_A(x+y) - \varepsilon \leq \max\{\nu_A(x) - \varepsilon, \nu_A(y) - \varepsilon\} = \max\{\nu_A(x), \nu_A(y)\} - \varepsilon$,
4- $\nu_A(x-y) - \varepsilon \leq \max\{\nu_A(x) - \varepsilon, \nu_A(y) - \varepsilon\} = \max\{\nu_A(x), \nu_A(y)\} - \varepsilon$,

**Example 3.3 :**

Let X={0,1,2, 3} in which (+, −) be a defined by the following table:

| + | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 2 | 3 | 0 |
| 2 | 2 | 3 | 0 | 1 |
| 3 | 3 | 0 | 1 | 2 |

| - | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 |
| 2 | 2 | 0 | 0 | 0 |
| 3 | 3 | 3 | 3 | 0 |

Then $(X;+,-,0)$ is an $\psi$ -algebra. It is easy to show that $S_1$ ={0,1}, $S_2$ ={0,2} and $S_3$={0,3}are $\psi$ -subalgebras of X. Define a fuzzy subset

$\mu_A$: X → [0,1] such that $\mu_A$ (0) = 0.7 , $\mu_A$ (1) = $\mu_A$ (2) =0.6, $\mu_A$ (3) = 0.4, $\alpha = 0.15 \in [0,0.3]$.
$\nu_A$: X → [0,1] such that $\nu_A$ (0) = 0.3 , $\nu_A$ (1) = $\nu_A$ (2) =0.4, $\nu_A$ (3) = 0.6, $\varepsilon = 0.25 \in [0,0.3]$.
Routine calculation gives that $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$ is a bifuzzy $\psi$ -subalgebra of X.

Also, gives that $A_{(\alpha,\varepsilon)}^{(T,K)} = \{(x, (\mu_A)_\alpha^T(x), (\nu_A)_\varepsilon^K(x)) \mid x \in X\}$ is a translation bifuzzy $\psi$ − subalgebra of X.

**Theorem 3.4:**

Let $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$ be an bifuzzy $\psi$ -subalgebra of a $\psi$-algebra $(X;+,-,0)$ and α ∈ [0,T], $\varepsilon$ ∈ [0,K], then $\mu_A$ is a fuzzy $\psi$-subalgebra of X and $\nu_A$is an anti-fuzzy $\psi$-subalgebra of X .

**Proof:**

Assume that A is a translation bifuzzy $\psi$-subalgebra of X, and $\alpha \in [0,T]$, $\varepsilon \in [0,K]$.

Let $x, y \in X$, then

$(\mu_A)^T_\alpha (x+y) \geq \min\{(\mu_A)^T_\alpha (x), (\mu_A)^T_\alpha(y)\}$, that mean $\mu_A (x+y) + \alpha \geq \min\{\mu_A (x) + \alpha, \mu_A(y) + \alpha\} = \min\{\mu_A (x), \mu_A(y)\} + \alpha$, implies that $\mu_A (x+y) \geq \min\{\mu_A (x), \mu_A(y)\}$.

Summarily, $\mu_A (x-y) \geq \min\{\mu_A (x), \mu_A(y)\}$.

$(\nu_A)^K_\varepsilon (x+y) \leq \max\{(\nu_A)^K_\varepsilon (x), (\nu_A)^K_\varepsilon(y)\}$, that mean $\nu_A (x+y) - \varepsilon \leq \max\{\nu_A (x) - \varepsilon, \nu_A(y) - \varepsilon\} = \max\{\nu_A (x), \nu_A(y)\} - \varepsilon$, implies that $\nu_A (x+y) \leq \max\{\nu_A (x), \nu_A(y)\}$.

Summarily, $\nu_A (x-y) \leq \max\{\nu_A (x), \nu_A(y)\}$.

Hence $\mu_A$ is a fuzzy $\psi$-subalgebra of X and $\nu_A$ is an anti-fuzzy $\psi$-subalgebra of X.□

**Proposition 3.5:**

Let $A = \{(x, \mu_A(x), \nu_A (x)) \mid x \in X\}$ be an bifuzzy subset of a $\psi$-algebra $(X; +, -, 0)$ and $\alpha \in [0,T]$, $\varepsilon \in [0,K]$ such that $\mu_A$ be a fuzzy $\psi$-subalgebra of X and $\nu_A$ be an anti-fuzzy $\psi$-subalgebra of X, then $(\mu_A)^T_\alpha$ is a translation fuzzy $\psi$-subalgebra of X and $(\nu_A)^K_\varepsilon$ is a translation anti-fuzzy $\psi$-subalgebra of X.

**Proof:**

Assume that $\mu_A$ is a fuzzy $\psi$-subalgebra of X and $\nu_A$ is an anti-fuzzy $\psi$-subalgebra of X, then for any $x, y \in X$ and $\alpha \in [0,T]$, $\varepsilon \in [0,K]$

$\mu_A (x+y) \geq \min\{\mu_A (x), \mu_A(y)\}$ and $\mu_A (x-y) \geq \min\{\mu_A (x), \mu_A(y)\}$ implies that

$\mu_A (x+y) + \alpha \geq \min\{\mu_A (x), \mu_A(y)\} + \alpha = \min\{\mu_A (x) + \alpha, \mu_A(y) + \alpha\}$ and

$\mu_A (x-y) + \alpha \geq \min\{\mu_A (x), \mu_A(y)\} + \alpha = \min\{\mu_A (x) + \alpha, \mu_A(y) + \alpha\}$. That mean

$(\mu_A)^T_\alpha (x+y) \geq \min\{(\mu_A)^T_\alpha (x), (\mu_A)^T_\alpha(y)\}$, and

$(\mu_A)^T_\alpha (x-y) \geq \min\{(\mu_A)^T_\alpha (x), (\mu_A)^T_\alpha(y)\}$.

$\nu_A (x+y) \leq \max\{\nu_A (x), \nu_A(y)\}$ and $\nu_A (x-y) \leq \max\{\nu_A (x), \nu_A(y)\}$ implies that

$\nu_A (x+y) - \varepsilon \leq \max\{\nu_A (x), \nu_A(y)\} - \varepsilon = \max\{\nu_A (x) - \varepsilon, \nu_A(y) - \varepsilon\}$,

$\nu_A (x-y) - \varepsilon \leq \max\{\nu_A (x), \nu_A(y)\} - \varepsilon = \max\{\nu_A (x) - \varepsilon, \nu_A(y) - \varepsilon\}$,

that mean $(\nu_A)^K_\varepsilon (x+y) \leq \max\{(\nu_A)^K_\varepsilon (x), (\nu_A)^K_\varepsilon(y)\}$and

$(\nu_A)^K_\varepsilon (x-y) \leq \max\{(\nu_A)^K_\varepsilon (x), (\nu_A)^K_\varepsilon(y)\}$,

Hence $\mu^T_\alpha$ is a fuzzy $\psi$-subalgebra of X and $\nu^K_\varepsilon$ is an anti-fuzzy $\psi$-subalgebra of X. □

**Collaroy 3.6:**

Let $A = \{(x, \mu_A(x), \nu_A (x)) \mid x \in X\}$ be an bifuzzy subset of a $\psi$-algebra $(X; +, -, 0)$ and $\alpha \in [0,T]$, $\varepsilon \in [0,K]$ such that $\mu_A$ be a fuzzy $\psi$-subalgebra of X and $\nu_A$ be an anti-fuzzy $\psi$-subalgebra of X, then

$A^{(T,K)}_{(\alpha,\varepsilon)} = \{(x, (\mu_A)^T_\alpha(x), (\nu_A)^K_\varepsilon (x)) \mid x \in X\}$ is a translation bifuzzy $\psi$ -subalgebra of $X$.

**Proof:**

By Theorem (3.5) and Definition (3.2). □

**Definition 3.7.**

For a fuzzy subset $\mu$ of a $\psi$ -algebra X, $\alpha \in [0,T]$ and $t \in Im(\mu)$ with $t \geq \alpha$, let $U_\alpha (\mu;\ t) = \{x \in X \mid \mu(x) \geq t - \alpha\}$ and a fuzzy subset $\nu$ of a $\psi$ -algebra X, $\varepsilon \in [0,K]$ and $s \in Im(\mu)$ with $s \leq \varepsilon$, $L_\varepsilon(\nu;\ s) = \{x \in X \mid \nu(x) \leq s - \varepsilon\}$.

**Remark 3.8.**

1- If $(\mu_A)^T_\alpha$ is a translation fuzzy $\psi$ -subalgebra of X, then it is that $U_\alpha (\mu_A; t)$ is a $\psi$ -subalgebra of X, for all $t \in Im(\mu_A)$ with $t \geq \alpha$. Let $x, y \in U_\alpha (\mu_A; t)$, then $\mu_A (x) \geq t - \alpha$, and $\mu_A (y) \geq t - \alpha$, then $\min\{\mu_A (x), \mu_A (y)\} \geq t - \alpha$, since $(\mu_A)^T_\alpha$ is a translation fuzzy $\psi$ -subalgebra, then $\mu_A (x+y) \geq min\{\mu_A(x), \mu_A(y)\} \geq t - \alpha$, therefore $x+y \in U_\alpha (\mu_A; t)$ and

$\mu_A (x-y) \geq min\{\mu_A(x), \mu_A(y)\} \geq t - \alpha$, therefore $x-y \in U_\alpha (\mu_A; t)$.

2- If $(\nu_A)^K_\varepsilon$ is a translation anti-fuzzy $\psi$ -subalgebra of X, then it is that $L_\varepsilon(\nu_A;\ s)$is a $\psi$ -subalgebra of X, for all $s \in Im(\nu_A)$ with $s \leq \varepsilon$. Let $x, y \in L_\varepsilon(\nu;\ s)$, then $\nu_A (x) \leq s - \varepsilon$, and $\nu_A (y) \leq s - \varepsilon$, then $\max\{\nu_A (x), \nu_A (y)\} \leq s - \varepsilon$, since $\nu_A$ is anti-fuzzy $\psi$ -subalgebra, then $\nu_A (x+y) \leq max\{\nu_A(x), \nu_A(y)\} \leq s - \varepsilon$ ,therefore $x+y \in L_\varepsilon(\nu_A;\ s)$ and $\nu_A (x-y) \leq max\{\nu_A(x), \nu_A(y)\} \leq s - \varepsilon$ , therefore $x-y \in L_\varepsilon(\nu_A;\ s)$.

3- But if we do not give a condition that $(\mu_A)^T_\alpha$ is a translation fuzzy $\psi$ -subalgebra of X, then $U_\alpha (\mu_A; t)$ is not a $\psi$ -subalgebra of X or $(\nu_A)^K_\varepsilon$ is anti-fuzzy $\psi$ -subalgebra of X, then $L_\varepsilon(\nu_A;\ s)$is not a $\psi$ -subalgebra of X as seen in the following example.

**Example 3.9:**
Consider X = {0, 1, 2, 3 } is a $\psi$ -algebra which is given in Example (3.3) . Define a fuzzy subset $\mu_A$ of X:

| X | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $\mu_A$ | 0.7 | 0.6 | 0.4 | 0.3 |

Then $(\mu_A)^T_\alpha$ is not a translation fuzzy $\psi$ -subalgebra of X.

Since $\mu_A$ (1+2) = 0.3 < 0.4 = min{ $\mu_A$ (1), $\mu_A$ (2)}. For α = 0.1 and t = 0.5, we obtain $U_\alpha$ ($\mu_A$; t) = {0, 1, 2} which is not an $\psi$ -subalgebra of X since 1+2 = 3 $\notin U_\alpha(\mu_A;$ t).

**Proposition 3.10.**

Let $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$ be an bifuzzy subset of a $\psi$-algebra $(X; +, -, 0)$ and $\alpha \in$ [0,T], $\varepsilon \in$ [0,K] such that $A^{(T,K)}_{(\alpha,\varepsilon)} = \{(x, (\mu_A)^T_\alpha(x), (\nu_A)^K_\varepsilon(x)) \mid x \in X\}$ is a translation bifuzzy $\psi$-subalgebra of X, then $U_\alpha(\mu_A;\ t)$ and $L_\varepsilon(\nu_A;\ s)$ are fuzzy $\psi$ -subalgebras of $X$, for any t∈ Im($\mu_A$), s∈ Im($\nu_A$) with t ≥ α and s ≤ $\varepsilon$.

**Proof:**

Assume that $A$ is a bifuzzy $\psi$-subalgebra, then by Theorem (3.4) $\mu_A$ is a fuzzy $\psi$-subalgebra of X and $\nu_A$ is an anti-fuzzy $\psi$-subalgebra of X, then by Remark (3.8), $U_\alpha(\mu_A;\ t)$and $L_\varepsilon(\nu_A;\ s)$ are fuzzy $\psi$ -subalgebras of $X$, for any t∈ Im($\mu_A$), s∈ Im($\nu$) with t ≥ α and s ≤ $\varepsilon$. □

**Theorem 3.11.**

Let $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$ be a bifuzzy subset of a $\psi$-algebra $(X; +, -, 0)$ and $\alpha \in$ [0,T], $\varepsilon \in$ [0,K] such that $U_\alpha(\mu_A;\ t)$ and $L_\varepsilon(\nu_A;\ s)$ are fuzzy $\psi$ -subalgebras of $X$, for all t∈ Im($\mu_A$), s∈ Im($\nu$) with t ≥ α and s ≤ $\varepsilon$, then $A^{(T,K)}_{(\alpha,\varepsilon)} = \{(x, (\mu_A)^T_\alpha(x), (\nu_A)^K_\varepsilon(x)) \mid x \in X\}$ is a translation bifuzzy $\psi$-subalgebra of X.

**Proof:**

Assume that $x, y \in U_\alpha(\mu_A;\ t)$ and $(\mu_A)^T_\alpha$ of μ is not a fuzzy $\psi$-subalgebra of $X$, therefore $(\mu_A)^T_\alpha(x+y) < t \le min\{(\mu_A)^T_\alpha(x), (\mu_A)^T_\alpha(y)\}$, then $(\mu_A)(x) \ge t - \alpha$ and $(\mu_A)(y) \ge t - \alpha$ , but $(\mu_A)(x+y) < t - \alpha$ . This shows that $x + y \notin U_\alpha(\mu_A;\ t)$. This is a contradiction, and so $(\mu_A)^T_\alpha(x+y) \ge min\{(\mu_A)^T_\alpha(x), (\mu_A)^T_\alpha(y)\}$, for all x, y ∈ $X$.

Summarily, $(\mu_A)^T_\alpha(x-y) \ge \min\{(\mu_A)^T_\alpha(x), (\mu_A)^T_\alpha(y)\}$.

Hence $(\mu_A)^T_\alpha$ is a translation fuzzy $\psi$-subalgebra of $X$.

$(\nu_A)^K_\varepsilon(x+y) > s \ge max\{(\nu_A)^K_\varepsilon(x), (\nu_A)^K_\varepsilon(y)\}$, then $(\nu_A)(x) \le s - \varepsilon$ and $(\nu_A)(y) \le s - \varepsilon$, but $(\nu_A)(x+y) > s - \varepsilon$ . This shows that $x + y \notin L_\varepsilon(\nu_A; s)$. This is a contradiction, and so $(\nu_A)^K_\varepsilon(x+y) \le max\{(\nu_A)^K_\varepsilon(x), (\nu_A)^K_\varepsilon(y)\}$, for all x, y ∈ $X$.

Summarily, $(\nu_A)^K_\varepsilon(x-y) \le \max\{(\nu_A)^K_\varepsilon(x), (\nu_A)^K_\varepsilon(y)\}$.

Therefore, $(\nu_A)^K_\varepsilon$ is a translation anti-fuzzy $\psi$-subalgebra of $X$.

Hence $A^{(T,K)}_{(\alpha,\varepsilon)}$ is a translation bifuzzy $\psi$-subalgebra of X. □

**Definition 3.12.**

Let $(X; +, -, 0)$ be a $\psi$-algebra, $\mu_1$ and $\mu_2$ be fuzzy subsets of $X$, then $\mu_2$ is called **a fuzzy extension** of $\mu_1$. If $\mu_2(x) \ge \mu_1(x)$, for all $x \in X$ .

**Proposition 3.13.**

Let $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$ be an bifuzzy subset of a $\psi$-algebra $(X; +, -, 0)$, then the translation bifuzzy subset $A^{(T,K)}_{(\alpha,\varepsilon)} = \{(x, (\mu_A)^T_\alpha(x), (\nu_A)^K_\varepsilon(x)) \mid x \in X\}$ of X is a fuzzy extension of A.

**Proof:**

Since $(\mu_A)^T_\alpha$(x) $= (\mu_A)(x) + \alpha \ge (\mu_A)(x)$, then $(\mu_A)^T_\alpha$(x) is a fuzzy extension of $(\mu_A)$(x), for all $x \in X$ and $(\nu_A)^K_\varepsilon$(x) $= (\nu_A)(x) - \varepsilon \le (\nu_A)(x)$, then $(\nu_A)^K_\varepsilon$(x) is a fuzzy extension of $(\nu_A)$(x), for all $x \in X$. □

**Proposition 3.14.**

Let $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$ be an bifuzzy $\psi$ −subalgebra of a $\psi$-algebra $(X; +, -, 0)$, then $A^{(T,K)}_{(\alpha,\varepsilon)} = \{(x, (\mu_A)^T_\alpha(x), (\nu_A)^K_\varepsilon(x)) \mid x \in X\}$ is a translation translation bifuzzy $\psi$ −subalgebra of X.

**Proof:**
Since $A$ is a bifuzzy $\psi$-subalgebra of $X$, then $(\mu_A)^T_\alpha$ is a translation fuzzy $\psi$-subalgebra of X and $(\nu_A)^K_\varepsilon$ is a translation anti-fuzzy $\psi$-subalgebra of X by Proposition (3.5).
then $A^{(T,K)}_{(\alpha,\varepsilon)}$ of X is a translation bifuzzy $\psi$ −subalgebra of X by Definition (3.2). □
In general, the converse of Proposition (3.13) is not true as seen in the following example.

**Example 3.15.** Let $X$={0, 1, 2, 3} be a $\psi$-algebra which is given in Example (3.3).

Define a fuzzy $\psi$-subalgebras $(\mu_A)$ and $(\nu_A)$ of $X$ by:

| | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $\mu_A$ | 0.8 | 0.5 | 0.7 | 0.5 |
| $\nu_A$ | 0.3 | 0.4 | 0.5 | 0.3 |

Then $\mu_A$ is a fuzzy $\psi$-subalgebra of $X$ and $\nu_A$ is an anti-fuzzy $\psi$-subalgebra of $X$. Let $(\mu_A)^{T}_{\alpha}$ be a fuzzy subsets of $X$ where $\alpha$ =0.1 and $(\nu_A)^{K}_{\varepsilon}$ be fuzzy subsets of $X$ where $\varepsilon$ =0.2 $\mu_A$ is a fuzzy $\psi$-subalgebra of $X$ given by:

| $X$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $(\mu_A)^{T}_{\alpha}$ | 0.9 | 0.6 | 0.8 | 0.6 |
| $(\nu_A)^{K}_{\varepsilon}$ | 0.1 | 0.2 | 0.3 | 0.1 |

Then $(\mu_A)^{T}_{\alpha}$ is a fuzzy extension of $\mu_A$, but the $\mu_A$ is not a fuzzy extension of $(\mu_A)^{T}_{\alpha}$ and $\nu_A$ is a fuzzy extension of $(\nu_A)^{K}_{\varepsilon}$, but the $(\nu_A)^{K}_{\varepsilon}$ is not a fuzzy extension of $\nu_A$ .

**Proposition 3.16.**

The intersection of translation bifuzzy $\psi$ -subalgebras of X of a fuzzy subset $(\mu_A)_i$ of $\psi$ -algebra $(X;+,-,0)$ is a translation bifuzzy $\psi$ -subalgebra of $X$.

**Proof:**

Let $\{((\mu_A)_i)^{T}_{\alpha} | i\in\Lambda\}$ be a family of translation bifuzzy $\psi$ -subalgebra of $(\mu_A)_i$ of $\psi$ -algebra $X$, then for any $x, y\in X$ , $i\in\Lambda$,

$$(\cap_{i\in\Lambda}((\mu_A)_i)^{T}_{\alpha})(x+y) = \inf (((\mu_A)_i)^{T}_{\alpha}(x+y))$$

$$= \inf ((\mu_A)_i (x+y)) + \alpha$$

$$\geq \inf \{\min\{ (\mu_A)_i(x), (\mu_A)_i(y) \}\} + \alpha$$

$$= \min \{\inf ( \mu_i(x)), \inf (\mu_i(y)\} + \alpha$$

$$= \min \{\inf ( \mu_i(x)) + \alpha, \inf (\mu_i(y) + \alpha\}$$

$$= \min \{(\cap_{i\in\Lambda}((\mu_A)_i)^{T}_{\alpha})(x), (\cap_{i\in\Lambda}((\mu_A)_i)^{T}_{\alpha})(y)\}.$$

Similarity, $(\cap_{i\in\Lambda}((\mu_A)_i)^{T}_{\alpha})(x-y) \geq \min \{(\cap_{i\in\Lambda}((\mu_A)_i)^{T}_{\alpha})(x), (\cap_{i\in\Lambda}((\mu_A)_i)^{T}_{\alpha})(y)\}$.

Let $\{((\nu_A)_i)^{T}_{\alpha} | i\in\Lambda\}$ be a family of translation bifuzzy $\psi$ -subalgebra of $(\nu_A)_i$, then for any $x, y\in X$ , $i\in\Lambda$,

$$(\cup_{i\in\Lambda}((\nu_A)_i)^{K}_{\varepsilon})(x+y) = \sup (((\nu_A)_i)^{K}_{\varepsilon}(x+y))$$

$$= \sup ((\nu_A)_i (x+y)) - \varepsilon$$

$$\leq \sup \{\max\{ (\nu_A)_i(x), (\nu_A)_i(y) \}\} - \varepsilon$$

$$= \max \{\sup ( (\nu_A)_i(x)), \sup ((\nu_A)_i(y)\} - \varepsilon$$

$$= \max \{\sup ( (\nu_A)_i(x)) - \varepsilon, \sup ((\nu_A)_i(y) - \varepsilon\}$$

$$= \max\{ \cup_{i\in\Lambda} (\nu_A)^{K}_{\varepsilon}(x), \cup_{i\in\Lambda} (\nu_A)^{K}_{\varepsilon}(y) \}.$$

Similarity, $(\cup_{i\in\Lambda}((\nu_A)_i)^{K}_{\varepsilon})(x-y) \leq \max \{(\cup_{i\in\Lambda}((\nu_A)_i)^{K}_{\varepsilon})(x), (\cup_{i\in\Lambda}((\nu_A)_i)^{K}_{\varepsilon})(y)\}$. □

**Remark 3.17.**

The union of the translation bifuzzy $\psi$ -subalgebra of $X$, is not a translation bifuzzy $\psi$ -subalgebra of X as seen in the following example.

**Example 3.18.**

Let $X$ = {0, a, b, c, d} be a set with the following table :

| + | 0 | a | b | c | d |
|---|---|---|---|---|---|
| 0 | 0 | a | b | c | d |
| a | a | b | c | d | 0 |
| b | b | c | d | 0 | a |
| c | c | d | 0 | a | b |
| d | d | 0 | a | b | c |

| - | 0 | a | b | c | d |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| a | a | 0 | 0 | 0 | a |
| b | b | b | 0 | 0 | a |
| c | c | b | d | 0 | a |
| d | d | d | d | d | 0 |

Then $(X; +, -, 0)$ is a $\psi$-algebra. It is easy to show that I ={0, c} and J ={0, d} are $\psi$-subalgebras of $X$ .

We defined two cubic set $A_1 = \{(x, \mu_{A1}(x), \nu_{A1}(x)) \mid x \in X\}$ and $A_2 = \{(x, \mu_{A2}(x), \nu_{A2}(x)) \mid x \in X\}$ of X by :-

$$\mu_{A1}(x) = \begin{cases} 0.8, & \text{if } x \in \{0, c\}, \\ 0.5, & \text{if } x \in \{a, b\}, \\ 0.3, & \text{otherwise} \end{cases} \qquad \nu_{A1}(x) = \begin{cases} 0.2, & \text{if } x \in \{0, c\}, \\ 0.6, & \text{if } x \in \{a, b\}, \\ 0.4, & \text{otherwise} \end{cases}$$

$\mu_{A2}(x) = \begin{cases} 0.7, & \text{if} x \in \{0, d\}, \\ 0.3, & \text{otherwise.} \end{cases}$ and $v_{A2}(x) = \begin{cases} 0.1, & \text{if} x \in \{0, d\}, \\ 0.4, & \text{otherwise.} \end{cases}$

Then $A_1$ and $A_2$ are bifuzzy $\psi$-subalgebra of X , but the intersection of $A_1$ and $A_2$ are not bifuzzy $\psi$-subalgebras of X . Since

$(\cup \mu_{Ai})(c-d) = \max\{0.5,0.3\} = 0.5 \ngeq 0.7 = \min\{(\cup \mu_{Ai})(c), (\cup \mu_{Ai})(d)\} = \min\{\max\{0.8,0.3\}, \max\{0.7,0.3\}\}$

and

$(\cup v_{Ai})(c-d) = \max\{0.6,0.4\} = 0.6 \nleq 0.4 = \max\{(\cup v_{Ai})(c), (\cup v_{Ai})(d)\} = \max\{\max\{0.2,0.4\}, \max\{0.4,0.1\}\}$.

**Definition 3.19.**

Let $A_i = \{(x, (\mu_A)_i(x), (v_A)_i(x)) \mid x \in X\}$ be bifuzzy subsets of a $\psi$-algebra $(X; +, -, 0)$ and $\alpha \in [0, T]$, $\varepsilon \in [0, K]$ where i ∈ Λ such that $(\mu_A)_i$ is a fuzzy $\psi$ -subalgebras of X and $(v_A)_i$ is anti-fuzzy $\psi$-subalgebras of $X$, for any $x \in X$, then

1-The R-intersection of any set of bifuzzy subset of $X$ is $(\cap_{i\in\Lambda}((\mu_A)_i)^{T}_{\alpha})(x) = \inf (\mu_A)_i(x)$ and $(\cup_{i\in\Lambda} ((v_A)_i)^{K}_{\varepsilon})(x) = \sup (v_A)_i(x)$.

2-The P-intersection of any set of bifuzzy subset of $X$ is $(\cap_{i\in\Lambda}((\mu_A)_i)^{T}_{\alpha})(x) = \inf (\mu_A)_i(x)$ and $\cap_{i\in\Lambda} ((v_A)_i)^{K}_{\varepsilon}(x) = \inf ((v_A)_i)(x)$.

3-The R-union of any set of bifuzzy subset of $X$ is $(\cup_{i\in\Lambda} ((\mu_A)_i)^{T}_{\alpha})(x) = \sup((\mu_A)_i)(x)$ and $(\cap_{i\in\Lambda} ((v_A)_i)^{K}_{\varepsilon})(x) = \inf ((v_A)_i)(x)$.

4-The P-union of any set of bifuzzy subset of $X$ is $(\cup_{i\in\Lambda} ((\mu_A)_i)^{T}_{\alpha})(x) = \sup((\mu_A)_i)(x)$ and $(\cup_{i\in\Lambda} ((v_A)_i)^{K}_{\varepsilon})(x) = \sup (v_A)_i(x)$.

**Proposition 3.20.**

The R-intersection of any set of translation bifuzzy $\psi$-subalgebra of $(X; +, -, 0)$ is also translation bifuzzy $\psi$-subalgebra of $X$.

| + | 0 | a | b | c | d |
|---|---|---|---|---|---|
| 0 | 0 | a | b | c | d |
| a | a | b | c | d | 0 |
| b | b | c | d | 0 | a |
| c | c | d | 0 | a | b |
| d | d | 0 | a | b | c |

| - | 0 | a | b | c | d |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| a | a | 0 | 0 | 0 | a |
| b | b | b | 0 | 0 | a |
| c | c | b | d | 0 | a |
| d | d | d | d | d | 0 |

**Proof.**

Let $A_i = \{(x, (\mu_A)_i(x), (v_A)_i(x)) \mid x \in X\}$ where i ∈ Λ, be a set of bifuzzy $\psi$-subalgebra of $X$ and x, y ∈ $X$, then

$(\cap_{i\in\Lambda}((\mu_A)_i)^{T}_{\alpha})(x+y) = \inf ((\mu_A)_i)^{T}_{\alpha}(x+y)$

$= \inf (\mu_A)_i(x+y) + \alpha$

$\geq \inf \{\min\{(\mu_A)_i(x), (\mu_A)_i(y)\}\} + \alpha$

$= \min\{\inf((\mu_A)_i(x)), \inf((\mu_A)_i(y))\} + \alpha$

$= \min\{\inf((\mu_A)_i(x)) + \alpha, \inf((\mu_A)_i(y)) + \alpha\}$

$= \min\{(\cap_{i\in\Lambda}((\mu_A)_i)^{T}_{\alpha})(x), (\cap_{i\in\Lambda}((\mu_A)_i)^{T}_{\alpha})(y)\}$

and

Summarily, $(\cap_{i\in\Lambda}((\mu_A)_i)^{T}_{\alpha})(x-y) \geq \min\{((\mu_A)_i)(x), (\cap_{i\in\Lambda}((\mu_A)_i)^{T}_{\alpha})(y)\}$.

Hence $(\cap_{i\in\Lambda}((\mu_A)_i)^{T}_{\alpha})$ is a translation fuzzy $\psi$-subalgebra of $X$.

$(\cup_{i\in\Lambda} ((v_A)_i)^{K}_{\varepsilon})(x+y) = \sup (((v_A)_i)^{K}_{\varepsilon})(x+y)$

$\leq \sup\{\max\{((v_A)_i)(x), ((v_A)_i)(y)\}\} - \varepsilon$

$= \max\{\sup ((v_A)_i(x)), \sup ((v_A)_i(y))\} - \varepsilon$

$= \max\{\sup ((v_A)_i(x)) - \varepsilon, \sup ((v_A)_i(y)) - \varepsilon\}$

$= \max\{(\cup_{i\in\Lambda} ((v_A)_i)^{K}_{\varepsilon})(x), (\cup_{i\in\Lambda} ((v_A)_i)^{K}_{\varepsilon})(y)\}$.

Summarily,

$(\cup_{i\in\Lambda} ((v_A)_i)^{K}_{\varepsilon})(x-y) \leq \max\{(\cup_{i\in\Lambda} ((v_A)_i)^{K}_{\varepsilon})(x), (\cup_{i\in\Lambda} ((v_A)_i)^{K}_{\varepsilon})(y)\}$.

Hence $\cup_{i\in\Lambda} ((v_A)_i)^{K}_{\varepsilon}$ is a translation anti-fuzzy $\psi$-subalgebra of $X$.

Hence, R-intersection of $A_{i(\alpha,\varepsilon)}^{(T,K)}$ is a translation bifuzzy $\psi$-subalgebra of $X$. □

**Remark 3.21.**

The P-intresection of any sets of translation bifuzzy $\psi$-subalgebra need not be a translation bifuzzy $\psi$-subalgebra, for example:

**Example 3.22.**

Let $X$ = {0, a, b, c, d} be a set with the following table:

Then $(X;*,0)$ is an $\psi$-algebra. It is easy to show that I ={0, c} and J ={0, d} are $\psi$-subalgebras of $X$ .

We defined two cubic set $A_1 = \{(x, \mu_{A1}(x), v_{A1}(x)) \mid x \in X\}$ and $A_2 = \{(x, \mu_{A2}(x), v_{A2}(x)) \mid x \in X\}$ of X by :-

$$\mu_{A1}(x) = \begin{cases} 0.8, & \text{if } x \in \{0,c\}, \\ 0.7, & \text{if } x \in \{a,b\}, \\ 0.6, & \text{otherwise} \end{cases} \qquad v_{A1}(x) = \begin{cases} 0.2, & \text{if } x \in \{0,c\}, \\ 0.6, & \text{if } x \in \{a,b\}, \\ 0.4, & \text{otherwise} \end{cases}$$

$$\mu_{A2}(x) = \begin{cases} 0.7, & \text{if } x \in \{0,d\}, \\ 0.2, & \text{otherwise.} \end{cases} \quad \text{and} \quad v_{A2}(x) = \begin{cases} 0.1, & \text{if } x \in \{0,d\}, \\ 0.4, & \text{otherwise.} \end{cases}$$

Then $A_1$ and $A_2$ are bifuzzy $\psi$-subalgebra of X , but P-intersection of $A_1 \cap A_2$ are not bifuzzy $\psi$-subalgebras of X . Since

$$(\cap_{i\in\Lambda}((\mu_A)_i)^T_\alpha)(c-d) = \min\{0.7,0.2\} = 0.2 \ngeq 0.6 = \min\{\cap_{i\in\Lambda}((\mu_A)_i)^T_\alpha(c), \cap_{i\in\Lambda}((\mu_A)_i)^T_\alpha(d)\} = \min\{\min\{0.8,0.2\}, \min\{0.7,0.6\}\}$$

and

$$(\cap_{i\in\Lambda}((v_A)_i)^K_\varepsilon)(c-d) = \min\{0.6,0.4\} = 0.4 \nleq 0.2 = \max\{\cap_{i\in\Lambda}((v_A)_i)^K_\varepsilon(c), \cap_{i\in\Lambda}((v_A)_i)^K_\varepsilon(d)\} = \max\{\min\{0.2,0.4\}, \min\{0.4,0.1\}\}.$$

**Proposition 3.23.**

Let $A_i = \{(x, \mu_{Ai}(x), v_{Ai}(x)) \mid x \in X\}$ where $i \in \Lambda$, be a set of bifuzzy $\psi$-subalgebra of $\psi$-algebra $(X;+,-,0)$, where $i \in \Lambda$, $inf\{max\{(\mu_A)_i(x), (\mu_A)_i(y)\}\} = max\{inf\ (\mu_A)_i(x), inf\ (\mu_A)_i(y)\}$, for all $x \in X$, then the P-intresection of $A_{i}{}^{(T,K)}_{(\alpha,\varepsilon)}$ is also a translation bifuzzy $\psi$-subalgebra of $X$.

**Proof.**

Let $A_i = \{(x, (\mu_A)_i(x), ((v_A)_i(x)) \mid x \in X\}$ where $i \in \Lambda$, be a set of bifuzzy $\psi$-subalgebra of $X$ and $x, y \in X$, then

$$(\cap_{i\in\Lambda}((\mu_A)_i)^T_\alpha)(x+y) = inf\ ((\mu_A)_i)^T_\alpha(x+y)$$
$$= inf\ ((\mu_A)_i)(x+y) + \alpha$$
$$\geq inf\ \{min\{((\mu_A)_i(x), (\mu_A)_i(y)\}\} + \alpha$$
$$= min\{inf((\mu_A)_i(x)), inf((\mu_A)_i(y))\} + \alpha$$
$$= min\{inf((\mu_A)_i(x)) + \alpha, inf((\mu_A)_i(y)) + \alpha\}$$
$$= \min\{(\cap_{i\in\Lambda}((\mu_A)_i)^T_\alpha)(x), (\cap_{i\in\Lambda}((\mu_A)_i)^T_\alpha)(y)\}$$

Summarily,

$$(\cap_{i\in\Lambda}((\mu_A)_i)^T_\alpha)\ (x-y) \geq \min\{(\cap_{i\in\Lambda}((\mu_A)_i)^T_\alpha)\ (x), (\cap_{i\in\Lambda}((\mu_A)_i)^T_\alpha)(y)\}.$$

Hence $\cap_{i\in\Lambda}((\mu_A)_i)^T_\alpha)$ is a translation fuzzy $\psi$-subalgebra of $X$.

$$(\cap_{i\in\Lambda}((v_A)_i)^K_\varepsilon)(x+y) = inf\ (\cap_{i\in\Lambda}((v_A)_i)^K_\varepsilon)(x+y)$$
$$= inf\ (\cap_{i\in\Lambda}((v_A)_i)(x+y) - \varepsilon$$
$$\leq \inf\{\max\{((v_A)_i)(x), ((v_A)_i)(y)\}\} - \varepsilon$$
$$= \max\{\inf((v_A)_i(x)), \inf((v_A)_i(y))\} - \varepsilon$$
$$= \max\{\inf((v_A)_i(x)) - \varepsilon, \inf((v_A)_i(y)) - \varepsilon\}$$
$$= \max\{(\cap_{i\in\Lambda}((v_A)_i)^K_\varepsilon)(x), (\cap_{i\in\Lambda}((v_A)_i)^K_\varepsilon)(y)\}.$$

Summarily,

$$(\cap_{i\in\Lambda}((v_A)_i)^K_\varepsilon)\ (x-y) \leq \max\{(\cap_{i\in\Lambda}((v_A)_i)^K_\varepsilon)\ (x), (\cap_{i\in\Lambda}((v_A)_i)^K_\varepsilon)(y)\}.$$

Hence $(\cap_{i\in\Lambda}((v_A)_i)^K_\varepsilon)$ is translation anti-fuzzy $\psi$-subalgebra of $X$. Hence, P-intersection of $A_{i}{}^{(T,K)}_{(\alpha,\varepsilon)}$ is a translation bifuzzy $\psi$-subalgebra of $X$. □

**Proposition 3.24.**

Let $A_i = \{(x, (\mu_A)_i(x), (v_A)_i(x)) \mid x \in X\}$ where $i \in \Lambda$, be a set of bifuzzy $\psi$-subalgebra of $\psi$-algebra $(X;+,-,0)$, where $i \in \Lambda$, $sup\{min\{(\mu_A)_i(x), (\mu_A)_i(y)\}\} = min\{sup\ (\mu_A)_i(x), sup(\mu_A)_i(y)\}$, for all $x \in X$, then the P-union of $A_{i}{}^{(T,K)}_{(\alpha,\varepsilon)}$ is also a translation bifuzzy $\psi$-subalgebra of $X$.

**Proof.**

Let $A_i = \{(x, (\mu_A)_i(x), (v_A)_i(x)) \mid x \in X\}$ where $i \in \Lambda$, be a set of bifuzzy $\psi$-subalgebra of $X$ and $x, y \in X$, then

$$(\cup_{i\in\Lambda}((\mu_A)_i)^T_\alpha)(x+y) = sup\ (((\mu_A)_i)^T_\alpha)(x+y)$$
$$= sup\ ((\mu_A)_i)(x+y) + \alpha$$
$$\geq sup\ \{min\{(\mu_A)_i(x), (\mu_A)_i(y)\}\} + \alpha$$
$$= min\{sup((\mu_A)_i(x)), sup((\mu_A)_i(y))\} + \alpha$$
$$= min\{sup((\mu_A)_i(x)) + \alpha, sup((\mu_A)_i(y)) + \alpha\} = \min\{(\cup_{i\in\Lambda}((\mu_A)_i)^T_\alpha)(x), (\cup_{i\in\Lambda}((\mu_A)_i)^T_\alpha)(y)\}$$

and

Summarily,

$(\cup_{i\in\Lambda}((\mu_A)_i)^{T}_{\alpha})(x-y) \geq \min\{(\cup_{i\in\Lambda}((\mu_A)_i)^{T}_{\alpha})(x), (\cup_{i\in\Lambda}((\mu_A)_i)^{T}_{\alpha})(y)\}.$

Hence $(\cup_{i\in\Lambda}((\mu_A)_i)^{T}_{\alpha})$ is a translation fuzzy $\psi$-subalgebra of $X$.

$$\begin{aligned}(\cup_{i\in\Lambda}((v_A)_i)^{K}_{\varepsilon})(x+y) &= \sup(((v_A)_i)^{K}_{\varepsilon})(x+y)\\ &= \sup((v_A)_i)(x+y)-\varepsilon\\ &\leq \sup\{\max\{((v_A)_i)(x),((v_A)_i)(y)\}\}-\varepsilon\\ &= \max\{\sup((v_A)_i)(x),\sup((v_A)_i)(y)\}-\varepsilon\\ &= \max\{\sup((v_A)_i)(x)-\varepsilon,\sup((v_A)_i)(y)-\varepsilon\}\\ &= \max\{(\cup_{i\in\Lambda}((v_A)_i)^{K}_{\varepsilon})(x),\cup_{i\in\Lambda}((v_A)_i)^{K}_{\varepsilon}(y)\}.\end{aligned}$$

Summarily,

$(\cup_{i\in\Lambda}((v_A)_i)^{K}_{\varepsilon})(x-y) \leq \max\{(\cup_{i\in\Lambda}((v_A)_i)^{K}_{\varepsilon})(x), (\cup_{i\in\Lambda}((v_A)_i)^{K}_{\varepsilon})(y)\}.$

Hence $(\cup_{i\in\Lambda}((v_A)_i)^{K}_{\varepsilon})$ is a translation anti-fuzzy $\psi$-subalgebra of $X$.

Hence, P-union of $A_{i}{}^{(T,K)}_{(\alpha,\varepsilon)}$ is a translation bifuzzy $\psi$-subalgebra of $X$. □

**Remark 3.25.**

The R-union of any sets of translation bifuzzy $\psi$-subalgebra need not be a translation bifuzzy $\psi$-subalgebra, for example:

**Example 3.26.**

Let $X = \{0, a, b, c, d\}$ be a set with the following table:

| + | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 |
| 1 | 1 | 2 | 3 | 4 | 0 |
| 2 | 2 | 3 | 4 | 0 | 1 |
| 3 | 3 | 4 | 0 | 1 | 2 |
| 4 | 4 | 0 | 1 | 2 | 3 |

| - | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 | 1 |
| 2 | 2 | 2 | 0 | 0 | 1 |
| 3 | 3 | 2 | 4 | 0 | 1 |
| 4 | 4 | 4 | 4 | 4 | 0 |

Then $(X;*,0)$ is an $\psi$-algebra. It is easy to show that I ={0, 3} and J ={0, 4} are $\psi$-subalgebras of $X$.

We defined two cubic set $A_1 = \{(x,\mu_{A1}(x),v_{A1}(x)) \mid x \in X\}$ and $A_2 = \{(x,\mu_{A2}(x),v_{A2}(x)) \mid x \in X\}$ of X by :-

$$\mu_{A1}(x)=\begin{cases}0.7, & \text{if } x\in\{0,3\},\\ 0.6, & \text{if } x\in\{1,2\},\\ 0.5, & \text{otherwise}\end{cases}\qquad v_{A1}(x)=\begin{cases}0.1, & \text{if } x\in\{0,3\},\\ 0.5, & \text{if } x\in\{1,2\},\\ 0.3, & \text{otherwise}\end{cases}$$

$$\mu_{A2}(x)=\begin{cases}0.6, & \text{if} x\in\{0,4\},\\ 0.1, & \text{otherwise.}\end{cases}\quad\text{and}\quad v_{A2}(x)=\begin{cases}0.1, & \text{if} x\in\{0,4\},\\ 0.3, & \text{otherwise.}\end{cases}$$

Then $A_1$ and $A_2$ are bifuzzy $\psi$-subalgebra of X , but R-union of $A_1 \cup A_2$ are not bifuzzy $\psi$-subalgebras of X .

Since

$(\cup_{i\in\Lambda}((\mu_A)_i)^{T}_{\alpha})(3-4) = \max\{0.6,0.1\} = 0.6 \ngeq 0.7 = \max\{\cup_{i\in\Lambda}((\mu_A)_i)^{T}_{\alpha}(3),\cup_{i\in\Lambda}((\mu_A)_i)^{T}_{\alpha}(4)\} = \max\{\max\{0.7,0.1\},\max\{0.6,0.5\}\}$ and

$(\cap_{i\in\Lambda}((v_A)_i)^{K}_{\varepsilon})(3-4) = \max\{0.5,0.3\} = 0.5 \nleq 0.1 = \max\{\cap_{i\in\Lambda}((v_A)_i)^{K}_{\varepsilon}(3),\cap_{i\in\Lambda}((v_A)_i)^{K}_{\varepsilon}(4)\} = \max\{\min\{0.1,0.3\},\min\{0.3,0.1\}\}.$

**Proposition 3.28.**

Let $A_i = \{(x,(\mu_A)_i(x),(v_A)_i(x)) \mid x \in X\}$ where $i \in \Lambda$, be a set of bifuzzy $\psi$-subalgebra of $\psi$-algebra $(X;+,-,0)$, where $i \in \Lambda$, $sup\{min\{(\mu_A)_i(x),(\mu_A)_i(y)\}\} = min\{sup\,(\mu_A)_i(x), sup\,(\mu_A)_i(y)\}$ and $inf\{max\{(\mu_A)_i(x),(\mu_A)_i(y)\}\} = max\{inf\,(\mu_A)_i(x), inf\,(\mu_A)_i(y)\}$, for all x ∈ X, then the R-union of $A_i$ is also a translation bifuzzy $\psi$-subalgebra of $X$.

**Proof.**

Let $A_i = \{(x,(\mu_A)_i(x),(v_A)_i(x)) \mid x \in X\}$ where $i \in \Lambda$, be a set of bifuzzy $\psi$-subalgebra of $X$ and $x,y \in X$, then

$$\begin{aligned}(\cup_{i\in\Lambda}((\mu_A)_i)^{T}_{\alpha})(x+y) &= sup\,(((\mu_A)_i)^{T}_{\alpha})(x+y)\\ &= sup\,((\mu_A)_i)(x+y)+\alpha\\ &\geq sup\,\{min\{((\mu_A)_i)(x),((\mu_A)_i)(y)\}\}+\alpha\\ &= min\{sup((\mu_A)_i(x)),sup((\mu_A)_i(y))\}+\alpha\\ &= min\{sup((\mu_A)_i(x))+\alpha, sup((\mu_A)_i(y))+\alpha\}\\ &= \min\{(\cup_{i\in\Lambda}((\mu_A)_i)^{T}_{\alpha})(x),(\cup_{i\in\Lambda}((\mu_A)_i)^{T}_{\alpha})(y)\}\end{aligned}$$ and

Summarily,

$(\cup_{i\in\Lambda}((\mu_A)_i)^{T}_{\alpha})(x-y) \geq \min\{(\cup_{i\in\Lambda}((\mu_A)_i)^{T}_{\alpha})(x), (\cup_{i\in\Lambda}((\mu_A)_i)^{T}_{\alpha})(y)\}.$

Hence $(\cup_{i\in\Lambda}((\mu_A)_i)^{T}_{\alpha})$ is a translation fuzzy $\psi$-subalgebra of $X$.

$(\cap_{i\in\Lambda}\ ((\nu_A)_i)^{K}_{\varepsilon})(x+y) = \inf(((\nu_A)_i)^{K}_{\varepsilon})(x+y)$

$= \inf((\nu_A)_i)(x+y) - \varepsilon$

$\leq \inf\{\max\{(\nu_{Ai})(x), (\nu_{Ai})(y)\}\} - \varepsilon$

$= \max\{\inf((\nu_A)_i)(x), \inf((\nu_A)_i)(y)\} - \varepsilon$

$= \max\{\inf((\nu_A)_i)(x) - \varepsilon, \inf((\nu_A)_i)(y) - \varepsilon\}$

$= \max\{(\cup_{i\in\Lambda}\ ((\nu_A)_i)^{K}_{\varepsilon})(x), (\cup_{i\in\Lambda}\ ((\nu_A)_i)^{K}_{\varepsilon})(y)\}$.

Summarily,

$$(\cup_{i\in\Lambda}\ ((\nu_A)_i)^{K}_{\varepsilon})\ (x-y) \leq \max\{(\cup_{i\in\Lambda}\ ((\nu_A)_i)^{K}_{\varepsilon})\ (x), (\cup_{i\in\Lambda}\ ((\nu_A)_i)^{K}_{\varepsilon})(y)\}.$$

Hence $(\cup_{i\in\Lambda}\ ((\nu_A)_i)^{K}_{\varepsilon})$ is a translation anti-fuzzy $\psi$-subalgebra of $X$.

Hence, R-union of ${A_i}^{(T,K)}_{(\alpha,\varepsilon)}$ is a translation bifuzzy $\psi$-subalgebra of $X$. □

## 4. The Translation Bifuzzy $\psi$- ideals of $\psi$-algebra

In this section, we shall define the notion of translation of bifuzzy $\psi$-ideals, and we study some of the relations, theorems, propositions and examples of translation of bifuzzy $\psi$-ideals of $\psi$-algebra.

**Definition 4.1:**

Let $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$ be a bifuzzy subset of a $\psi$-algebra and $\alpha \in [0,T]$, $\varepsilon \in [0,K]$ of $(X; +, -, 0)$, then $A^{(T,K)}_{(\alpha,\varepsilon)} = \{(x, (\mu_A)^{T}_{\alpha}(x), (\nu_A)^{K}_{\varepsilon}(x)) \mid x \in X\}$ is called a **translation bifuzzy $\psi$ −ideal of X**, if for all $x, y, z \in X$,

1- $(\mu_A)^{T}_{\alpha}(0) \geq (\mu_A)^{T}_{\alpha}(x)$ and $(\nu_A)^{K}_{\varepsilon}(0) \leq (\nu_A)^{K}_{\varepsilon}(x)$ ,

2- $(\mu_A)^{T}_{\alpha}(y+x) \geq \min\{(\mu_A)^{T}_{\alpha}(y+z), (\mu_A)^{T}_{\alpha}(x-z)\}$, and

3- $(\nu_A)^{K}_{\varepsilon}(y+x) \leq \max\{(\nu_A)^{K}_{\varepsilon}(y+z), (\nu_A)^{K}_{\varepsilon}(x-z)\}$. i.e,

1- $\mu_A(0) + \alpha \geq \mu_A(x) + \alpha$ and $\nu_A(0) - \varepsilon \leq \nu_A(x) - \varepsilon$ ,

2- $\mu_A(y+x) + \alpha \geq \min\{\mu_A(y+z) + \alpha, \mu_A(x-z) + \alpha\}$

$= \min\{\mu_A(y+z), \mu_A(x-z)\} + \alpha$, and

3- $\nu_A(y+x) - \varepsilon \leq \max\{\nu_A(y+z) - \varepsilon, \nu_A(x-z) - \varepsilon\} = \max\{\nu_A(y+z), \nu_A(x-z)\} - \varepsilon$.

**Example 4.2 :**

Let X={0,1,2, 3} in which $(+, -)$ be a defined by the following table:

| + | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 2 | 3 | 0 |
| 2 | 2 | 3 | 0 | 1 |
| 3 | 3 | 0 | 1 | 2 |

| - | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 |
| 2 | 2 | 0 | 0 | 0 |
| 3 | 3 | 3 | 3 | 0 |

Then $(X; +, -, 0)$ is an $\psi$ -algebra. It is easy to show that $I_1$ ={0,1}, $I_2$ ={0,2}and $I_3$={0,3} are $\psi$ -ideals of X. Define a fuzzy subset

$\mu_A$: X → [0,1] such that $\mu_A(0) = 0.7$ , $\mu_A(1) = \mu_A(2) = 0.6$, $\mu_A(3) = 0.4$, $\alpha = 0.15 \in [0,0.3]$.

$\nu_A$: X → [0,1] such that $\nu_A(0) = 0.3$ , $\nu_A(1) = \nu_A(2) = 0.4$, $\nu_A(3) = 0.6$, $\varepsilon = 0.25 \in [0,0.3]$.

Routine calculation gives that $(\mu_A)^{T}_{\alpha}$ is a translation fuzzy $\psi$-ideal of $X$ and $(\nu_A)^{K}_{\varepsilon}$ is a translation anti-fuzzy $\psi$-ideal of $X$.

**Theorem 4.3:**

Let $A^{(T,K)}_{(\alpha,\varepsilon)} = \{(x, (\mu_A)^{T}_{\alpha}(x), (\nu_A)^{K}_{\varepsilon}(x)) \mid x \in X\}$ be an bifuzzy $\psi$ -ideal of a $\psi$-algebra $(X; +, -, 0)$ and $\alpha \in [0,T]$, $\varepsilon \in [0,K]$, then $\mu_A$ is a fuzzy $\psi$-ideal of X and $\nu_A$ is an anti-fuzzy $\psi$- ideal of X .

**Proof:**

Assume that A is a bifuzzy $\psi$ − ideal of X, and $\alpha \in [0,T]$, $\varepsilon \in [0,K]$.

Let $x, y \in X$, then

1- $(\mu_A)^{T}_{\alpha}(0) \geq (\mu_A)^{T}_{\alpha}(x)$, and $(\nu_A)^{K}_{\varepsilon}(0) \leq (\nu_A)^{K}_{\varepsilon}(x)$

$\mu_A(0) + \alpha \geq \mu_A(x) + \alpha$ implies that $\mu_A(0) \geq \mu_A(x)$, for any $x \in X$, and

$\nu_A(0) - \varepsilon \leq \nu_A(x) - \varepsilon$ implies that $\nu_A(0) \leq \nu_A(x)$, for any $x \in X$.

2- $(\mu_A)^{T}_{\alpha}(y+x) \geq \min\{(\mu_A)^{T}_{\alpha}(y+z), (\mu_A)^{T}_{\alpha}(x-z)\}$, that mean

$$\mu_A(y+x)+\alpha \geq \min\{\mu_A(y+z)+\alpha, \mu_A(x-z)+\alpha\}$$

$= \min\{\mu_A(y+z), \mu_A(x-z)\} + \alpha$, implies that

$$\mu_A(y+x) \geq \min\{\mu_A(y+z), \mu_A(x-z)\}.$$

3- $(\nu_A)^{K}_{\varepsilon}(y+x) \leq \max\{(\nu_A)^{K}_{\varepsilon}(y+z), (\nu_A)^{K}_{\varepsilon}(x-z)\}$, that mean

$$\nu_A(y+x)-\varepsilon \leq \max\{\nu_A(y+z)-\varepsilon, \nu_A(x-z)-\varepsilon\}$$

$= \max\{\nu_A(y+z), \nu_A(x-z)\} - \varepsilon$, implies that

$$\nu_A(y+x) \leq \max\{\nu_A(y+z), \nu_A(x-z)\}.$$

Hence $\mu_A$ is a fuzzy $\psi$-ideal of X and $\nu_A$ is an anti-fuzzy $\psi$-ideal of X.□

**Theorem 4.4.**

Let $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$ be an bifuzzy subset of a $\psi$-algebra $(X; +, -, 0)$ and $\alpha \in [0,T]$, $\varepsilon \in [0,K]$ such that $\mu_A$ be a fuzzy $\psi$-ideal of X and $\nu_A$ be an anti-fuzzy $\psi$-ideal of X, then $A^{(T,K)}_{(\alpha,\varepsilon)} = \{(x, (\mu_A)^{T}_{\alpha}(x), (\nu_A)^{K}_{\varepsilon}(x)) \mid x \in X\}$ is a translation bifuzzy $\psi$-ideal of X.

**Proof:**

Assume that $\mu_A$ is a fuzzy $\psi$-ideal of X and $\nu_A$ is an anti-fuzzy $\psi$-ideal of X, then for any $x, y, z \in$ X and $\alpha \in [0,T]$, $\varepsilon \in [0,K]$

1- $\mu_A(0) \geq \mu_A(x)$ implies that $\mu_A(0)+\alpha \geq \mu_A(x)+\alpha$, for any $x \in X$, then $(\mu_A)^{T}_{\alpha}(0) \geq (\mu_A)^{T}_{\alpha}(x)$, and $\nu_A(0) \leq \nu_A(x)$, implies that

$\nu_A(0)-\varepsilon \leq \nu_A(x)-\varepsilon$, for any $x \in X$, then $(\nu_A)^{K}_{\varepsilon}(0) \leq (\nu_A)^{K}_{\varepsilon}(x)$.

2- $\mu_A(y+x) \geq \min\{\mu_A(y+z), \mu_A(x-z)\}$ implies that $\mu_A(y+x)+\alpha \geq \min\{\mu_A(y+z), \mu_A(x-z)\} + \alpha = \min\{\mu_A(y+z)+\alpha, \mu_A(x-z)+\alpha\}$ that mean $(\mu_A)^{T}_{\alpha}(y+x) \geq \min\{(\mu_A)^{T}_{\alpha}(y+z), (\mu_A)^{T}_{\alpha}(x-z)\}$.

3- $\nu_A(y+x) \leq \max\{\nu_A(y+z), \nu_A(x-z)\}$ implies that

$$\nu_A(y+x)-\varepsilon \leq \max\{\nu_A(y+z), \nu_A(x-z)\} - \varepsilon$$

$= \max\{\nu_A(y+z)-\varepsilon, \nu_A(x-z)-\varepsilon\}$ that mean

$$(\nu_A)^{K}_{\varepsilon}(y+x) \leq \max\{(\nu_A)^{K}_{\varepsilon}(y+z), (\nu_A)^{K}_{\varepsilon}(x-z)\}.$$

Hence $(\mu_A)^{T}_{\alpha}$ is a translation fuzzy $\psi$-ideal of X and $(\nu_A)^{K}_{\varepsilon}$ is a translation anti-fuzzy $\psi$-ideal of X.

Therefore $A^{(T,K)}_{(\alpha,\varepsilon)} = \{(x, (\mu_A)^{T}_{\alpha}(x), (\nu_A)^{K}_{\varepsilon}(x)) \mid x \in X\}$ is a translation bifuzzy $\psi$-ideal of X. □

**Collaroy 4.5.**

Let $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$ be an bifuzzy subset of a $\psi$-algebra $(X; +, -, 0)$ and $\alpha \in [0,T]$, $\varepsilon \in [0,K]$ such that $(\mu_A)^{T}_{\alpha}$ be a translation fuzzy $\psi$-ideal of X and $(\nu_A)^{K}_{\varepsilon}$ be a translation anti-fuzzy $\psi$-ideal of X, then A is a bifuzzy $\psi$-ideal of $X$.

**Proof:**

By Definition (4.1) and by Theorem (4.4). □

**Remark 4.6.**

1- If $(\mu_A)^{T}_{\alpha}$ is a translation fuzzy $\psi$ - ideal of X, then it is that $U_{\alpha}((\mu_A)^{T}_{\alpha}; t)$ is a $\psi$ - ideal of X, for all $t \in \text{Im}((\mu_A)^{T}_{\alpha})$ with $t \geq \alpha$. Let $x, y, z \in U_{\alpha}((\mu_A)^{T}_{\alpha}; t)$, then $\mu_A(y+z) \geq t-\alpha$, and $(\mu_A)^{T}_{\alpha}(x-z) \geq t-\alpha$, then $\min\{(\mu_A)^{T}_{\alpha}(y+z), (\mu_A)^{T}_{\alpha}(x-z)\} \geq t-\alpha$, since $(\mu_A)^{T}_{\alpha}$ is a translation fuzzy $\psi$ - ideal, then $(\mu_A)^{T}_{\alpha}(y+x) \geq min\{(\mu_A)^{T}_{\alpha}(y+z), (\mu_A)^{T}_{\alpha}(x-z)\} \geq t-\alpha$, therefore $y+x \in U_{\alpha}((\mu_A)^{T}_{\alpha}; t)$.

2- If $(\nu_A)^{K}_{\varepsilon}$ is translation anti-fuzzy $\psi$ - ideal of X, then it is that $L_{\varepsilon}((\nu_A)^{K}_{\varepsilon}; s)$ is a $\psi$ - ideal of X, for all $s \in \text{Im}((\nu_A)^{K}_{\varepsilon})$ with $s \leq \varepsilon$. Let $x, y, z \in L_{\varepsilon}((\nu_A)^{K}_{\varepsilon}; s)$, then $(\nu_A)^{K}_{\varepsilon}(y+z) \leq s-\varepsilon$, and $(\nu_A)^{K}_{\varepsilon}(x-z) \leq s-\varepsilon$, then $\max\{(\nu_A)^{K}_{\varepsilon}(y+z), (\nu_A)^{K}_{\varepsilon}(x-z)\} \leq s-\varepsilon$, since $(\nu_A)^{K}_{\varepsilon}$ is translation anti-fuzzy $\psi$ - ideal, then $(\nu_A)^{K}_{\varepsilon}(y+x) \leq max\{(\nu_A)^{K}_{\varepsilon}(y+z), (\nu_A)^{K}_{\varepsilon}(x-z)\} \leq s-\varepsilon$, therefore $y+x \in L_{\varepsilon}((\nu_A)^{K}_{\varepsilon}; s)$.

3- But if we do not give a condition that $(\mu_A)^{T}_{\alpha}$ is a translation fuzzy $\psi$ - ideal of X, then $U_{\alpha}((\mu_A)^{T}_{\alpha}; t)$ is not a $\psi$ - ideal of X or $(\nu_A)^{K}_{\varepsilon}$ is translation anti-fuzzy $\psi$ - ideal of X, then $L_{\varepsilon}((\nu_A)^{K}_{\varepsilon}; s)$ is not a $\psi$ - ideal of X as seen in the following example.

**Example 4.7:**

Consider X = {0, 1, 2, 3 } is a $\psi$ -algebra which is given in Example (3.3) .

Define a fuzzy subset $\mu_A$ of X:

| X | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $\mu_A$ | 0.7 | 0.6 | 0.4 | 0.3 |

Then $\mu_A$ is not a fuzzy $\psi$ - ideal of X.

Since $(\mu_A)^{T}_{\alpha}$ (1+2) = 0.3 ≱ 0.4 = min{$(\mu_A)^{T}_{\alpha}$ (1), $(\mu_A)^{T}_{\alpha}$ (2)}. For α = 0.1 and t = 0.5, we obtain $U_\alpha((\mu_A)^{T}_{\alpha}; t)$ = {0, 1, 2} which is not an $\psi$ - ideal of X since 1+2 = 3 ∉ $U_\alpha(\mu_A; t)$.

**Proposition 4.8.**

Let $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$ be a bifuzzy subset of a $\psi$-algebra $(X; +, -, 0)$ and α ∈ [0,T], ε ∈ [0,K] such that $A^{(T,K)}_{(\alpha,\varepsilon)} = \{(x, (\mu_A)^{T}_{\alpha}(x), (\nu_A)^{K}_{\varepsilon}(x)) \mid x \in X\}$ is a translation bifuzzy $\psi$- ideal of X, then $U_\alpha((\mu_A)^{T}_{\alpha}; t)$ and $L_\varepsilon((\nu_A)^{K}_{\varepsilon}; s)$ are fuzzy $\psi$ - ideals of $X$, for any t∈ Im($(\mu_A)^{T}_{\alpha}$), s∈ Im($(\nu_A)^{K}_{\varepsilon}$) with t ≥ α and s ≤ $\varepsilon$.

**Proof:**

Assume that $A$ is a bifuzzy $\psi$- ideal, then by Theorem (4.4) $(\mu_A)^{T}_{\alpha}$ is a fuzzy $\psi$- ideal of X and $(\nu_A)^{K}_{\varepsilon}$ is an anti-fuzzy $\psi$- ideal of X, then by Remark (4.6), $U_\alpha((\mu_A)^{T}_{\alpha}; t)$and $L_\varepsilon((\nu_A)^{K}_{\varepsilon}; s)$ are fuzzy $\psi$ - ideals of $X$, for any t∈ Im($(\mu_A)^{T}_{\alpha}$), s∈ Im($(\nu_A)^{K}_{\varepsilon}$) with t ≥ α and s ≤ $\varepsilon$. □

**Theorem 4.9.**

Let $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$ be an bifuzzy subset of a $\psi$-algebra $(X; +, -, 0)$ and α ∈ [0,T], ε ∈ [0,K] such that $U_\alpha(\mu_A; t)$ and $L_\varepsilon(\nu_A; s)$ are fuzzy $\psi$ - ideals of $X$, for all t∈ Im($\mu_A$), s∈ Im($\nu_A$) with t ≥ α and s ≤ $\varepsilon$, then $A^{(T,K)}_{(\alpha,\varepsilon)} = \{(x, (\mu_A)^{T}_{\alpha}(x), (\nu_A)^{K}_{\varepsilon}(x)) \mid x \in X\}$ is a translation bifuzzy $\psi$- ideal of X.

**Proof:**

1- $\mu_A(0) \geq \mu_A(x)$ implies that $\mu_A(0) + \alpha \geq \mu_A(x) + \alpha$, for any $x \in X$, then $(\mu_A)^{T}_{\alpha}(0) \geq (\mu_A)^{T}_{\alpha}(x)$, and $\nu_A(0) \leq \nu_A(x)$, implies that

$\nu_A(0) - \varepsilon \leq \nu_A(x) - \varepsilon$ ,for any $x \in X$, then $(\nu_A)^{K}_{\varepsilon}(0) \leq (\nu_A)^{K}_{\varepsilon}(x)$.

Assume that $x, y, z \in U_\alpha(\mu; t)$ and

2- $(\mu_A)^{T}_{\alpha}$ is not a translation fuzzy $\psi$- ideal of $X$, therefore $(\mu_A)^{T}_{\alpha}(y + x) < t \leq min\{(\mu_A)^{T}_{\alpha}(y + z), (\mu_A)^{T}_{\alpha}(x - z)\}$, then $(\mu_A)(y + z) \geq t - \alpha$ and $(\mu_A)(x - z) \geq t - \alpha$ , b $(\mu_A)(y + x) < t - \alpha$ . This shows that $y + x \notin U_\alpha(\mu; t)$. This is a contradiction, and so $(\mu_A)^{T}_{\alpha}(y + x) \geq min\{(\mu_A)^{T}_{\alpha}(y + z), (\mu_A)^{T}_{\alpha}(x - z)\}$, for all x, y ∈ $X$. Hence $(\mu_A)^{T}_{\alpha}$ is a translation fuzzy $\psi$- ideal of $X$.

3- $(\nu_A)^{K}_{\varepsilon}(y + x) > s \geq max\{(\nu_A)^{K}_{\varepsilon}(y + z), (\nu_A)^{K}_{\varepsilon}(x - z)\}$, then $(\nu_A)(y + z) \leq s - \varepsilon$ and $(\nu_A)(x - z) \leq s - \varepsilon$, but $(\nu_A)(y + x) > s - \varepsilon$ . This shows that $y + x \notin L_\varepsilon(\nu; s)$. This is a contradiction, and so $(\nu_A)^{K}_{\varepsilon}(y + x) \leq max\{(\nu_A)^{K}_{\varepsilon}(y + z), (\nu_A)^{K}_{\varepsilon}(x - z)\}$, for all x, y, z ∈ $X$. Therefore, $(\nu_A)^{K}_{\varepsilon}$ is a translation anti-fuzzy $\psi$- ideal of $X$.

Hence $A^{(T,K)}_{(\alpha,\varepsilon)}$ is a translation bifuzzy $\psi$- ideal of X. □

**Theorem 4.10.**

Every translation bifuzzy $\psi$ -ideal of $\psi$ -algebra $(X; +, -, 0)$ is a translation bifuzzy $\psi$ -subalgebra of a $\psi$ -algebra $X$ .

**Proof:**

Let $(X; +, -, 0)$ be a $\psi$ -algebra and $A^{(T,K)}_{(\alpha,\varepsilon)} = \{(x, (\mu_A)^{T}_{\alpha}(x), (\nu_A)^{K}_{\varepsilon}(x)) \mid x \in X\}$ is a translation bifuzzy $\psi$- ideal of $X$.

Since A is an bifuzzy $\psi$ -ideal of $X$, then by Theorem (4.8), $U_\alpha(\mu_A; t)$ and $L_\varepsilon(\nu_A; s)$ are fuzzy $\psi$ - ideals of $X$, for all t∈ Im($\mu_A$), s∈ Im($\nu_A$) with t ≥ α and s ≤ $\varepsilon$. By Proposition (3.11), $U_\alpha(\mu_A; t)$ and $L_\varepsilon(\nu_A; s)$ are fuzzy $\psi$ - subalgebras of $X$, for all t∈ Im($\mu_A$), s∈ Im($\nu_A$) with t ≥ α and s ≤ $\varepsilon$,.

Hence $A$ is a bifuzzy $\psi$-subalgebra of $X$ by Theorem (3.10). □

**Collaroy 4.11.**

Let $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$ be an bifuzzy $\psi$ −ideal of $\psi$-algebra $(X; +, -, 0)$, then the translation bifuzzy subset A of X is bifuzzy $\psi$ − ideal of X.

**Proof:**

Since $A$ is a bifuzzy $\psi$- ideal of $X$, then $(\mu_A)^{T}_{\alpha}$ is a translation fuzzy $\psi$- ideal of X and $(\nu_A)^{K}_{\varepsilon}$ is a translation anti-fuzzy $\psi$- ideal of X by theorem (4.4), then the translation bifuzzy subset A of X is bifuzzy $\psi$ − ideal of X by Definition (4.1). □

**Proposition 4.12.**

Let $A^{(T,K)}_{(\alpha,\varepsilon)} = \{(x, (\mu_A)^{T}_{\alpha}(x), (\nu_A)^{K}_{\varepsilon}(x)) \mid x \in X\}$ be an bifuzzy $\psi$ −ideal of $\psi$-algebra $(X; +, -, 0)$, then the bifuzzy subset $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$ of X is bifuzzy $\psi -$ ideal of X.

**Proof:**

Since $A^{(T,K)}_{(\alpha,\varepsilon)}$ is a fuzzy $\psi$- ideal of $X$, then $(\mu_A)^{T}_{\alpha}$ is a translation fuzzy $\psi$- ideal of X and $(\nu_A)^{K}_{\varepsilon}$ is a translation anti-fuzzy $\psi$- ideal of X by theorem (4.3), then the bifuzzy subset A of X is bifuzzy $\psi -$ ideal of X by Definition (4.1). □

**Proposition 4.13.**

The R-intersection of any set of translation bifuzzy $\psi$- ideal of $(X; +, -, 0)$ is also translation bifuzzy $\psi$- ideal of $X$.

**Proof.**

Let $A_{i}{}^{(T,K)}_{(\alpha,\varepsilon)} = \{(x, ((\mu_A)_i)^{T}_{\alpha}(x), ((\nu_A)_i)^{K}_{\varepsilon}(x)) \mid x \in X\}$ where i ∈ Λ, be a set of a translation bifuzzy $\psi$- ideal of $X$, then

1- For any $x \in X$, $\bigcap_{i\in\Lambda}((\mu_A)_i)^{T}_{\alpha}(0) = \inf\{((\mu_A)_i)^{T}_{\alpha}(0)\} \geq \inf\{((\mu_A)_i)^{T}_{\alpha}(x)\} = \bigcap_{i\in\Lambda}((\mu_A)_i)^{T}_{\alpha}(x)$ and , then $\bigcap_{i\in\Lambda}((\mu_A)_i)^{T}_{\alpha}(0) \geq \bigcap_{i\in\Lambda}((\mu_A)_i)^{T}_{\alpha}(x)$. And

$\cup_{i\in\Lambda}((\nu_A)_i)^{K}_{\varepsilon}(0) = \sup\{((\nu_A)_i)^{K}_{\varepsilon}(0)\} \leq \sup\{((\nu_A)_i)^{K}_{\varepsilon}(x)\} = \cup_{i\in\Lambda}((\mu_A)_i)^{T}_{\alpha}(x)$ implies that, then $\cup_{i\in\Lambda}((\nu_A)_i)^{K}_{\varepsilon}(0) \leq \cup_{i\in\Lambda}((\nu_A)_i)^{K}_{\varepsilon}(x)$.

2- $x, y, z \in X$,

$\bigcap_{i\in\Lambda}((\mu_A)_i)^{T}_{\alpha}(y+x) = inf\ ((\mu_A)_i)^{T}_{\alpha}(y+x) \geq inf\{min\{((\mu_A)_i)^{T}_{\alpha}(y+z), ((\mu_A)_i)^{T}_{\alpha}(x-z)\}\} = min\{inf(((\mu_A)_i)^{T}_{\alpha}(y+z)), inf(((\mu_A)_i)^{T}_{\alpha}(x-z))\} = \min\{\bigcap_{i\in\Lambda}((\mu_A)_i)^{T}_{\alpha}(y+z), \bigcap_{i\in\Lambda}((\mu_A)_i)^{T}_{\alpha}(x-z)\}$ and

Hence $\bigcap_{i\in\Lambda}((\mu_A)_i)^{T}_{\alpha}$ is a translation fuzzy $\psi$-ideal of $X$.

3- $(\cup_{i\in\Lambda}((\nu_A)_i)^{K}_{\varepsilon})(y+x) = \sup((\nu_A)_i)^{K}_{\varepsilon})(y+x) \leq \sup\{\max\{((\nu_A)_i)^{K}_{\varepsilon})(y+z), ((\nu_A)_i)^{K}_{\varepsilon})(x-z)\}\} = \max\{\sup((\nu_A)_i)^{K}_{\varepsilon}(y+z)), \sup((\nu_A)_i)^{K}_{\varepsilon}(x-z))\} = \max\{(\cup_{i\in\Lambda}((\nu_A)_i)^{K}_{\varepsilon})(y+z), (\cup_{i\in\Lambda}((\nu_A)_i)^{K}_{\varepsilon})(x-z)\}$.

Hence $\cup_{i\in\Lambda}((\nu_A)_i)^{K}_{\varepsilon}$ is a translationanti-fuzzy $\psi$-ideal of $X$.

Hence, R-intersection of $A_{i}{}^{(T,K)}_{(\alpha,\varepsilon)}$ is a translation bifuzzy $\psi$-ideal of $X$. □

**Remark 4.14.**

The P-intresection of any sets of a translation bifuzzy $\psi$-ideal need not be a translation bifuzzy $\psi$-ideal, for example:

**Example 4.15.**

Let $X = \{0, a, b, c, d\}$ be a set with the following table:

| + | 0 | a | b | c | d |
|---|---|---|---|---|---|
| 0 | 0 | a | b | c | d |
| a | a | b | c | d | 0 |
| b | b | c | d | 0 | a |
| c | c | d | 0 | a | b |
| d | d | 0 | a | b | c |

| - | 0 | a | b | c | d |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| a | a | 0 | 0 | 0 | a |
| b | b | b | 0 | 0 | a |
| c | c | b | d | 0 | a |
| d | d | d | d | d | 0 |

Then $(X; *, 0)$ is an $\psi$-algebra. It is easy to show that I ={0, c} and J ={0, d} are $\psi$- ideals of $X$.

We defined two cubic set $A_1 = \{(x, \mu_{A1}(x), \nu_{A1}(x)) \mid x \in X\}$ and $A_2 = \{(x, \mu_{A2}(x), \nu_{A2}(x)) \mid x \in X\}$ of X by :-

$$\mu_{A1}(x) = \begin{cases} 0.8, & \text{if } x \in \{0, c\}, \\ 0.7, & \text{if } x \in \{a, b\}, \\ 0.6, & \text{otherwise} \end{cases} \qquad \nu_{A1}(x) = \begin{cases} 0.2, & \text{if } x \in \{0, c\}, \\ 0.6, & \text{if } x \in \{a, b\}, \\ 0.4, & \text{otherwise} \end{cases}$$

$$\mu_{A2}(x) = \begin{cases} 0.7, & \text{if} x \in \{0, d\}, \\ 0.2, & \text{otherwise.} \end{cases} \quad \text{and} \quad \nu_{A2}(x) = \begin{cases} 0.1, & \text{if} x \in \{0, d\}, \\ 0.4, & \text{otherwise.} \end{cases}$$

Then $A_1$ and $A_2$ are bifuzzy $\psi$- ideal of X , but P-intersection of $A_1 \cap A_2$ are not a bifuzzy $\psi$- ideals of X .

Since

$(\cap (\mu_A)^{T}_{\alpha})(c+d) = (\cap (\mu_A)^{T}_{\alpha})(b) = \min\{0.7, 0.2\} = 0.2 \ngeq 0.6 = \min\{(\cap (\mu_A)^{T}_{\alpha})(c+b), (\cap (\mu_A)^{T}_{\alpha})(d-b)\} = \min\{(\cap (\mu_A)^{T}_{\alpha})(0), (\cap (\mu_A)^{T}_{\alpha})(d)\} = \{\min\{\min\{0.8, 0.7\}, \min\{0.6, 0.7\}\}$ and

$(\cap (\nu_A)^{K}_{\varepsilon})(c+d) = (\cup \nu_{Ai})(b) = \min\{0.3, 0.5\} = 0.3 \nleq 0.1 = \max\{(\cup (\nu_A)^{K}_{\varepsilon})(c+b), (\cup (\nu_A)^{K}_{\varepsilon})(d-b)\} = \max\{(\cup (\nu_A)^{K}_{\varepsilon})(0), (\cup (\nu_A)^{K}_{\varepsilon})(d)\} = \max\{\min\{0.2, 0.1\}, \min\{0.4, 0.1\}\}$.

**Proposition 4.16.**

Let $A_{i}{}^{(T,K)}_{(\alpha,\varepsilon)} = \{(x, ((\mu_A)_i)^{T}_{\alpha}(x), ((\nu_A)_i)^{K}_{\varepsilon}(x)) \mid x \in X\}$ where i ∈ Λ, be a set of a translation bifuzzy $\psi$-ideal of $\psi$-algebra $(X; +, -, 0)$, where i ∈ Λ, $inf\{max\{(\mu_A)_i(x), (\mu_A)_i(y)\}\} = max\{inf\ (\mu_A)_i(x), inf\ (\mu_A)_i(y)\}$, for all x ∈ X, then the P-intresection of $A_{i}{}^{(T,K)}_{(\alpha,\varepsilon)}$ is also a translation bifuzzy $\psi$-ideal of $X$.

**Proof.**

Let $A_{i_{(\alpha,\varepsilon)}}^{(T,K)} = \{(x, ((\mu_A)_i)_\alpha^T(x), ((v_A)_i)_\varepsilon^K(x)) \mid x \in X\}$ where $i \in \Lambda$, be a set of a translation bifuzzy $\psi$-ideal of $X$ and $x, y, z \in X$, then

1- $\bigcap_{i\in\Lambda}((\mu_A)_i)_\alpha^T(0) \geq \bigcap_{i\in\Lambda}((\mu_A)_i)_\alpha^T(x)$ and $\cap_{i\in\Lambda}((v_A)_i)_\varepsilon^K(0) \leq \cap_{i\in\Lambda}((v_A)_i)_\varepsilon^K(x)$.

2- $(\bigcap_{i\in\Lambda}((\mu_A)_i)_\alpha^T)(y+x) = inf\,(((\mu_A)_i)_\alpha^T)(y+x)$
$\geq inf\,\{min\{(((\mu_A)_i)_\alpha^T)(y+z), (((\mu_A)_i)_\alpha^T)(x-z)\}\}$
$= min\{inf(((\mu_A)_i)_\alpha^T(y+z)), inf(((\mu_A)_i)_\alpha^T(x-z))\}$
$= \min\{\bigcap_{i\in\Lambda}((\mu_A)_i)_\alpha^T(y+z), \bigcap_{i\in\Lambda}((\mu_A)_i)_\alpha^T(x-z)\}$.

Hence $\bigcap_{i\in\Lambda}((\mu_A)_i)_\alpha^T$ is a fuzzy $\psi$-subalgebra of $X$.

3- $(\cap_{i\in\Lambda}((v_A)_i)_\varepsilon^K)(y+x) = inf\,((v_A)_i)_\varepsilon^K)(y+x)$
$\leq \inf\{\max\{(v_A)_i)_\varepsilon^K(y+z), (v_A)_i)_\varepsilon^K(x-z)\}\}$
$= \max\{\inf((v_A)_i)_\varepsilon^K(y+z)), \inf((v_A)_i)_\varepsilon^K(x-z))\}$
$= \max\{\cap_{i\in\Lambda}((v_A)_i)_\varepsilon^K(y+z), \cap_{i\in\Lambda}((v_A)_i)_\varepsilon^K(x-z)\}$.

Hence $\cap_{i\in\Lambda}((v_A)_i)_\varepsilon^K$ is a translation anti-fuzzy $\psi$-ideal of $X$.
Hence, P-intersection of $A_{i_{(\alpha,\varepsilon)}}^{(T,K)}$ is a translation bifuzzy $\psi$-ideal of $X$. □

**Proposition 4.17.**

Let $A_{i_{(\alpha,\varepsilon)}}^{(T,K)} = \{(x, ((\mu_A)_i)_\alpha^T(x), ((v_A)_i)_\varepsilon^K(x)) \mid x \in X\}$ where $i \in \Lambda$, be a set of the translation bifuzzy $\psi$-ideal of $\psi$-algebra $(X;+,-,0)$, where $i \in \Lambda$, $sup\{min\{(\mu_A)_i(x), (\mu_A)_i(y)\}\} = min\{sup\,(\mu_A)_i(x), sup\,(\mu_A)_i(y)\}$, for all $x \in X$, then the P-union of $A_{i_{(\alpha,\varepsilon)}}^{(T,K)}$ is also a translation bifuzzy $\psi$-ideal of $X$.

**Proof.**

Let $A_{i_{(\alpha,\varepsilon)}}^{(T,K)} = \{(x, ((\mu_A)_i)_\alpha^T(x), ((v_A)_i)_\varepsilon^K(x)) \mid x \in X\}$ where $i \in \Lambda$, be a set of a translation bifuzzy $\psi$-ideal of $X$ and $x, y, z \in X$, then

1- $\cup_{i\in\Lambda}((\mu_A)_i)_\alpha^T\mu_A(0) \geq \cup_{i\in\Lambda}((\mu_A)_i)_\alpha^T(x)$ and $\cup_{i\in\Lambda}((v_A)_i)_\varepsilon^K(0) \leq \cup_{i\in\Lambda}((v_A)_i)_\varepsilon^K(x)$.

2- $(\cup_{i\in\Lambda}((\mu_A)_i)_\alpha^T)(y+x) = sup\,(((\mu_A)_i)_\alpha^T)(y+x)$
$\geq sup\,\{min\{(((\mu_A)_i)_\alpha^T)(y+z), (((\mu_A)_i)_\alpha^T)(x-z)\}\}$
$= min\{sup(((\mu_A)_i)_\alpha^T(y+z)), sup(((\mu_A)_i)_\alpha^T(x-z))\}$
$= \min\{(\cup_{i\in\Lambda}((\mu_A)_i)_\alpha^T)(y+z), (\cup_{i\in\Lambda}((\mu_A)_i)_\alpha^T)(x-z)\}$ and

Hence $(\cup_{i\in\Lambda}((\mu_A)_i)_\alpha^T)$ is a translation fuzzy $\psi$-ideal of $X$.

3- $(\cup_{i\in\Lambda}((v_A)_i)_\varepsilon^K)(y+x) = \sup((v_A)_i)_\varepsilon^K)(y+x)$
$\leq \sup\{\max\{((v_A)_i)_\varepsilon^K(y+z), ((v_A)_i)_\varepsilon^K(x-z)\}\}$
$= \max\{\sup((v_A)_i)_\varepsilon^K(y+z), \sup((v_A)_i)_\varepsilon^K(x-z)\}$
$= \max\{(\cup_{i\in\Lambda}((v_A)_i)_\varepsilon^K)(y+z), (\cup_{i\in\Lambda}((v_A)_i)_\varepsilon^K v_{Ai})(x-z)\}$.

Hence $(\cup_{i\in\Lambda}((v_A)_i)_\varepsilon^K)$ is a translation anti-fuzzy $\psi$-ideal of $X$.

Hence, P-union of $A_{i_{(\alpha,\varepsilon)}}^{(T,K)}$ is a translation bifuzzy $\psi$-ideal of $X$. □

**Remark 4.18.**

The R-union of any sets of translation bifuzzy $\psi$ -ideal need not be a translation bifuzzy $\psi$ -ideal, for example:

**Example 4.20.**

Let $X$ = {0, a, b, c, d} be a set with the following table:

| + | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 |
| 1 | 1 | 2 | 3 | 4 | 0 |
| 2 | 2 | 3 | 4 | 0 | 1 |
| 3 | 3 | 4 | 0 | 1 | 2 |
| 4 | 4 | 0 | 1 | 2 | 3 |

| - | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 | 1 |
| 2 | 2 | 2 | 0 | 0 | 1 |
| 3 | 3 | 2 | 4 | 0 | 1 |
| 4 | 4 | 4 | 4 | 4 | 0 |

Then $(X;*,0)$ is an $\psi$-algebra. It is easy to show that I ={0, 3} and J ={0, 4} are $\psi$- ideals of $X$.

Then $A_1$ and $A_2$ are bifuzzy $\psi$- ideal of X , but R-union of $A_1 \cup A_2$ are not bifuzzy $\psi$- ideals of X . Since We defined two cubic set $A_1 = \{(x, \mu_{A1}(x), v_{A1}(x)) \mid x \in X\}$ and $A_2 = \{(x, \mu_{A2}(x), v_{A2}(x)) \mid x \in X\}$ of X by :-

$$\mu_{A1}(x) = \begin{cases} 0.7, & \text{if } x \in \{0,3\}, \\ 0.4, & \text{if } x \in \{1,2\}, \\ 0.5, & \text{otherwise} \end{cases} \quad v_{A1}(x) = \begin{cases} 0.1, & \text{if } x \in \{0,3\}, \\ 0.5, & \text{if } x \in \{1,2\}, \\ 0.3, & \text{otherwise} \end{cases}$$

$$\mu_{A2}(x) = \begin{cases} 0.6, & \text{if } x \in \{0,4\}, \\ 0.1, & \text{otherwise.} \end{cases} \quad \text{and} \quad v_{A2}(x) = \begin{cases} 0.1, & \text{if } x \in \{0,4\}, \\ 0.3, & \text{otherwise.} \end{cases}$$

$(\cup ((\mu_A)_i)^{T}_{\alpha})(2+4) = (\cup ((\mu_A)_i)^{T}_{\alpha})(1) = \max\{0.4, 0.1\} = 0.4 \ngeq 0.6 = \min\{(\cup ((\mu_A)_i)^{T}_{\alpha})(2+1), (\cup ((\mu_A)_i)^{T}_{\alpha})(4-1)\} = \min\{(\cup ((\mu_A)_i)^{T}_{\alpha})(3), (\cup ((\mu_A)_i)^{T}_{\alpha})(4)\} = \{\min\{\max\{0.7, 0.1\}, \max\{0.5, 0.6\}\}$ and

$(\cap (v_A)_i)^{K}_{\varepsilon})(2+4) = (\cap (v_A)_i)^{K}_{\varepsilon})(1) = \min\{0.5, 0.3\} = 0.3 \nleq 0.1 = \max\{(\cap (v_A)_i)^{K}_{\varepsilon})(2+1), (\cap (v_A)_i)^{K}_{\varepsilon})(4-1)\} = \max\{(\cap (v_A)_i)^{K}_{\varepsilon})(3), (\cap (v_A)_i)^{K}_{\varepsilon})(4)\} = \max\{\min\{0.1, 0.3\}, \min\{0.3, 0.1\}\}$.

**Proposition 4.21.**
Let $A_{i(\alpha,\varepsilon)}^{(T,K)} = \{(x, ((\mu_A)_i)^{T}_{\alpha}(x), ((v_A)_i)^{K}_{\varepsilon}(x)) \mid x \in X\}$ where i ∈ Λ, be a set of a translation bifuzzy $\psi$-ideal of $\psi$-algebra $(X; +, -, 0)$, where i ∈ Λ,
$sup\{min\{((\mu_A)_i)^{T}_{\alpha}(x), ((\mu_A)_i)^{T}_{\alpha}(y)\}\} = min\{sup\, (\mu_A)^{T}_{\alpha}(x), sup\, (\mu_A)^{T}_{\alpha}(y)\}$ and $inf\{max\{(v_A)_i)^{K}_{\varepsilon}(x), (v_A)_i)^{K}_{\varepsilon}(y)\}\} = max\{inf\ (v_A)_i)^{K}_{\varepsilon}(x), inf\ (v_A)_i)^{K}_{\varepsilon}(y)\}$, for all x ∈ X, then the R-union of $A_{i(\alpha,\varepsilon)}^{(T,K)}$ is also a translation bifuzzy $\psi$-ideal of $X$.

**Proof.**

Let $A_{i(\alpha,\varepsilon)}^{(T,K)} = \{(x, ((\mu_A)_i)^{T}_{\alpha}(x), ((v_A)_i)^{K}_{\varepsilon}(x)) \mid x \in X\}$ where i ∈ Λ, be a set of a translation bifuzzy $\psi$-ideal of $X$ and x, y, z ∈ $X$, then

1- $\cup_{i\in\Lambda} ((\mu_A)_i)^{T}_{\alpha}(0) \geq \cup_{i\in\Lambda} ((\mu_A)_i)^{T}_{\alpha}(x)$ and $\cap_{i\in\Lambda} ((v_A)_i)^{K}_{\varepsilon}(0) \leq \cap_{i\in\Lambda} ((v_A)_i)^{K}_{\varepsilon}(x)$.

2- $(\cup_{i\in\Lambda} ((\mu_A)_i)^{T}_{\alpha})(y+x) = sup\,(((\mu_A)_i)^{T}_{\alpha})(y+x)$
$\geq sup\,\{min\{(((\mu_A)_i)^{T}_{\alpha})(y+z), (((\mu_A)_i)^{T}_{\alpha})(x-z)\}\}$
$= min\{sup(((\mu_A)_i)^{T}_{\alpha}(y+z)), sup(((\mu_A)_i)^{T}_{\alpha}(x-z))\}$
$= \min\{(\cup_{i\in\Lambda} ((\mu_A)_i)^{T}_{\alpha})(y+z), (\cup_{i\in\Lambda} ((\mu_A)_i)^{T}_{\alpha})(x-z)\}$ and

Hence $(\cup_{i\in\Lambda} ((\mu_A)_i)^{T}_{\alpha})$ is a translation fuzzy $\psi$-ideal of $X$.

3- $(\cap_{i\in\Lambda} ((v_A)_i)^{K}_{\varepsilon})(y+x) = \inf(((v_A)_i)^{K}_{\varepsilon})(y+x)$
$\leq \inf\{\max\{((v_A)_i)^{K}_{\varepsilon})(y+z), ((v_A)_i)^{K}_{\varepsilon})(x-z)\}\}$
$= \max\{\inf((v_A)_i)^{K}_{\varepsilon})(y+z), \inf((v_A)_i)^{K}_{\varepsilon})(x-z)\}$
$= \max\{(\cap_{i\in\Lambda} ((v_A)_i)^{K}_{\varepsilon})(y+z), (\cap_{i\in\Lambda} ((v_A)_i)^{K}_{\varepsilon})(x-z)\}$.

Hence $(\cap_{i\in\Lambda} ((v_A)_i)^{K}_{\varepsilon})$ is a translation anti-fuzzy $\psi$-ideal of $X$.
Hence, R-union of $A_{i(\alpha,\varepsilon)}^{(T,K)}$ is a translation bifuzzy $\psi$-ideal of $X$. □

## 5. Homomorphism of Translation Bifuzzy $\psi$ -algebras

In this section, we will present some results on images and pre-images of translation bifuzzy $\psi$- subalgebras and translation $\psi$-ideals of $\psi$-algebra.

**Definition 5.1.**

Let $f: (X; +, -, 0) \to (Y; +', -'0')$ be a function of two $\psi$-algebras, let A and B are two bifuzzy subsets of X and Y where, $A_{(\alpha,\varepsilon)}^{(T,K)} = \{(x, (\mu_A)^{T}_{\alpha}(x), (v_A)^{K}_{\varepsilon}(x)) \mid x \in X\}$ and $B_{(\alpha,\varepsilon)}^{(T,K)} = \{(y, (\mu_B)^{T}_{\alpha}(y), (v_B)^{K}_{\varepsilon}(y)) \mid y \in Y\}$, then the image of $A_{(\alpha,\varepsilon)}^{(T,K)}$ under $f$ is defined as

$f(A_{(\alpha,\varepsilon)}^{(T,K)}) = \{(y, (\mu_A)^{T}_{\alpha f(A)}(y), (v_A)^{K}_{\varepsilon f(A)}(y)) \mid y \in Y\}$ such that

$$(\mu_A)^{T}_{\alpha f(A)}(y) = \begin{cases} sup_{x\in f^{-1}(y)}\, (\mu_A)^{T}_{\alpha}(x) & \text{if } f^{-1}(y) = \{x \in X, f(x) = y\} \neq \emptyset \\ 0 & otherwise \end{cases},$$

$$(\nu_A)^{K}_{\varepsilon f(A)}(y) = \begin{cases} inf_{x\in f^{-1}(y)}\,(\nu_A)^{K}_{\varepsilon}(x) & \text{if } f^{-1}(y)=\{x\in X, f(x)=y\}\neq\emptyset \\ 0 & otherwise \end{cases}.$$

The pre-image of $B^{(T,K)}_{(\alpha,\varepsilon)}$ is

$$f^{-1}\left(B^{(T,K)}_{(\alpha,\varepsilon)}\right) = \{(x, (\mu_B)^{T}_{\alpha f^{-1}(B)}(x), (\nu_B)^{K}_{\varepsilon f^{-1}(B)}(x)) \mid x\in X\}$$
such that

$\mu_{f^{-1}(B^{(T,K)}_{(\alpha,\varepsilon)})}(x) = (\mu_B)^{T}_{\alpha}(f(x))$ and $(\nu_A)^{K}_{\varepsilon f^{-1}(B)}(x) = (\nu_A)^{K}_{\varepsilon}(f(x))$, for all x ∈ X.

**Theorem 5.2.**

Let $f: (X; +, -, 0) \rightarrow (Y; +', -'0')$ be a homomorphism of two $\psi$ −algebras and $B^{(T,K)}_{(\alpha,\varepsilon)}$ be a bifuzzy $\psi$-subalgebra of Y, then the pre-image $f^{-1}\left(B^{(T,K)}_{(\alpha,\varepsilon)}\right) = \{(x, (\mu_B)^{T}_{\alpha f^{-1}(B)}(x), (\nu_B)^{K}_{\varepsilon f^{-1}(B)}(x)) \mid x\in X\}$ is a translation bifuzzy $\psi$-subalgebra of X.

**Proof :** Let x , y ∈ X,

$(\mu_B)^{T}_{\alpha f^{-1}(B)}(x+y) = (\mu_B)^{T}_{\alpha}(f(x+y))$

$= (\mu_B)^{T}_{\alpha}(f(x)+'f(y))$

$\geq \min\{(\mu_B)^{T}_{\alpha}(f(x)), (\mu_B)^{T}_{\alpha}(f(y))$

$= \min\{(\mu_B)^{T}_{\alpha f^{-1}(B)}(x), (\mu_B)^{T}_{\alpha f^{-1}(B)}(y)\}$.

Similarity, $\mu_{f^{-1}(B)}(x-y) \geq \min\{\mu_{f^{-1}(B)}(x), \mu_{f^{-1}(B)}(y)\}$.

Also, $(\nu_B)^{K}_{\varepsilon f^{-1}(B)}(x+y) = (\nu_B)^{K}_{\varepsilon}(f(x+y))$

$= (\nu_B)^{K}_{\varepsilon}(f(x)+'f(y))$

$\leq \max\{(\nu_B)^{K}_{\varepsilon}(f(x)), (\nu_B)^{K}_{\varepsilon}(f(y))$

$=\max\{(\nu_B)^{K}_{\varepsilon f^{-1}(B)}(x), (\nu_B)^{K}_{\varepsilon f^{-1}(B)}(y)\}$.

Similarity, $(\nu_B)^{K}_{\varepsilon f^{-1}(B)}(x-y) \geq \min\{(\nu_B)^{K}_{\varepsilon f^{-1}(B)}(x), (\nu_B)^{K}_{\varepsilon f^{-1}(B)}(y)\}$.

Then $f^{-1}\left(B^{(T,K)}_{(\alpha,\varepsilon)}\right) = \{(x, (\mu_B)^{T}_{\alpha f^{-1}(B)}(x), (\nu_B)^{K}_{\varepsilon f^{-1}(B)}(x)) \mid x\in X\}$ is a translation bifuzzy $\psi$-subalgebra of X. ⌂

**Theorem 5.3.**

Let $f: (X; +, -, 0) \rightarrow (Y; +', -'0')$ be an epimorphism of two $\psi$ -algebras and let $A^{(T,K)}_{(\alpha,\varepsilon)} = \{(x, (\mu_A)^{T}_{\alpha}(x), (\nu_A)^{K}_{\varepsilon}(x)) \mid x\in X\}$ be a translation bifuzzy $\psi$-subalgebra of X with sup-inf property, then the image

$f(A^{(T,K)}_{(\alpha,\varepsilon)}) = \{(y, (\mu_A)^{T}_{\alpha f(A)}(y), (\nu_A)^{K}_{\varepsilon f(A)}(y)) \mid y\in Y\}$ is a bifuzzy $\psi$-subalgebra of Y.

**Proof :**

Let a , b ∈ Y with $x = f^{-1}$ (a) and $y = f^{-1}$ (b) such that

$(\mu_A)^{T}_{\alpha}(x) = \sup_{t\in f^{-1}(a)}(\mu_A)^{T}_{\alpha}(t)$ , $(\mu_A)^{T}_{\alpha}(y) = \sup_{t\in f^{-1}(b)}(\mu_A)^{T}_{\alpha}(t)$,

$(\nu_A)^{K}_{\varepsilon}(x) = \inf_{t\in f^{-1}(a)}(\nu_A)^{K}_{\varepsilon}(t)$ , $(\nu_A)^{K}_{\varepsilon}(y) = \inf_{t\in f^{-1}(b)}(\nu_A)^{K}_{\varepsilon}(t)$.

Now, by Definition (5.1),

$(\mu_A)^{T}_{\alpha f(A)}(a+b) = \sup_{t\in f^{-1}(a+b)}(\mu_A)^{T}_{\alpha}(t)$

$= (\mu_A)^{T}_{\alpha}(x+y)$

$\geq \min\{(\mu_A)^{T}_{\alpha}(x), (\mu_A)^{T}_{\alpha}(y)\}$

$= \min\{\sup_{t\in f^{-1}(a)}(\mu_A)^{T}_{\alpha}(t), \sup_{t\in f^{-1}(b)}(\mu_A)^{T}_{\alpha}(t)\}$

$= \min\{(\mu_A)^{T}_{\alpha f(A)}(a), (\mu_A)^{T}_{\alpha f(A)}(b)\}$.

Similarity, $(\mu_A)^{T}_{\alpha f(A)}(a-b) \geq \min\{(\mu_A)^{T}_{\alpha f(A)}(a), (\mu_A)^{T}_{\alpha f(A)}(b)\}$.

Also, $(\nu_A)^{K}_{\varepsilon f(A)}(a+b) = \inf_{t\in f^{-1}(a+b)}(\nu_A)^{K}_{\varepsilon A}(t)$

$= (\nu_A)^{K}_{\varepsilon}(x+y)$

$\leq \max\{(\nu_A)^{K}_{\varepsilon}(x), (\nu_A)^{K}_{\varepsilon}(y)\}$

$= \max\{\inf_{t\in f^{-1}(a)}(\nu_A)^{K}_{\varepsilon}(t), \inf_{t\in f^{-1}(b)}(\nu_A)^{K}_{\varepsilon}(t)\}$

$= \max\{(\nu_A)^{K}_{\varepsilon f(A)}(a), \nu(\nu_A)^{K}_{\varepsilon f(A)}(b)\}$.

Similarity, $(\nu_A)^{K}_{\varepsilon f(A)}(a-b) \leq \max\{(\nu_A)^{K}_{\varepsilon f(A)}(a), (\nu_A)^{K}_{\varepsilon f(A)}(b)\}$.

Hence the image $f(A_{(\alpha,\varepsilon)}^{(T,K)}) = \{(y, (\mu_A)_{\alpha f(A)}^{T}(y), (\nu_A)_{\varepsilon f(A)}^{K}(y)) \mid y \in Y\}$ is a bifuzzy a translation $\psi$-subalgebra of Y. ⌂

**Theorem 5.4:**

Let $f: (X; +, -, 0) \to (Y; +', -'0')$ be a homomorphism of two $\psi$ −algebras and let $B_{(\alpha,\varepsilon)}^{(T,K)}$ be a translation bifuzzy $\psi$- ideal in Y, then the pre-image

$f^{-1}\left(B_{(\alpha,\varepsilon)}^{(T,K)}\right) = \{(x, (\mu_B)_{\alpha f^{-1}(B)}^{T}(x), (\nu_B)_{\varepsilon f^{-1}(B)}^{K}(x)) \mid x \in X\}$ is a bifuzzy $\psi$-ideal of X.

**Proof :**

Let x , y , z ∈ X, $(\mu_B)_{\alpha}^{T}(f(0)) = (\mu_B)_{\alpha}^{T}(0')$ and $(\nu_B)_{\varepsilon}^{K}(y) = (\nu_B)_{\varepsilon}^{K}(f(x))$

$(\mu_B)_{\alpha f^{-1}(B)}^{T}(0) = (\mu_B)_{\alpha}^{T}(f(0)) = (\mu_B)_{\alpha}^{T}(0') \geq (\mu_B)_{\alpha}^{T}(f(x)) = (\mu_B)_{\alpha f^{-1}(B)}^{T}(x)$,

$(\nu_B)_{\varepsilon f^{-1}(B)}^{K}(0) = (\nu_B)_{\varepsilon}^{K}(f(0)) = (\nu_B)_{\varepsilon}^{K}(0') \leq (\nu_B)_{\varepsilon}^{K}(f(x)) = (\nu_B)_{\varepsilon f^{-1}(B)}^{K}(x)$.

$(\mu_B)_{\alpha f^{-1}(B)}^{T}(y+x) = (\mu_B)_{\alpha}^{T}(f(y+x))$

$\geq \min\{(\mu_B)_{\alpha}^{T}(f(y+z), (\mu_B)_{\alpha}^{T}f(x-z))\}$

$= \min\{(\mu_B)_{\alpha f^{-1}(B)}^{T}(y+z), (\mu_B)_{\alpha f^{-1}(B)}^{T}(x-z)\}$.

Also, $(\nu_B)_{\varepsilon f^{-1}(B)}^{K}(y+x) = (\nu_B)_{\varepsilon}^{K}(f(y+x))$

$\leq \max\{(\nu_B)_{\varepsilon}^{K}(f(y+z), (\nu_B)_{\varepsilon}^{K}f(x-z))\}$

$= \max\{(\nu_B)_{\varepsilon f^{-1}(B)}^{K}(y+z), (\nu_B)_{\varepsilon f^{-1}(B)}^{K}(x-z)\}$.

Then the pre-image $f^{-1}\left(B_{(\alpha,\varepsilon)}^{(T,K)}\right) = \{(x, (\mu_B)_{\alpha f^{-1}(B)}^{T}(x), (\nu_B)_{\varepsilon f^{-1}(B)}^{K}(x)) \mid x \in X\}$ is a translation bifuzzy $\psi$- ideal in X. ⌂

**Theorem 5.5.**

Let $f: (X; +, -, 0) \to (Y; +', -'0')$ be an epimorphism of two $\psi$ -algebras and let $A_{(\alpha,\varepsilon)}^{(T,K)} = \{(x, (\mu_A)_{\alpha}^{T}(x), (\nu_A)_{\varepsilon}^{K}(x)) \mid x \in X\}$ be a translation bifuzzy $\psi$-ideal of X with sup-inf property, then the image $f(A_{(\alpha,\varepsilon)}^{(T,K)}) = \{(y, (\mu_A)_{\alpha f(A)}^{T}(y), (\nu_A)_{\varepsilon f(A)}^{K}(y)) \mid y \in Y\}$ is a translation bifuzzy $\psi$- ideal in Y.

**Proof:**

Let a, b, c ∈ Y with $x = f^{-1}(a)$, $y = f^{-1}(b)$ and $z = f^{-1}(c)$ such that

$(\mu_A)_{\alpha}^{T}(x) = \sup_{t\in f^{-1}(a)} (\mu_A)_{\alpha}^{T}(t)$ , $(\mu_A)_{\alpha}^{T}(y) = \sup_{t\in f^{-1}(b)} (\mu_A)_{\alpha}^{T}(t)$ and $(\mu_A)_{\alpha}^{T}(z) = \sup_{t\in f^{-1}(c)} (\mu_A)_{\alpha}^{T}(t)$,

$(\nu_A)_{\varepsilon}^{K}(x) = \inf_{t\in f^{-1}(a)} (\nu_A)_{\varepsilon}^{K}(t)$ , $(\nu_A)_{\varepsilon}^{K}(y) = \inf_{t\in f^{-1}(b)} (\nu_A)_{\varepsilon}^{K}(t)$ and $(\nu_A)_{\varepsilon}^{K}(z) = \inf_{t\in f^{-1}(c)} (\nu_A)_{\varepsilon}^{K}(t)$.

$(\mu_A)_{\alpha f(A)}^{T}(0) = \sup_{t\in f^{-1}(0')} (\mu_A)_{\alpha}^{T}(t) = (\mu_A)_{\alpha}^{T}(f(0)) \geq (\mu_A)_{\alpha}^{T}(f(x)) = (\mu_A)_{\alpha f(A)}^{T}(x)$,

$(\nu_A)_{\varepsilon f(A)}^{K}(0) = \inf_{t\in f^{-1}(0')} (\nu_A)_{\varepsilon}^{K}(t) = (\nu_A)_{\varepsilon}^{K}(f(0)) \leq (\nu_A)_{\varepsilon}^{K}(f(x)) = (\nu_A)_{\varepsilon f(A)}^{K}(x)$.

Now, by Definition (5.1),

$(\mu_A)_{\alpha f(A)}^{T}(b+a) = \sup_{t\in f^{-1}(b+a)} (\mu_A)_{\alpha}^{T}(t)$

$= (\mu_A)_{\alpha}^{T}(y+x)$

$\geq \min\{(\mu_A)_{\alpha}^{T}(y+z), (\mu_A)_{\alpha}^{T}(x-z)\}$

$= \min\left\{\sup_{t\in f^{-1}(b+c)} (\mu_A)_{\alpha}^{T}(t), \sup_{t\in f^{-1}(a-c)} (\mu_A)_{\alpha}^{T}(t)\right\}$

$= \min\{(\mu_A)_{\alpha f(A)}^{T}(b+c), (\mu_A)_{\alpha f(A)}^{T}(a+c)\}$.

Also, $(\nu_A)_{\varepsilon f(A)}^{K}(b+a) = \inf_{t\in f^{-1}(b+a)} (\nu_A)_{\varepsilon}^{K}(t)$

$= (\nu_A)_{\varepsilon}^{K}(y+x)$

$\leq \max\{(\nu_A)_{\varepsilon}^{K}(y+z), (\nu_A)_{\varepsilon}^{K}(x-z)\} =$

$\max\left\{\inf_{t\in f^{-1}(b+c)} (\nu_A)_{\varepsilon}^{K}(t), \inf_{t\in f^{-1}(a-c)} (\nu_A)_{\varepsilon}^{K}(t)\right\}$

$= \max\{(\nu_A)_{\varepsilon f(A)}^{K}(b+c), (\nu_A)_{\varepsilon f(A)}^{K}(a-c)\}$.

Then the image $f(A_{(\alpha,\varepsilon)}^{(T,K)}) = \{(y, (\mu_A)_{\alpha f(A)}^{T}(y), (\nu_A)_{\varepsilon f(A)}^{K}(y)) \mid y \in Y\}$ is a translation bifuzzy $\psi$- ideal in Y. ⌂

## 6. Cartesian product on bifuzzy $\psi$ -algebra

In this section we introduce the notions of Cartesian product of bifuzzy $\psi$ -subalgebras and bifuzzy ideals in a $\psi$ -algebra.

**Definition 6.1.**

Let $A^{(T,K)}_{(\alpha,\varepsilon)} = \{(x, (\mu_A)^T_\alpha(x), (\nu_A)^K_\varepsilon(x)) \mid x \in X\}$ be a translation bifuzzy subset of X and $B^{(T,K)}_{(\alpha,\varepsilon)} = \{(y, (\mu_B)^T_\alpha(y), (\nu_B)^K_\varepsilon(y)) \mid y \in Y\}$ be a translation bifuzzy subset of Y. The Cartesian product of A and B is defined as A×B = (X×Y, $(\mu_A)^T_\alpha \times (\mu_B)^T_\alpha$, $(\nu_A)^K_\varepsilon \times (\nu_B)^K_\varepsilon$) where $(\mu_A)^T_\alpha \times (\mu_B)^T_\alpha$: X×Y→[0,1] and $(\nu_A)^K_\varepsilon \times (\nu_B)^K_\varepsilon$: X×Y→[0,1] ∀ x ∈ X, y ∈ Y, such that $((\mu_A)^T_\alpha \times (\mu_B)^T_\alpha)(x,y) = \min\{(\mu_A)^T_\alpha(x), (\mu_B)^T_\alpha(y)\}$ and $((\nu_A)^K_\varepsilon \times (\nu_B)^K_\varepsilon)(x,y) = \max\{(\nu_A)^K_\varepsilon(x), (\nu_B)^K_\varepsilon(y)\}$

**Theorem 6.2.**

Let $A^{(T,K)}_{(\alpha,\varepsilon)} = \{(x, (\mu_A)^T_\alpha(x), (\nu_A)^K_\varepsilon(x)) \mid x \in X\}$ be a translation bifuzzy $\psi$ -subalgebra of X and $B^{(T,K)}_{(\alpha,\varepsilon)} = \{(y, (\mu_B)^T_\alpha(y), (\nu_B)^K_\varepsilon(y)) \mid y \in Y\}$ be a translation bifuzzy $\psi$ -subalgebra of Y, then $A^{(T,K)}_{(\alpha,\varepsilon)} \times B^{(T,K)}_{(\alpha,\varepsilon)}$ is a translation bifuzzy $\psi$ -subalgebras of X×Y.

**Proof:**

Let $(x_1, y_1) \in X \times Y$ and $(x_2, y_2) \in X \times Y$, then

$((\mu_A)^T_\alpha \times (\mu_B)^T_\alpha)((x_1, y_1) + (x_2, y_2)) = ((\mu_A)^T_\alpha \times (\mu_B)^T_\alpha)((x_1+x_2), (y_1+y_2))$

$= \min\{(\mu_A)^T_\alpha(x_1+x_2), (\mu_B)^T_\alpha(y_1+y_2)\}$

$\geq \min\{\min((\mu_A)^T_\alpha(x_1), (\mu_A)^T_\alpha(x_2)), \min((\mu_B)^T_\alpha(y_1), (\mu_B)^T_\alpha(y_2))\}$

$= \min(\min((\mu_A)^T_\alpha(x_1), (\mu_B)^T_\alpha(y_1)), \min((\mu_A)^T_\alpha(x_2), (\mu_B)^T_\alpha(y_2)))$

$= \min\{((\mu_A)^T_\alpha \times (\mu_B)^T_\alpha)(x_1, y_1), ((\mu_A)^T_\alpha \times (\mu_B)^T_\alpha)(x_2, y_2)\}$,

Similarity, $((\mu_A)^T_\alpha \times (\mu_B)^T_\alpha)((x_1, y_1) - (x_2, y_2)) \geq min((\mu_A)^T_\alpha \times (\mu_B)^T_\alpha)(x_1, y_1), ((\mu_A)^T_\alpha \times (\mu_B)^T_\alpha)(x_2, y_2)\}$,

And $((\nu_A)^T_\alpha \times (\nu_B)^T_\alpha)((x_1, y_1) + (x_2, y_2)) = ((\nu_A)^T_\alpha \times (\nu_B)^T_\alpha)((x_1+x_2), (y_1+y_2))$

$= \max\{(\nu_A)^K_\varepsilon(x_1+x_2), (\nu_B)^K_\varepsilon(y_1+y_2)\}$

$\leq \max(\max((\mu_A)^T_\alpha(x_1), (\nu_A)^K_\varepsilon(x_2)), \max((\nu_B)^K_\varepsilon(y_1), (\nu_B)^K_\varepsilon(y_2)))$

$= \max(\max((\nu_A)^K_\varepsilon(x_1), (\nu_B)^K_\varepsilon(y_1)), \max((\nu_A)^K_\varepsilon(x_2), (\nu_B)^K_\varepsilon(y_2)))$

$= \max\{((\nu_A)^T_\alpha \times (\nu_B)^T_\alpha)(x_1, y_1), ((\nu_A)^T_\alpha \times (\nu_B)^T_\alpha)(x_2, y_2)\}$,

Similarity, $((\nu_A)^T_\alpha \times (\nu_B)^T_\alpha)((x_1, y_1) - (x_2, y_2)) \leq \max\{((\nu_A)^T_\alpha \times (\nu_B)^T_\alpha)(x_1, y_1), ((\nu_A)^T_\alpha \times (\nu_B)^T_\alpha)(x_2, y_2)\}$,

Hence $A^{(T,K)}_{(\alpha,\varepsilon)} \times B^{(T,K)}_{(\alpha,\varepsilon)}$ is a bifuzzy $\psi$ -subalgebra of X×Y.

**Theorem 6.3.**

Let $A^{(T,K)}_{(\alpha,\varepsilon)} = \{(x, (\mu_A)^T_\alpha(x), (\nu_A)^K_\varepsilon(x)) \mid x \in X\}$ be a bifuzzy $\psi$ -subset of X and $B^{(T,K)}_{(\alpha,\varepsilon)} = \{(y, (\mu_B)^T_\alpha(y), (\nu_B)^K_\varepsilon(y)) \mid y \in Y\}$ be a bifuzzy $\psi$ -subset of Y. If $A^{(T,K)}_{(\alpha,\varepsilon)} \times B^{(T,K)}_{(\alpha,\varepsilon)}$ is a bifuzzy $\psi$ -subalgebra of X×Y , then $A^{(T,K)}_{(\alpha,\varepsilon)}$ is a bifuzzy $\psi$ -subalgebra of X and $B^{(T,K)}_{(\alpha,\varepsilon)}$ is a translation bifuzzy $\psi$ -subalgebra of Y.

**Proof:**

Assume that $A^{(T,K)}_{(\alpha,\varepsilon)} \times B^{(T,K)}_{(\alpha,\varepsilon)}$ is a translation bifuzzy $\psi$ -subalgebra of X×Y, then

$((\mu_A)^T_\alpha \times (\mu_B)^T_\alpha)((x_1, y_1) + (x_2, y_2)) \geq \min\{((\mu_A)^T_\alpha \times (\mu_B)^T_\alpha)(x_1, y_1), ((\mu_A)^T_\alpha \times (\mu_B)^T_\alpha)(x_2, y_2)\}$ ………….(1)

Putting $x_1 = x_2 = 0$ in (1) we get,

$((\mu_A)^T_\alpha \times (\mu_B)^T_\alpha)((0, y_1) + (0, y_2)) \geq \min\{((\mu_A)^T_\alpha \times (\mu_B)^T_\alpha)(0, y_1), ((\mu_A)^T_\alpha \times (\mu_B)^T_\alpha)(0, y_2)\}$ ,

$((\mu_A)^T_\alpha \times (\mu_B)^T_\alpha)(0, y_1 + y_2) \geq \min\{((\mu_A)^T_\alpha \times (\mu_B)^T_\alpha)(0, y_1), ((\mu_A)^T_\alpha \times (\mu_B)^T_\alpha)(0, y_2)\}$, then we have

$(\mu_B)^T_\alpha(y_1 + y_2) \geq \min\{(\mu_B)^T_\alpha(y_1), (\mu_B)^T_\alpha(y_2)\}$, and

$((\mu_A)^T_\alpha \times (\mu_B)^T_\alpha)((0, y_1) - (0, y_2)) \geq \min\{((\mu_A)^T_\alpha \times (\mu_B)^T_\alpha)(0, y_1), ((\mu_A)^T_\alpha \times (\mu_B)^T_\alpha)(0, y_2)\}$ ,

$((\mu_A)^T_\alpha \times (\mu_B)^T_\alpha)(0, y_1 - y_2) \geq \min\{((\mu_A)^T_\alpha \times (\mu_B)^T_\alpha)(0, y_1), ((\mu_A)^T_\alpha \times (\mu_B)^T_\alpha)(0, y_2)\}$, then we have

$(\mu_B)^T_\alpha(y_1 - y_2) \geq \min\{(\mu_B)^T_\alpha(y_1), (\mu_B)^T_\alpha(y_2)\}$.

Hence $A^{(T,K)}_{(\alpha,\varepsilon)}$ is a translation bifuzzy $\psi$ -subalgebra of $Y$.

Also,

$((\nu_A)^T_\alpha \times (\nu_B)^T_\alpha)((x_1, y_1) + (x_2, y_2)) \leq \max\{((\nu_A)^T_\alpha \times (\nu_B)^T_\alpha)(x_1, y_1), ((\nu_A)^T_\alpha \times (\nu_B)^T_\alpha)(x_2, y_2)\}$ ………….(2)

Putting $x_1 = x_2 = 0$ in (2) we get,

$((\nu_A)^T_\alpha \times (\nu_B)^T_\alpha)((0, y_1) + (0, y_2)) \leq \max\{((\nu_A)^T_\alpha \times (\nu_B)^T_\alpha)(0, y_1), ((\nu_A)^T_\alpha \times (\nu_B)^T_\alpha)(0, y_2)\}$ ,

$((\nu_A)^T_\alpha \times (\nu_B)^T_\alpha)(0, y_1 + y_2) \leq \max\{((\nu_A)^T_\alpha \times (\nu_B)^T_\alpha)(0, y_1), ((\nu_A)^T_\alpha \times (\nu_B)^T_\alpha)(0, y_2)\}$, then we have

$(\nu_B)^{\mathrm{K}}_{\varepsilon}(y_1+y_2) \le \max\{(\nu_B)^{\mathrm{K}}_{\varepsilon}(y_1), (\nu_B)^{\mathrm{K}}_{\varepsilon}(y_2)\}$,

$((\nu_A)^{\mathrm{T}}_{\alpha} \times (\nu_B)^{\mathrm{T}}_{\alpha})((0,y_1)-(0,y_2)) \le \max\{((\nu_A)^{\mathrm{T}}_{\alpha} \times (\nu_B)^{\mathrm{T}}_{\alpha})(0,y_1), ((\nu_A)^{\mathrm{T}}_{\alpha} \times (\nu_B)^{\mathrm{T}}_{\alpha})(0,y_2)\}$,

$((\nu_A)^{\mathrm{T}}_{\alpha} \times (\nu_B)^{\mathrm{T}}_{\alpha})(0,y_1-y_2) \le \max\{((\nu_A)^{\mathrm{T}}_{\alpha} \times (\nu_B)^{\mathrm{T}}_{\alpha})(0,y_1), ((\nu_A)^{\mathrm{T}}_{\alpha} \times (\nu_B)^{\mathrm{T}}_{\alpha})(0,y_2)\}$, then we have

$(\nu_B)^{\mathrm{K}}_{\varepsilon}(y_1-y_2) \le \max\{(\nu_B)^{\mathrm{K}}_{\varepsilon}(y_1), (\nu_B)^{\mathrm{K}}_{\varepsilon}(y_2)\}$.

Hence $B^{(T,K)}_{(\alpha,\varepsilon)}$ is a translation bifuzzy $\psi$ -subalgebra of $Y$. ⌂

**Theorem 6.4.**

Let $A^{(T,K)}_{(\alpha,\varepsilon)} = \{(x, (\mu_A)^{\mathrm{T}}_{\alpha}(x), (\nu_A)^{\mathrm{K}}_{\varepsilon}(x)) \mid x \in X\}$ be a bifuzzy $\psi$ -subset of X and $B^{(T,K)}_{(\alpha,\varepsilon)} = \{(y, (\mu_B)^{\mathrm{T}}_{\alpha}(y), (\nu_B)^{\mathrm{K}}_{\varepsilon}(y)) \mid y \in Y\}$ be a bifuzzy $\psi$ -subset of Y. If $A^{(T,K)}_{(\alpha,\varepsilon)} \times B^{(T,K)}_{(\alpha,\varepsilon)}$ is a bifuzzy $\psi$ -subalgebra of X×Y , then $A^{(T,K)}_{(\alpha,\varepsilon)}$ is a bifuzzy $\psi$ -ideal of X and $B^{(T,K)}_{(\alpha,\varepsilon)}$ is a bifuzzy $\psi$ -ideal of Y.

**Proof:**

For any x = $(x_1,x_2) \in$ X×X, we have

1- $((\mu_A)^{\mathrm{T}}_{\alpha} \times (\mu_B)^{\mathrm{T}}_{\alpha})(0) = ((\mu_A)^{\mathrm{T}}_{\alpha} \times (\mu_B)^{\mathrm{T}}_{\alpha})(0,0) = \min((\mu_A)^{\mathrm{T}}_{\alpha}(0), (\mu_B)^{\mathrm{T}}_{\alpha}(0)) \ge \min((\mu_A)^{\mathrm{T}}_{\alpha}(x_1), (\mu_B)^{\mathrm{T}}_{\alpha}(x_2))$

$= ((\mu_A)^{\mathrm{T}}_{\alpha} \times (\mu_B)^{\mathrm{T}}_{\alpha})(x_1,x_2) = ((\mu_A)^{\mathrm{T}}_{\alpha} \times (\mu_B)^{\mathrm{T}}_{\alpha})(x)$.

$((\nu_A)^{\mathrm{T}}_{\alpha} \times (\nu_B)^{\mathrm{T}}_{\alpha})(0) = ((\nu_A)^{\mathrm{T}}_{\alpha} \times (\nu_B)^{\mathrm{T}}_{\alpha})(0,0) = \max((\nu_A)^{\mathrm{K}}_{\varepsilon}(0), (\nu_B)^{\mathrm{K}}_{\varepsilon}(0)) \le \max((\nu_A)^{\mathrm{K}}_{\varepsilon}(x_1), (\nu_B)^{\mathrm{K}}_{\varepsilon}(x_2))$

$= ((\nu_A)^{\mathrm{T}}_{\alpha} \times (\nu_B)^{\mathrm{T}}_{\alpha})(x_1,x_2) = ((\nu_A)^{\mathrm{T}}_{\alpha} \times (\nu_B)^{\mathrm{T}}_{\alpha})(x)$.

2- Let $x=(x_1,x_2)$ , $y=(y_1,y_2)$ and $z=(z_1,z_2)$

$((\mu_A)^{\mathrm{T}}_{\alpha} \times (\mu_B)^{\mathrm{T}}_{\alpha})(y+x) = ((\mu_A)^{\mathrm{T}}_{\alpha} \times (\mu_B)^{\mathrm{T}}_{\alpha})((y_1,y_2)+(x_1,x_2))$

$= ((\mu_A)^{\mathrm{T}}_{\alpha} \times (\mu_B)^{\mathrm{T}}_{\alpha})((y_1+x_1),(y_2+x_2))$

$= \min\{(\mu_A)^{\mathrm{T}}_{\alpha}(y_1+x_1), (\mu_B)^{\mathrm{T}}_{\alpha}((y_2+x_2)\}$

$\ge \min\{\min\{(\mu_A)^{\mathrm{T}}_{\alpha}(y_1+z_1)\ (\mu_A)^{\mathrm{T}}_{\alpha}(x_1-z_1)\}, \min\{(\mu_B)^{\mathrm{T}}_{\alpha}(y_2+z_2), (\mu_B)^{\mathrm{T}}_{\alpha}(x_2-z_2)\}\}$

$= \min\{\min\{(\mu_A)^{\mathrm{T}}_{\alpha}(y_1+z_1), (\mu_B)^{\mathrm{T}}_{\alpha}(y_2+z_2)\}, \min\{(\mu_A)^{\mathrm{T}}_{\alpha}(x_1-z_1), (\mu_B)^{\mathrm{T}}_{\alpha}(x_2-z_2)\}\}$

$= \min\{((\mu_A)^{\mathrm{T}}_{\alpha} \times (\mu_B)^{\mathrm{T}}_{\alpha})((y_1,y_2)+(z_1,z_2)), ((\mu_A)^{\mathrm{T}}_{\alpha} \times (\mu_B)^{\mathrm{T}}_{\alpha})((x_1,x_2)-(z_1,z_2))\}$

$= \min\{((\mu_A)^{\mathrm{T}}_{\alpha} \times (\mu_B)^{\mathrm{T}}_{\alpha})(y+z), ((\mu_A)^{\mathrm{T}}_{\alpha} \times (\mu_B)^{\mathrm{T}}_{\alpha})(x-z)\}$,

$((\nu_A)^{\mathrm{T}}_{\alpha} \times (\nu_B)^{\mathrm{T}}_{\alpha})(y+x) = ((\nu_A)^{\mathrm{T}}_{\alpha} \times (\nu_B)^{\mathrm{T}}_{\alpha})((y_1,y_2)+(x_1,x_2))$

$= ((\nu_A)^{\mathrm{T}}_{\alpha} \times (\nu_B)^{\mathrm{T}}_{\alpha})((y_1+x_1),(y_2+x_2))$

$= \max\{(\nu_A)^{\mathrm{K}}_{\varepsilon}(y_1+x_1), (\nu_B)^{\mathrm{K}}_{\varepsilon}((y_2+x_2)\}$

$\le \max\{\max\{(\nu_A)^{\mathrm{K}}_{\varepsilon}(y_1+z_1), (\nu_A)^{\mathrm{K}}_{\varepsilon}(x_1-z_1)\}, \max\{(\nu_B)^{\mathrm{K}}_{\varepsilon}(y_2+z_2), (\nu_B)^{\mathrm{K}}_{\varepsilon}(x_2-z_2)\}\}$

$= \max\{\max\{(\nu_A)^{\mathrm{K}}_{\varepsilon}(y_1+z_1), (\nu_B)^{\mathrm{K}}_{\varepsilon}(y_2+z_2)\}, \max\{(\nu_A)^{\mathrm{K}}_{\varepsilon}(x_1-z_1), (\nu_B)^{\mathrm{K}}_{\varepsilon}(x_2-z_2)\}\}$

$= \max\{((\nu_A)^{\mathrm{T}}_{\alpha} \times (\nu_B)^{\mathrm{T}}_{\alpha})((y_1,y_2)+(z_1,z_2)), ((\nu_A)^{\mathrm{T}}_{\alpha} \times (\nu_B)^{\mathrm{T}}_{\alpha})((x_1,x_2)-(z_1,z_2))\}$

$= \max\{((\nu_A)^{\mathrm{T}}_{\alpha} \times (\nu_B)^{\mathrm{T}}_{\alpha})(y+z), ((\nu_A)^{\mathrm{T}}_{\alpha} \times (\nu_B)^{\mathrm{T}}_{\alpha})(x-z)\}$.

Hence $A^{(T,K)}_{(\alpha,\varepsilon)} \times B^{(T,K)}_{(\alpha,\varepsilon)}$ is a bifuzzy $\psi$ -ideal of X×Y. ⌂

**Theorem 6.5.**

Let $A^{(T,K)}_{(\alpha,\varepsilon)} = \{(x, (\mu_A)^{\mathrm{T}}_{\alpha}(x), (\nu_A)^{\mathrm{K}}_{\varepsilon}(x)) \mid x \in X\}$ be a translation bifuzzy $\psi$ -subset of X and $B^{(T,K)}_{(\alpha,\varepsilon)} = \{(y, (\mu_B)^{\mathrm{T}}_{\alpha}(y), (\nu_B)^{\mathrm{K}}_{\varepsilon}(y)) \mid y \in Y\}$ be a translation bifuzzy $\psi$ -subset of Y. If $A^{(T,K)}_{(\alpha,\varepsilon)} \times B^{(T,K)}_{(\alpha,\varepsilon)}$ is a bifuzzy $\psi$ -subalgebra of X×Y , then $A^{(T,K)}_{(\alpha,\varepsilon)}$ is a translation bifuzzy $\psi$-ideal of X and $B^{(T,K)}_{(\alpha,\varepsilon)}$ is a translation bifuzzy $\psi$ -ideal of Y.

**Proof:**

Let $x=(x_1,x_2)$ , $y=(y_1,y_2)$ , $z=(z_1,z_2) \in$ X×X

$((\mu_A)^{\mathrm{T}}_{\alpha} \times (\mu_B)^{\mathrm{T}}_{\alpha})((0,0),(0,0)) = \min(((\mu_A)^{\mathrm{T}}_{\alpha})(0,0), ((\mu_B)^{\mathrm{T}}_{\alpha})(0,0))$

$\ge \min\{((\mu_A)^{\mathrm{T}}_{\alpha})(x_1,y_1), ((\mu_B)^{\mathrm{T}}_{\alpha})(x_2,y_2)\}$

Putting $y=(y_1,y_2)$= (0,0), we have $(\mu_A)^{\mathrm{T}}_{\alpha}(0,0) \ge (\mu_A)^{\mathrm{T}}_{\alpha}(x_1,x_2)$,

$((\nu_A)^{\mathrm{T}}_{\alpha} \times (\nu_B)^{\mathrm{T}}_{\alpha})((0,0),(0,0)) = \max\{((\nu_A)^{\mathrm{T}}_{\alpha} \times (\nu_B)^{\mathrm{T}}_{\alpha})(0,0), ((\nu_A)^{\mathrm{T}}_{\alpha} \times (\nu_B)^{\mathrm{T}}_{\alpha})(0,0)\}$

$\le \max\{((\nu_A)^{\mathrm{K}}_{\varepsilon})(x_1,y_1), ((\nu_B)^{\mathrm{K}}_{\varepsilon})(x_2,y_2)\}$

Putting $y=(y_1,y_2)$= (0,0), we have $(\nu_A)^{\mathrm{K}}_{\varepsilon}(0,0) \le (\nu_A)^{\mathrm{K}}_{\varepsilon}(x_1,x_2)$.

Assume that $A^{(T,K)}_{(\alpha,\varepsilon)} \times B^{(T,K)}_{(\alpha,\varepsilon)}$ is a translation bifuzzy $\psi$-ideal of X×Y, then

$((\mu_A)^{\mathrm{T}}_{\alpha} \times (\mu_B)^{\mathrm{T}}_{\alpha})((y_1,y_2)+(x_1,x_2)) \ge \min\{((\mu_A)^{\mathrm{T}}_{\alpha} \times (\mu_B)^{\mathrm{T}}_{\alpha})((y_1,y_2)+(z_1,z_2)), ((\mu_A)^{\mathrm{T}}_{\alpha} \times (\mu_B)^{\mathrm{T}}_{\alpha})((x_1,x_2)-(z_1,z_2))\}$………………….. (1)

Putting $x_2=y_2=z_2=0$, then we have

$((\mu_A)^{\mathrm{T}}_{\alpha})((y_1,0)+(x_1,0)) \ge \min\{((\mu_A)^{\mathrm{T}}_{\alpha})((y_1,0)+(z_1,0)), ((\mu_A)^{\mathrm{T}}_{\alpha})((x_1,0)-(z_1,0))\}$, thus

$(\mu_A)^{\mathrm{T}}_{\alpha}(y_1+x_1) \ge \min\{(\mu_A)^{\mathrm{T}}_{\alpha}((y_1+z_1), (\mu_A)^{\mathrm{T}}_{\alpha}(x_1-z_1)\}$.

And

$((v_A)_\alpha^{T} \times (v_B)_\alpha^{T})((y_1, y_2)+ (x_1, x_2)) \leq \max\{((v_A)_\alpha^{T} \times (v_B)_\alpha^{T})((y_1, y_2) + (z_1, z_2)),((v_A)_\alpha^{T} \times (v_B)_\alpha^{T})((x_1, x_2) - (z_1, z_2))\}$………………….. (2)

Putting $x_2 = y_2 = z_2 = 0$, then we have

$((v_A)_\varepsilon^{K})((y_1, 0)+ (x_1, 0)) \leq \max\{((v_A)_\varepsilon^{K})((y_1, 0) + (z_1, 0)),(v_A)_\varepsilon^{K})\ ((x_1, 0) - (z_1, 0))\}$, thus

$(v_A)_\varepsilon^{K}\ (y_1 + x_1) \leq \max\{(v_A)_\varepsilon^{K}((y_1 + z_1), (v_A)_\varepsilon^{K}(x_1 - z_1)\}$.

Hence $A_{(\alpha,\varepsilon)}^{(T,K)}$ is a bifuzzy $\psi$ -ideal of $X$.

Similarity, $B_{(\alpha,\varepsilon)}^{(T,K)}$ is a translation bifuzzy $\psi$ -subalgebra of $Y$.

**References**


[1] A.T. Hameed and N.H. Malik, (2021), **(β, α)-Fuzzy Magnified Translations of AT-algebra**, Journal of Physics: Conference Series (IOP Publishing), 2021, pp:1-13.

[2] A.T. Hameed and N.H. Malik, (2021), **Magnified translation of intuitionistic fuzzy AT-ideals on AT-algebra,** Journal of Discrete Mathematical Sciences and Cryptography, (2021), pp:1-7.

[3] A.T. Hameed and N.H. Jaber (2023), **On The $\psi$-algebra**, to appear.

[4] A.T. Hameed and N.H. Jaber, (2023), **On The fuzzy $\psi$-ideals of $\psi$-algebra**, to appear.

[5] A.T. Hameed and N.H. Jaber, (2023), **On The bifuzzy $\psi$-ideals of $\psi$-algebra**, to appear.

[6] J. Meng and Y. B. Jun, **BCK-algebras** , Kyung Moon Sa Co. , Korea, 1994.

[7] L.A. Zadeh, **Fuzzy Sets** , Inform . and Control ,vol.8 (1965), pp:338-353 .