Four Points Block Method with Second Derivative for Solving First Order Ordinary Differential Equations

Mohammed Yousif Turki
Department of Mathematics
University of Anbar, Faculty of Education for Pure Sciences
Anbar, Iraq
moh9883@uoanbar.edu.iq
Orcid.org/0000-0003-4211-6493

Niran Sabah Jasim
Department of Mathematics
University of Baghdad, College of Education for Pure Science Ibn Al-Haitham
Baghdad, Iraq.
niraan.s.j@ihcoedu.uoanbar.edu.iq
Orcid.org/0000-0003-4211-6493

Mohammed S. Mechee
Information Technology Research and Development Center (ITRDC)
University of kufa
Najaf, Iraq
mohammeds.abed@uokufa.edu.iq
Orcid.org/0000-0003-4211-6493

Shrooq Bahjat Smeein
Section Mathematics
University of Technology and Applied science - Muscat, Information Department
Sultanate of Oman
shrooq.Smeein@hct.edu.om
Orcid.org/0000-0003-4211-6493

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Abstract— *In this study, we devised 4-point implicit block (New 4pb) approaches that take advantage of second derivatives. The aim provides is more accurate and quicker numerical solutions to first - order ordinary differential equations (ODEs). Whereas the properties of the (New 4pb) method such as the order and zero-stability have been discussed. After that, the approaches are applied to a collection of first-order (ODEs). In addressing the set of test problems, numerical results clearly indicate that the newly proposed systems outperformed previous, well-known existing methods.*

Keywords— A four points, Block Method, first Order Ordinary Differential Equations

I. INTRODUCTION

The differential equation is created when physical phenomena in science and engineering are mathematically formulated. This formulation provides a thorough explanation of the phenomenon's behavior. Solving ordinary differential equations (ODEs) is necessary for finding answers to practical issues, making it a crucial area of mathematical research. Block method comprises two types: one-step and multi-step methods. Regarding the one-step method, it is new block value y_{n+i}, $i = 1, 2, \dots, k$ has been obtained from information at y_n. On the other hand, using the previous values to complete the next one is referred to as a multi-step. Majid et al. (2006), 3-point fully implicit block method has been derived for solving the first order ODEs. Ibrahim et al. (2007) derived the r- points block method by using a backward difference formula to solve first - order ODEs. The extra derivatives implicit block methods have been derived by Sahi et al. (2012), proposed Simpson's1/3 block method with an additional second derivative for solving ODEs. generalized a four - steps Adam's type block method of the second derivative has been modified by Kumleng and Sirisena (2014). Akinfenwa et al. (2015), derived a new family of continuous third derivate order ODEs. Order and Convergence of Order 7th Numerical Scheme for the solving system of First Order initial value problems (IVPs) have been discussed by Muhammad Abdullahi et al.(2023). Additionally, the numerical results were examined with equivalent numerical results produced using the Sahi et al. (2012) and Turki et al. (2018) methods that are already in use. Moreover, for future works, can be modified the studying of the authors Jaber A. Alrammahi A (2020) and Hasan NN, Hussien DA (2018), for the second type of Fredholm and

Volterra integral equations by using the suggested technique. Hussain KA, Hasan WJ.(2023), improved the Runge-Kutta method to solve a system of first-order ODEs. Mohammed AS (2022), developed an iterative method for solving nonlinear differential equations. Fawzi FA, Jumaa MH (2022) and Fawzi FA, Jaleel NW (2023), the implicit Runge Kutta type have been derived to solve first and third order ODEs. The effectiveness and accuracy of the developed method are compared with the more common methods using numerical results in order to get impressive numerical results for the new 4-point approach, applications of IVPs are also introduced. (New4pb) is created by using Hermite polynomials to incorporate the second derivative of $f(z,y)$ to improve the accuracy of approximation solutions to IVPs.

$$y' = f(z,y), \quad y(z_0) = y_0 \qquad \ldots(1)$$

The second derivative with respect to z gives

$$g(z,y) = f'(z,y) = f_z + f f_y$$

In this work, a four point implicit block method with extra derivative has been derived. The (New4pb) method is derived by using Hermite Interpolating Polynomial p, which can be defined by:

$$p(z) = \sum_{i=0}^{n}\sum_{k=0}^{m_{i-1}} f_i^{(k)} L_{i,k}(z), \qquad \ldots(2)$$

Where $f_j = f(z_j)$, $z_j = a + jh$, $j = 0,1,\ldots,n$ and $h = \frac{b-a}{n}$, n is positive integer. $L_{j,k}(z)$ is the definition of the generalized Lagrange polynomial.

$L_{j,m_j}(z) = l_{j,m_j}(z)$, $j = 0,1,\ldots,n$,

$l_{j,k}(z) = \frac{(z-z_j)^k}{k!}\prod_{i=0,j\neq i}^{n}\left(\frac{z-z_i}{z_j-z_i}\right)^{m_i}$, $j = 0,1,2,\ldots,n$, $k = 0,1,\ldots,m_j$.

$L_{j,m_j}(z) = l_{j,m_j}(z) - \sum_{v=k+1}^{m_{i-1}} l_{j,k}^{(v)}(z_j)L_{j,v}(z)$, $k = m_j - 2, m_j - 3, \ldots, 0$.

II. DERIVATIVE OF THE METHOD

In the 4- point implicit block method, the close interval [a,b] is contains 4- point for each block with $4h$ step size . The four values of y_{n+1}, y_{n+2}, y_{n+3} and y_{n+4} are calculated at the same time in a block. To evaluate y_{n+1} we take $z_{n+1} = z_n + h$ and integrating Eq. (1) over the interval $[z_n, z_{n+1}]$ gives :

$$\int_{z_n}^{z_{n+1}} y'\,dx = \int_{z_n}^{z_{n+1}} f(z,y)\,dz,$$

$$y(z_{n+1}) = y(z_n) + \int_{z_n}^{z_{n+1}} f(z,y)\,dz. \qquad \ldots(3)$$

$f(z,y)$ in Eq.(3) can be exchange by Hermit Interpolating Polynomial in Eq.(2) which is given by:

$$p_4(z) = f_nL_{0,0}(z) + f_{n+1}L_{0,1}(z) + f_{n+2}L_{0,2}(z) + f_{n+3}L_{0,3}(z) + f_{n+4}L_{0,4}(z) + g_nL_{1,0}(z) + g_{n+1}L_{1,1}(z) + g_{n+2}L_{1,2}(z) + g_{n+3}L_{1,3}(z) + g_{n+4}L_{1,4}(z) . \qquad \ldots(4)$$

Let $z = z_{n+4} + sh$ and $s = \frac{z-z_{n+4}}{h}$ …(5)

If we change integration in Eq. (3) from -4 to -3 and replace $dz = h\,ds$, we get:

$$y(z_{n+1}) = y(z_n) + \int_{-4}^{-3} [f_nL_{0,0}(s) + f_{n+1}L_{0,1}(s) + f_{n+2}L_{0,2}(s) + f_{n+3}L_{0,3}(s) + f_{n+4}L_{0,4}(s) + g_nL_{1,0}(s) + +g_{n+1}L_{1,1}(s) + g_{n+2}L_{1,2}(s) + g_{n+3}L_{1,3}(s) + g_{n+4}L_{1,4}(s)]h\,ds. \qquad \ldots(6)$$

where

$$L_{0,0}(s) = \frac{(1+s)^2(2+s)^2(3+s)^2(s)^2}{576}\left[1 + \frac{25}{6}(s+4)\right]$$

$$L_{1,0}(s) = \frac{(1+s)^2(2+s)^2(4+s)^2(s)^2}{36}\left[1 + \frac{5}{3}(s+3)\right]$$

$$L_{2,0}(s) = \frac{(1+s)^2(2+s)^2(s)^2(4+s)^2}{16}$$

$$L_{3,0}(s) = \frac{(1+s)^2(s)^2(3+s)^2(4+s)^2}{36}\left[1 - \frac{2}{3}(s+1)\right]$$

$$L_{4,0}(s) = \frac{(s)^2(2+s)^2(3+s)^2(4+s)^2}{576}\left[1 - \frac{13}{6}(s)\right]$$

$$L_{0,1}(s) = \frac{(s)^2(1+s)^2(2+s)^2(3+s)^2(4+s)}{576}$$

$$L_{1,1}(s) = \frac{(s)^2(1+s)^2(2+s)^2(3+s)\ (4+s)^2}{36}$$

$$L_{2,1}(s) = \frac{(s)^2(1+s)^2(3+s)^2(2+s)\ (4+s)^2}{16}$$

$$L_{3,1}(s) = \frac{(s)^2(2+s)^2(1+s)\ (3+s)^2(4+s)^2}{36}$$

$$L_{4,1}(s) = \frac{(s+1)^2(s)\ (2+s)^2(3+s)^2(4+s)^2}{576}$$

by using MAPLE to evaluate the integral in (6) as follows:

$$y_{n+1} = y_n + \frac{h}{4354560}(1539551f_n + 680968f_{n+1} + 711936f_{n+2} + 613456f_{n+3} + 59681f_{n+4}) + \frac{h^2}{725760}(26051g_n - 249656g_{n+1} - 183708g_{n+2} - 49720g_{n+3} - 2237g_{n+4}) . \qquad \ldots(7)$$

For the evaluation y_{n+2} approximate solutions are obtained by integrating Eq.(1) over the interval $[z_n, z_{n+2}]$.

$$\int_{z_n}^{z_{n+2}} y'\,dx = \int_{z_n}^{z_{n+2}} f(z,y)\,dz,$$

$$y(z_{n+2}) = y(z_n) + \int_{z_n}^{z_{n+2}} f(z,y)\,dz. \qquad \ldots(8)$$

If we change integration in Eq. (8) from -4 to -2 and replace $dz = h\,ds$, we get:

$$y(z_{n+2}) = y(z_n) + \int_{-4}^{-2} [f_nL_{0,0}(s) + f_{n+1}L_{0,1}(s) + f_{n+2}L_{0,2}(s) + f_{n+3}L_{0,3}(s) + f_{n+4}L_{0,4}(s) + g_nL_{1,0}(s) + +g_{n+1}L_{1,1}(s) + g_{n+2}L_{1,2}(s) + g_{n+3}L_{1,3}(s) + g_{n+4}L_{1,4}(s)]h\,ds. \qquad \ldots(9)$$

By using MAPLE to evaluate the integral in Eq. (9) as follows:

$$y_{n+2} = y_n + \frac{h}{68040}(24463f_n + 43808f_{n+1} + 44928f_{n+2} + 12608f_{n+3} + 1153f_{n+4}) + \frac{h^2}{11340}(421g_n - 3040g_{n+1} - 4536g_{n+2} - 992g_{n+3} - 43g_{n+4}) \qquad \ldots(10)$$

To evaluation y_{n+3} approximate solutions are obtained by integrating Eq. (1) over the interval $[z_n, z_{n+3}]$.

$\int_{z_n}^{z_{n+3}} y'\, dz = \int_{z_n}^{z_{n+3}} f(z,y)\, dz,$

$$y(z_{n+3}) = y(z_n) + \int_{z_n}^{z_{n+3}} f(z,y)\, dz. \quad \ldots(11)$$

If we change integration in Eq. (11) from -4 to -1 and replace $dz = h\, ds$, we get:

$$y(z_{n+3}) = y(z_n) + \int_{-4}^{-1} [f_n L_{0,0}(s) + f_{n+1}L_{0,1}(s) + f_{n+2}L_{0,2}(s) + f_{n+3}L_{0,3}(s) + f_{n+4}L_{0,4}(s) + g_n L_{1,0}(s) + +g_{n+1}L_{1,1}(s) + g_{n+2}L_{1,2}(s) + g_{n+3}L_{1,3}(s) + g_{n+4}L_{1,4}(s)] h ds. \quad \ldots(12)$$

By using MAPLE to evaluate the integral in Eq. (12) as follows:

$$y_{n+3} = y_n + \frac{3h}{17920}(2167f_n + 4168f_{n+1} + 6912f_{n+2} + 3792f_{n+3} + 137f_{n+4}) + \frac{3h^2}{8960}(113g_n - 744g_{n+1} - 756g_{n+2} - 488g_{n+3} - 15g_{n+4}) \quad \ldots(13)$$

to evaluate y_{n+4} approximate solutions are obtained by integrating Eq. (1) over the interval $[z_n, z_{n+4}]$.

$\int_{z_n}^{z_{n+4}} y'\, dz = \int_{z_n}^{z_{n+4}} f(z,y)\, dz,$

$$y(z_{n+4}) = y(z_n) + \int_{z_n}^{z_{n+4}} f(z,y)\, dz. \quad \ldots(14)$$

If we change integration in Eq. (14) from -4 to 0 and replace $dz = h\, ds$, we get:

$$y(z_{n+4}) = y(z_n) + \int_{-4}^{0} [f_n L_{0,0}(s) + f_{n+1}L_{0,1}(s) + f_{n+2}L_{0,2}(s) + f_{n+3}L_{0,3}(s) + f_{n+4}L_{0,4}(s) + g_n L_{1,0}(s) + +g_{n+1}L_{1,1}(s) + g_{n+2}L_{1,2}(s) + g_{n+3}L_{1,3}(s) + g_{n+4}L_{1,4}(s)] h ds. \quad \ldots(15)$$

By using MAPLE to evaluate the integral in Eq. (15) as follows:

$$z_{n+4} = z_n + \frac{2h}{8505}(1601f_n + 3712f_{n+1} + 5616f_{n+2} + 4096f_{n+3} + 1601f_{n+4}) + \frac{4h^2}{2835}(29g_n - 128g_{n+1} + 128g_{n+3} - 29g_{n+4}) \quad \ldots(16)$$

III. ORDER OF THE METHOD

In this section, we define the order of the 4-point block approaches that were developed in this paper. According to Lambert and Fatunla (1991), the local truncation error associated with the (New 4pb) method normalized form may be characterized as the linear differential operator.

$$\mathcal{L}[y(z); h] = \sum_{i=0}^{k} \alpha_i y(z + ih) - \sum_{i=0}^{k} h\beta_i y'(z + ih) - \sum_{i=0}^{k} h^2 \gamma_i y''(z + ih) \quad \ldots(17)$$

Assuming that $y(z)$ is sufficiently differentiable, expanding Eq. (17) as a Taylor series expansion about the point z yields the formula:

$\mathcal{L}[y(z);\ h] = c_0 y(z) + c_1 h y'(z) + \ldots + c_p h^p y^p(z) + \ldots$.

The constant coefficients c_p and h are supplied as follows:

$c_0 = \sum_{i=0}^{k} \alpha_i,\ c_1 = \sum_{i=0}^{k}(i\alpha_i - \beta_i), \ldots,\ c_p = \sum_{i=0}^{k}(\frac{i^p}{p!}\alpha_i - \frac{i^{p-1}}{(p-1)!}\beta_i - \frac{i^{p-2}}{(p-2)!}\gamma_i), p = 2,3, \ldots$

According to Henrici [18], the (New 4pb) method has order $\mathcal{P}$ if $c_0 = c_1 = c_2 = \cdots = c_p = 0$, $c_{p+1} \neq 0$. Therefore, $c_{p+1} h^{p+1} y^{p+1}(z)$ is the largest local truncation error at the position z_n, where c_{p+1} is the error constant . The formulas for the (New 4pb) 7 method are given in paragraphs Eq. 7, 10, 13 and Eq. 17 and they are organized into a matrix as follows:

$$\alpha Y_m = +h\beta\mathcal{F}_m + h^2\gamma\mathcal{G}_m \quad \ldots(18)$$

where the coefficients with the m – vector Y_m, $\mathcal{F}_m$ and $\mathcal{G}_m$, α, β and γ, are defined as,

$$\alpha = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix};\quad \beta = \begin{bmatrix} \frac{1539551}{4354560} & \frac{1429936}{4354560} & \frac{711936}{4354560} & \frac{613456}{4354560} & \frac{59681}{4354560} \\ \frac{24463}{68040} & \frac{43808}{68040} & \frac{44928}{68040} & \frac{12608}{68040} & \frac{1153}{68040} \\ \frac{2167*3}{17920} & \frac{4912*3}{17920} & \frac{6912*3}{17920} & \frac{3792*3}{17920} & \frac{137*3}{17920} \\ \frac{1601*2}{8505} & \frac{3712*2}{8505} & \frac{5616*2}{8505} & \frac{4096*2}{8505} & \frac{1601*2}{8505} \end{bmatrix}$$

$$\gamma = \begin{bmatrix} \frac{26051}{725760} & \frac{-249656}{725760} & \frac{-183708}{725760} & \frac{-49720}{725760} & \frac{-2237}{725760} \\ \frac{421}{11340} & \frac{-3040}{11340} & \frac{-4536}{11340} & \frac{-992}{11340} & \frac{-43}{11340} \\ \frac{113*3}{8960} & \frac{-744*3}{8960} & \frac{-756*3}{8960} & \frac{-488*3}{8960} & \frac{-15*3}{8960} \\ \frac{29*4}{2835} & \frac{-128*4}{2835} & 0 & \frac{-128*4}{2835} & \frac{-29*4}{2835} \end{bmatrix}$$

$$\mathcal{Y}_m = \begin{bmatrix} y_n \\ y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \end{bmatrix}, \mathcal{F}_m = \begin{bmatrix} f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{bmatrix}, \mathcal{G}_m = \begin{bmatrix} g_n \\ g_{n+1} \\ g_{n+2} \\ g_{n+3} \\ g_{n+4} \end{bmatrix}$$

$c_0 = \sum_{i=0}^{k} \alpha_i = \bar{0}$,

$c_1 = \sum_{i=0}^{k}(i\alpha_i - \beta_i) = \bar{0}$,

$$c_2 = \sum_{i=0}^{k}(\frac{i^2}{2!}\alpha_i - \frac{i}{1!}\beta_i - \gamma_i) = \bar{0},$$

$$c_3 = \sum_{i=0}^{k}(\frac{i^3}{3!}\alpha_i - \frac{i^2}{2!}\beta_i - \frac{i}{1!}\gamma_i) = \bar{0},$$

$$c_4 = \sum_{i=0}^{k}(\frac{i^4}{4!}\alpha_i - \frac{i^3}{3!}\beta_i - \frac{i^2}{2!}\gamma_i) = \bar{0},$$

$$c_5 = \sum_{i=0}^{k}(\frac{i^5}{5!}\alpha_i - \frac{i^4}{4!}\beta_i - \frac{i^3}{3!}\gamma_i) = \bar{0},$$

$$c_6 = \sum_{i=0}^{k} \left(\frac{i^6}{6!}\alpha_i - \frac{i^5}{5!}\beta_i - \frac{i^4}{4!}\gamma_i\right) = \bar{0} \ ,$$

$$c_7 = \sum_{i=0}^{k} \left(\frac{i^7}{7!}\alpha_i - \frac{i^6}{6!}\beta_i - \frac{i^5}{5!}\gamma_i\right) = \bar{0} \ ,$$

$$c_8 = \sum_{i=0}^{k} \left(\frac{i^8}{8!}\alpha_i - \frac{i^7}{7!}\beta_i - \frac{i^6}{6!}\gamma_i\right) = \bar{0} \ ,$$

$$c_9 = \sum_{i=0}^{k} \left(\frac{i^9}{9!}\alpha_i - \frac{i^8}{8!}\beta_i - \frac{i^7}{7!}\gamma_i\right) = \bar{0} \ ,$$

$$c_{10} = \sum_{i=0}^{k} \left(\frac{i^{10}}{10!}\alpha_i - \frac{i^9}{9!}\beta_i - \frac{i^8}{8!}\gamma_i\right) = \bar{0} \ ,$$

$$c_{11} = \sum_{i=0}^{k} \left(\frac{i^{11}}{11!}\alpha_i - \frac{i^{10}}{10!}\beta_i - \frac{i^9}{9!}\gamma_i\right) \neq \bar{0} \ ,$$

$$\boldsymbol{c_{11}} = \begin{bmatrix} \frac{13038317}{7242504192000} \\ \frac{11729}{5658206400} \\ \frac{23381}{9934848000} \\ \frac{1811}{442047375} \end{bmatrix} \neq \bar{0}$$

IV. THE ZERO STABLE OF THE METHOD

The Zero stable of the 4-points implicit block method is discussed in this section. The general form of Eq. (7), (10), (13) and Eq. (17) may be expressed in matrix form as follows:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix}$$

$$+\boldsymbol{h} \begin{bmatrix} \frac{1539551}{4354560} & \frac{1429936}{4354560} & \frac{711936}{4354560} & \frac{613456}{4354560} & \frac{59681}{4354560} \\ \frac{24463}{68040} & \frac{43808}{68040} & \frac{44928}{68040} & \frac{12608}{68040} & \frac{1153}{68040} \\ \frac{2167*3}{17920} & \frac{4912*3}{17920} & \frac{6912*3}{17920} & \frac{3792*3}{17920} & \frac{137*3}{17920} \\ \frac{1601*2}{8505} & \frac{3712*2}{8505} & \frac{5616*2}{8505} & \frac{4096*2}{8505} & \frac{1601*2}{8505} \end{bmatrix} \begin{bmatrix} f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{bmatrix}$$

$$+\boldsymbol{h^2} \begin{bmatrix} \frac{26051}{725760} & \frac{-249656}{725760} & \frac{-183708}{725760} & \frac{-49720}{725760} & \frac{-2237}{725760} \\ \frac{421}{11340} & \frac{-3040}{11340} & \frac{-4536}{11340} & \frac{-992}{11340} & \frac{-43}{11340} \\ \frac{113*3}{8960} & \frac{-744*3}{8960} & \frac{-756*3}{8960} & \frac{-488*3}{8960} & \frac{-15*3}{8960} \\ \frac{29*4}{2835} & \frac{-128*4}{2835} & \boldsymbol{0} & \frac{-128*4}{2835} & \frac{-29*4}{2835} \end{bmatrix} \begin{bmatrix} g_n \\ g_{n+1} \\ g_{n+2} \\ g_{n+3} \\ g_{n+4} \end{bmatrix}$$

The first characteristic of the (New 4pb) is stated as follows:

$$\rho(\mathfrak{R}) = det\ [\mathfrak{R}\mathfrak{U}^{(0)} - \mathfrak{U}^{(1)}] = 0$$

$$\mathfrak{U}^{(0)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } \mathfrak{U}^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\rho(\mathfrak{R}) = det \begin{bmatrix} \mathfrak{R} & 0 & 0 & -1 \\ 0 & \mathfrak{R} & 0 & -1 \\ 0 & 0 & \mathfrak{R} & -1 \\ 0 & 0 & 0 & \mathfrak{R}-1 \end{bmatrix} = 0$$

$\mathfrak{R}^3(\mathfrak{R}-1) = 0\ ,\quad \mathfrak{R} = 0,\ 0,\ 0, 1 \quad , |\mathfrak{R}| \le 1$

The four point implicit block methods are zero - stable, according to Fatunla (1991) since the first characteristic polynomial $\rho(\mathfrak{R}) = 0$ satisfies, $|\mathfrak{R}_j| \le 1$; j = 0,... , k. Additionally, because the order p of the four point implicit block methods is greater than one, they are consistent.

Convergence

According to Lambert (1973) and Fatunla (1988) and Lambert (1991), the (New 4pb) method is convergence since it is zero stable and consistent.

V. NUMERICAL EXPERIMENTS:

In this section, we imply the proposed method for solving five problems and their numerical results introduced in Figure 1 to 5.

Problem 1: $y' = zy^3 - y\ ,\ y(0) = 1\ , 0 \le z \le 10$

Exact solution: $y(z) = \frac{2}{\sqrt{2+4z+2e^{2z}}}$

Source : Famurewa et al. (2011).

Problem 2: $y' = y - z^2 + 1;\ y(0) = \frac{1}{2}\ ;\ [0, 5]$

Exact solution: $y(z) = (1+z)^2 - \frac{1}{2}e^z$

Source: Yaacob and Sanugi (1995).

Problem 3:

$y'_i = -\beta_i y_i + y_i^2\ ;\ i = 1;\ 2;\ 3;\ 4;\ y_i(0) = -1;\ [0, 20]$

with $\beta_1 = 0{,}2;\ \beta_2 = 0{,}2;\ \beta_3 = 0{,}3;\ \beta_4 = 0{,}4$

Exact solution : $y_i(z) = \frac{\beta_i}{1+c_i e^{\beta_i z}}\ ,\ c_i = -(1+\beta_i)$

Source: Johnson and Barney (1976).

Problem 4: $y'_1 = y_3 \quad , \quad y_1(0) = 1, \quad [0, \pi]$

$y'_2 = y_4 \quad , \quad y_2(0) = 1,$

$y'_3 = -e^{-z}y_2\ , \quad y_3(0) = 0,$

$y'_4 = 2e^z y_3\ , \quad y_4(0) = 1,$

Exact solution : $y_1(z) = \cos(z)$

$y_2(z) = e^x\cos(z)$

$y(z) = -\sin(z)$

$y_4(z) = e^x\cos(z) - e^x\sin(z)$

Source : A. .Majid et al. (2012).

Problem 5: Given a non linear first (ODEs)

$y' = 2zy \quad ; y(0) = 1\ ,\ h = 0.1$

Exact solution : $y(z) = e^{z^2}$,

Source : Ibijola E. A. (2015).

Numerical results

Notations used are as follows.

- h : step - size.
- Time : in the seconds.

- Max Error : maximum error $|y_i(z) - y_i|$.
- New 4Pb : 4-point implicit block method with second derivative derived in this paper.
- M.Turki : 3-points implicit block method with additional derivative proposed by M.Turki et al(2018)
- Sahi : A simpson's1/3 block method proposed by Sahi et al. (2012).
- Ayinde S. O: New numerical method for solving first order ODEs proposed by Ayinde S. O et al.(2015).
- Nathaniel : Multistep hybrid method for the solving first order ODEs proposed by Nathaniel et al.(2021).

TABLE 1 : Performance of Problem 1.

h	Methods	Max Error	Time
0.1	New 4Pb M.Turki Sahi	1.146036(-9) 1.368469(-7) 1.350523(-7)	0.033 0.032 0.036
0.05	New 4Pb M.Turki Sahi	2.150196(-12) 5.097240(-9) 9.439226(-9)	0.066 0.072 0.080
0.025	New 4Pb M.Turki Sahi	2.735464(-15) 8.308907(-11) 6.071470(-10)	0.124 0.126 0.131
0.0125	New 4Pb M.Turki Sahi	7.813402(-16) 1.311735(-12) 3.812597(-11)	0.149 0.151 0.157

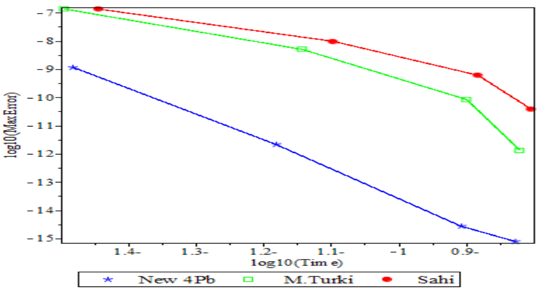

FIGURE 1: The efficiency curves for Problem1.

TABLE 2 : Performance of Problem 2.

h	Methods	Max Error	Time
0.1	New 4Pb M.Turki Sahi	5.069750(-12) 8.462243(-8) 1.605447(-6)	0.006 0.007 0.010
0.05	New 4Pb M.Turki Sahi	7.040700(-14) 1.309864(-9) 2.476227(-8)	0.020 0.025 0.034
0.025	New 4Pb M.Turki Sahi	7.018768(-15) 2.071335(-11) 3.812920(-10)	0.061 0.065 0.075
0.0125	New 4Pb M.Turki Sahi	2.446143(-16) 2.957443(-12) 1.740532(-11)	0.092 0.098 0.122

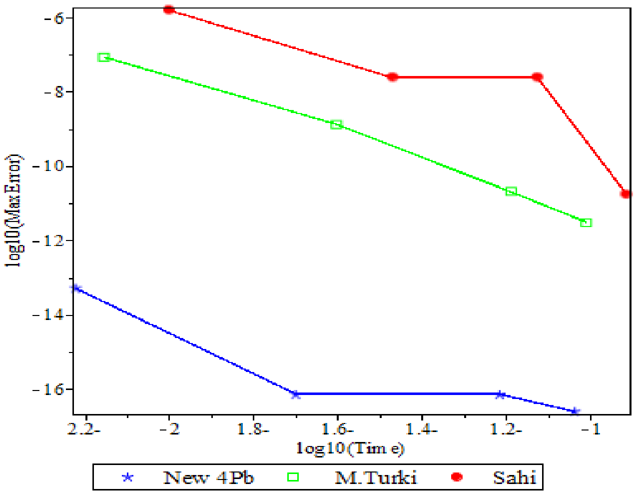

FIGURE 2: The efficiency curves for Problem 2.

TABLE 3 : Performance of Problem 3.

h	Methods	Max Error	Time
0.1	New 4Pb M.Turki Sahi	5.594472(-7) 5.100885(-6) 2.407790(-5)	0.043 0.046 0.062
0.05	New 4Pb M.Turki Sahi	6.463922(-9) 3.832369(-7) 2.257036(-6)	0.102 0.109 0.171
0.025	New 4Pb M.Turki Sahi	2.179912(-10) 1.772670(-8) 1.573703(-7)	0.208 0.218 0.296
0.0125	New 4Pb M.Turki Sahi	2.117508(-12) 6.679853(-10) 1.023594(-8)	0.401 0.405 0.438

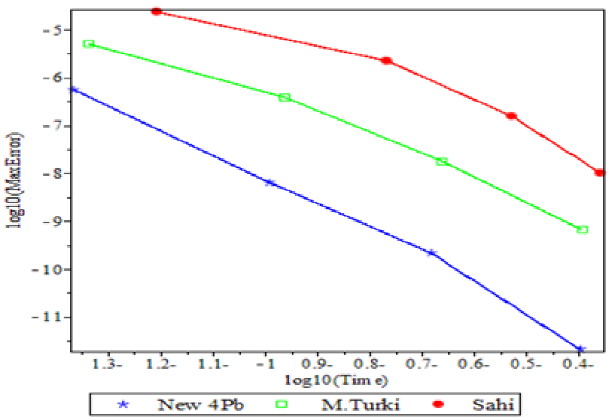

FIGURE 3: The efficiency curves for Problem 3.

TABLE 4: Performance of Problem 4.

h	Methods	Max Error	Time
0.1	New 4Pb M.Turki Sahi	8.417867(-7) 2.941061(-6) 7.541150(-5)	0.038 0.040 0.046
0.05	New 4Pb M.Turki Sahi	9.328705(-9) 8.019770(-8) 3.244254(-6)	0.074 0.089 0.095
0.025	New 4Pb M.Turki Sahi	8.502462(-10) 2.349645(-9) 1.087780(-7)	0.134 0.136 0.141
0.0125	New 4Pb M.Turki Sahi	6.150112(-12) 7.114069(-11) 3.662937(-9)	0.163 0.172 0.175

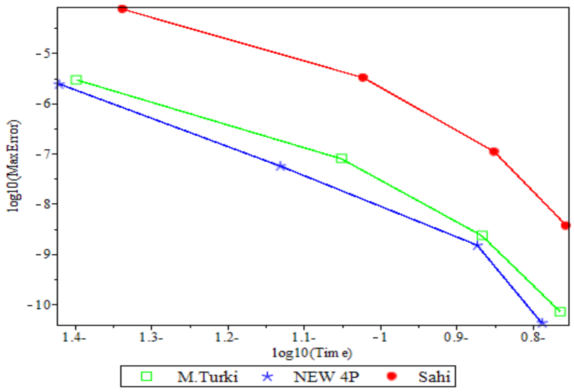

FIGURE 4: The efficiency curves for Problem 4.

TABLE 5. Performance of Problem 5.

z	Exact solution	Result of Proposed method
0.1	1.01005016708417	1.01005016703736
0.2	1.04081077419239	1.04081077411345
0.3	1.09417428370521	1.09417428352353
0.4	1.17351087099181	1.17351087103431
0.5	1.28402541668774	1.28402541686566
0.6	1.43332941456034	1.43332941485812
0.7	1.63231621995538	1.63231622044636
0.8	1.89648087930495	1.89648088013973
0.9	2.24790798667647	2.24790798812912
1.0	2.71828182845905	2.71828183093713

z	New 4Pb	Nathaniel	Ayinde S.O
0.1	7.815986 (-13)	4.3800 (-12)	1.8995(-11)
0.2	6.868656(-12)	1.8500(-11)	1.7145(-10)
0.3	4.241699 (-11)	4.5660 (-11)	1.5564(-10)
0.4	5.202326 (-11)	9.2410 (-11)	1.4150(-10)
0.5	2.314146 (-11)	1.7046(-10)	1.2803(-9)
0.6	3.451841 (-11)	2.9987 (-10)	1.1412(-9)
0.7	3.614966 (-11)	5.1498 (-10)	9.8392(-9)
0.8	4.187598 (-10)	8.7497 (-10)	7.9005(-9)
0.9	7.779973 (-10)	1.48263(-9)	5.3376(-8)
1.0	8.445608 (-10)	2.51810(-9)	1.7703(-8)

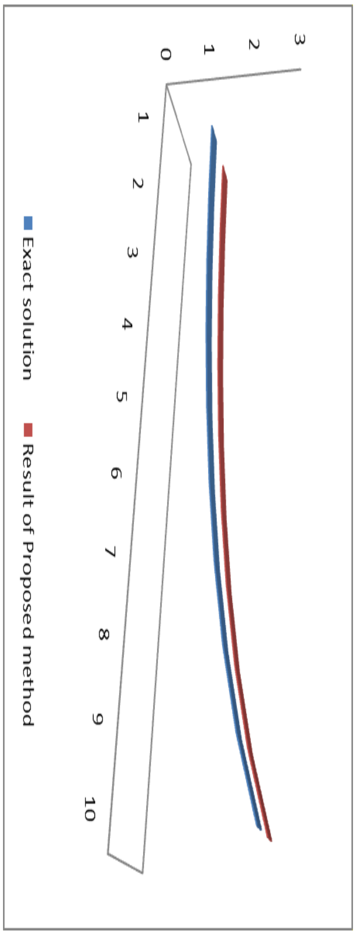

FIGURE 5 : The efficiency curves for Problem 5 (New 4pb method) with step-size z= 0.1, 0.2, 0.3, 0.4 , 0.5, 0.6, 0.7, 0.8, 0.9, 1 .

VI. RESULTS AND DISCUSSION

In this study, we propose 4 – point implicit block method with a second derivation for solving first - order Order Differential Equations. The numerical data are reported in Tables 1 to 5 and displayed in Figures 1 to 5. These figures depicted the efficiency curves, which displayed the common logarithm of the maximum errors in the calculation time. Figures 1-3 indicate that the New 4Pb (4-point block method with second derivative) developed in this research is more efficient than M .Turki (3-point implicit block multistep technique with second derivative) and Sahi's (order 6, Simpson's type block method with second derivative) and Ayinde S.O (New Numerical Method for Solving First Order Differential Equations) and Nathanie (order 6, hybrid multi - step method for solving first order initial value problems of (ODEs)). Tables 1-5 demonstrated that the novel (New 4Pb) approach has a lower maximum error and requires less computing time than M. Turki and Sahi methods and Nathaniel methods and Ayinde methods. Numerical findings showed that the New 4Pb techniques are more efficient than the current methods and that the new block method is more accurate and capable for solving first order ODEs.

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