Analysis of Backward Fuzzy Doubly Stochastic Differential Equations

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DOI: http://dx.doi.org/10.31642/JoKMC/2018/100205

Received Apr. 25, 2023. Accepted for publication Jun. 1, 2023

Abstract—*In this paper, we study formula for the backward doubly fuzzy stochastic differential equations (BFDSDEs), In the beginning, we study some basic concepts, definitions, and Hypotheses to obtain the numerical scheme for BFDSDEs, as our scheme depends on the partition of interval [0, T]. In our work, we prove that under Lipschitz conditions, the approximation solution for the backward fuzzy doubly stochastic differential equations converges to the exact solution by using mean square error, and prove the existence and uniqueness of approximations solutions to BFDSDEs.*

Keywords—: Backward Doubly Fuzzy Stochastic Differential Equations, Backward Fuzzy Stochastic Differential Equations,

1- INTRODUCTION

We are interested in studying the class of BFSDEs. We present some research on stochastic differential equations.

Bodo, et al. [1] presented fundamentals of stochastic calculus, stochastic differential equations, and applications. Mao [2] provided a basic introduction to stochastic differential equations and their applications to anyone who might be interested in finding out more about them. Cortes et al. [3] proposed the numerical solution of stochastic differential equations using Scheme's random Euler difference method. Nouri and Ranjbar [4] used the Rung-Kutta method to prove approximation solutions for stochastic differential equations (SDE) in the presence of initial conditions. Elbarrimi and Ouknine [5] proposed stability approximation solutions to stochastic differential equations driven by continuous semi-martingales.

We consider the BDSDEs that have the following

$$X(t)=\xi+\int_0^T \Psi\big(s, X(s), \Gamma(s)\big)ds+\int_0^T \Phi\big(s, X(s), \Gamma(s)\big)dW(s)$$

$$-\int_0^T \Gamma(s)dB(s) \qquad (1)$$

Where $\{W(t), 0 \le t \le T\}$ and $\{B(t), 0 \le t \le T\}$ are two independent Wiener processes defined on the complete probability spaces $(\Omega_1, \Upsilon_1, P_1)$ and $(\Omega_2, \Upsilon_2, P_2)$ respectively, and $T < \infty$ is a finite time horizon with the natural filtration $(\Upsilon_t, 0 \le t \le T)$ and ξ is a terminal condition such that $E|\xi|^2 < \infty$. Clearly, two stochastic processes $\{(X(t), \Gamma(t)), 0 \le t \le T\}$ with values in $R \times R$ which is Υ_t- adapted and satisfies equation (1), an adapted solution to the equation is said to be this pair. Pardoux and Peng [6] proved the existence and uniqueness of the adapted solution for BSDEs under the Lipschitz condition. Peng [7] proved that a comparison theorem can be used to solve one-dimensional backward stochastic differential equations under Lipschitz conditions. Cao and Yan [8] and Liu and Ren [9] also generalized this theorem. Zhao et al [10]introduced a new numerical approach for BSDEs is presented, which uses the Monte Carlo method to approximate mathematical expectation. Falah and Liu [11] proposed the numerical method for backward stochastic differential equations with non-Lipschitz coefficients. Ma et al [12] proposed an effective numerical technique that reduces the difficulty of building an approximate solution by preventing discretized filtration from convergent to Brownian filtration. Pardoux and Peng [13] presented a class for the backward doubly stochastic differential equations and proved the existence and uniqueness of the adapted solutions. Aman, [14] proposed a numerical method for the class of BDSDEs that may have terminal values that depend on a path. Owo [15] presented a result for the existence and uniqueness of backward doubly stochastic differential equations whose coefficients satisfy the stochastic Lipschitz condition. Matoussi and Sabbagh [16] proposed a numerical solution for backward doubly stochastic differential equations with random terminal time. Bao et al [17] presented a numerical scheme for the BDSDEs using the splitting-up method.

On the other hand, Stojakovic [18] demonstrated that fuzzy random variables with values in separable Banach space exist and are unique when the appropriate Lipschitz condition is met. Kim [19] proved that this solution exists and is unique under the appropriate Lipschitz condition. Malinowski [20] studied fuzzy stochastic integrals and described some of their characteristics. Also, he demonstrated the existence of solutions to stochastic fuzzy differential equations governed by m-dimensional Brownian motion. Nayak and Chakraverty [21]proposed that The fuzzy stochastic Ito integral concept has also been used to locate the precise method's solutions. To handle the fuzzy numbers associated with the fuzzy stochastic differential equations (FSDE), a numerical solution of the FSDE to a broad methodology has been described. Malinowski [22] demonstrated that a unique solution can be found for fuzzy stochastic differential equations of decreasing fuzziness when the necessary approximation sequences and coefficients are used. In this work, we present some fundamental ideas and presumptions for studying the backward fuzzy doubly stochastic differential equations. Additionally, we go over the approximate solution of BDFSDEs under Lipschitz conditions.

This paper is organized as follows: In Section 2, we introduce some basic premises, spaces, and assumptions used to study the approximate solution for BFDSDEs. In section 3, we give some basic definitions and concepts of fuzzy solutions of the backward stochastic differential equations, In section 4, we present a numerical scheme of BFDSDEs. In section 5, we discussed that the approximate solutions of BFDSDEs converge to the exact solution under the Lipschtiz condition and also the existence and uniqueness of the approximate solution to the above equation

2- *PRELIMINARIES AND BASIC HYPOTHESES SELECTING A TEMPLATE*

In this section, we present some of the basic concepts, spaces, and conditions used in the sequel [23] As a result, we assume two independent standard δ-dimensional Wiener Processes $\{W(t), 0 \le t \le T\}$ and $\{B(t), 0 \le t \le T\}$, defined on the complete probability spaces $(\Omega_1, \Upsilon_1, P_1)$ and $(\Omega_2, \Upsilon_2, P_2)$, respectively, and a finite time horizon $T < \infty$. We denote

$$\Upsilon^{B}_{s,t}=\sigma\{B(r)-B(s),\ s \le r \le t\},\ \Upsilon^{W}_{t}=\sigma\{W(r), 0 \le r \le t\}.$$

Moreover, we consider $\Omega = \Omega_1 \times \Omega_2$, $\Upsilon = \Upsilon_1 \otimes \Upsilon_2$ and $P = P_1 \otimes P_2$. In addition, we put $\Upsilon_t \triangleq \Upsilon^W_t \otimes \Upsilon^B_{s,t} \otimes \mathcal{M}$,
where $\mathcal{M}$ is the collection of $\wp$-null sets of Υ. That is to say, the σ-fields Υ_t , $0 \le t \le T$, be $\wp$-complete, and the family of σ-algebra $\Upsilon = \{\Upsilon_t\}_{t\in[0,T]}$ be neither decreasing nor increasing, it is not filtration. We assume the Euclidian norm $|\cdot|$ in R^λ and $R^{\lambda\times\delta}$, the following spaces are used

1- Let $H^2_T(R^\lambda)$ be the space of Υ- predictable processes $X:\Omega\times[0,T]\longrightarrow R^\lambda$ s.t. $E\left[\sup_{0\le t\le T}|X(t)|^2\right] < \infty$.
2- Let $S^2_T(R^{\lambda\times\delta})$ be the space of Υ-adapted processes Γ: $\Omega\times[0,T]\longrightarrow R^{\lambda\times\delta}$ s.t. $E\left[\int_0^T|\Gamma(t)|^2dt\right] < \infty$.

The spaces $H^2_T(R^\lambda)$ and $S^2_T(R^{\lambda\times\delta})$ are done with the norm $\|X\|^2_{H^2_T} = E\left[\sup_{0\le t\le T}|X(t)|^2\right]$ and $\|\Gamma\|^2_{S^2_T} = E\left[\int_0^T|\Gamma(t)|^2dt\right]$, respectively.

We consider the backward doubly stochastic differential equations:

$$d\big(X(t)\big) = \Psi\big(t, X(t), \Gamma(t)\big)dt + \Phi\big(t, X(t), \Gamma(t)\big)dW(t) - \Gamma(t)dB(t) \qquad (2)$$

Where Ψ and Φ are Borel measurable function and $\{W(t), 0 \le t \le T\}$ and $\{B(t), 0 \le t \le T\}$ be a δ- dimensional Wiener process, and ξ is given Υ_t-a measurable random variable with $E|\xi|^2 < \infty$.
Suppose that $\Psi: \Omega\times[0,T]\times R^\lambda\times R^{\lambda\times\delta} \longrightarrow R^\lambda$ and $\Phi: \Omega\times[0,T]\times R^\lambda \longrightarrow R^{\lambda\times\delta}$ are jointly measurable and $(X,\Gamma)\in R^\lambda\times R^{\lambda\times\delta}$.
The following conditions are used:

$$(H_1) \qquad |\Psi(X_1,\Gamma_1,t)-\Psi(X_2,\Gamma_2,t)|^2 \le K\left(|X_1-X_2|^2+|\Gamma_1-\Gamma_2|^2\right)$$

$$|\Phi(X_1,t)-\Phi(X_2,t)|^2 \le K\left(|X_1-X_2|^2\right)$$

where $K > 0$, and for all $t \in [0,T], X_1, X_2 \in R^\lambda$ and $\Gamma_1, \Gamma_2 \in R^{\lambda\times\delta}$

$$(H_2) \qquad |\Psi(X,\Gamma|^2 \vee |\Phi(X,\Gamma)|^2 \le K(1 + |X|^2 + |\Gamma|^2)$$

where $a \vee b = max\ \{a, b\}$

$$(H_3) \qquad E\left[|\xi(t) - \xi(s)|^2\right] \le K(t - s)$$

Theorem(2.1) [2] Let $p \ge 2$ and $\Phi \in \mathcal{M}^2([0,T];\ R^{\lambda\times\delta})$ such that

$$E\left[\int_0^T|\Phi(s)|^p ds\right] < \infty.$$

then

$$E\left(\sup_{0\le t\le T}\left|\int_0^t \Phi(s)dWs\right|^p\right) \le \left(\frac{p^3}{2(p-1)}\right)^{p/2} T^{\frac{p-2}{2}} E\int_o^T|\Phi(s)|^p ds$$

Theorem (2.2) [2] (Gronwall′s inequality) Let $T > 0$ and $C \ge 0$. Let $X(\cdot)$ be a Borel measurable bounded nonnegative function on [0, T], and let $\Gamma(\cdot)$ be a nonnegative integrable function on [0, T]. If

$$X(t) \le C + \int_0^t \Gamma(s)\, X(s)\, ds \qquad for\ all\ 0 \le t \le T$$

then

$$X(t) \le C\ exp\left(\int_0^t X(s)ds\right) \qquad for\ all\ 0 \le t \le T$$

3-Fuzzy Solution of the Backward Stochastic Differential Equations

Let $\eta(R^\lambda)$ be the family of all compact, convex, nonempty subsets R^λ. We denote the fuzzy set space of R^λ as $B(R^\lambda)$, i.e. the set of functions $\varphi: R^\lambda \longrightarrow [0,1]$ such that $[\varphi]^\beta \in \eta(R^\lambda)$ for every $\beta \in [0,1]$, where $[\varphi]^\beta = \{a \in R^\lambda: \varphi(a) \ge \beta\}$ for $\beta \in [0,1]$ and $[\varphi]^0 = \{a \in R^\lambda: \varphi(a) > 0\}$. Assume $(\Omega_1, \Upsilon_1, P_1)$ and

$(\Omega_2, \Upsilon_2, P_2)$ are probability spaces together with a filtration $\{\Upsilon_t\}_{t\in[0,T]}, T \in (0,\infty)$, observing the usual conditions.

A mapping $X: \Omega \longrightarrow B(R^\lambda)$ such a variable is called a fuzzy random variable, if $[X]^\beta: \Omega \longrightarrow \eta(R^\lambda)$ is an Υ-measurable multi-function for each $\beta \in [0, 1]$.

Definition (1): [20] Let (Ω, Υ, P) is a complete probability space, we can say $\mathcal{Y}: \Omega \longrightarrow \Upsilon(R^\lambda)$ be fuzzy random variable if for each $\beta \in [0,1]$, $[X]^\beta : \Omega \longrightarrow \eta(R^\lambda)$ be an Υ-measurable.

Definition (2): [20] A mapping $X: [0,1] \times \Omega \longrightarrow \Upsilon(R^\lambda)$ called a fuzzy stochastic process, if the mapping $X(t,.) = X(t): \Omega \longrightarrow \Upsilon(R^\lambda)$ be a fuzzy random variable.

Definition (3): [20] A fuzzy stochastic process X be δ_∞-continuous if the mappings $X(., W): [0,1] \longrightarrow \Upsilon(R^\lambda)$ be δ_∞-continuous functions.

We consider the backward fuzzy doubly stochastic differential equations as

$$dX(t)=\Psi(t, X(t),\Gamma(t))dt+\Phi(t,X(t),\Gamma(t))dW(t)-\Gamma(t)dB(t)$$
$$X(T)=\xi_T=\xi(X(T),\Gamma(T)) \quad (3)$$

where $\Psi: \Omega \times [0,1] \times R^\lambda \times R^{\lambda\times\delta} \times B(R^\lambda) \longrightarrow B(R^\lambda)$ and $X_T: \Omega \longrightarrow B(R^\lambda)$ is a fuzzy random variable

4-Numerical Scheme of the BDSDEs

In this section, we present a numerical scheme based on a discretization of (1). Therefore, for each integer $n \geq 1$ and $t \in [0, T]$. Assume $0 = t_0 < t_1 < \cdots . t_n = T$, be a partition of $[0, T]$ and denote

$$\pi=\Delta t_{i+1}=t_{i+1}-t_i=\frac{T}{n}, 1 \leq i \leq n, \quad \Delta W(t_{i+1})=W(t_{i+1})-W(t_i)$$
$$,\Delta B(t_{i+1})=B(t_{i+1})-B(t_i)$$

where $i = 0,1, \ldots \ldots, n-1$ and $\Delta t = \max\Delta t_i$. For the small interval $[t_i, t_{i+1}]$, the BFDSDEs is following:

$$X_{t_1}=X_{t_{i+1}}+\int_{t_i}^{t_{i+1}} [\Psi(s, X(s),\Gamma(s))]ds$$
$$+\int_{t_i}^{t_{t+1}} [\Phi(s, X(s),\Gamma(s))]dWs - \int_{t_i}^{t_{i+1}} \Gamma(s)\, dBs$$

Therefore, the approximation formula is

$$X_{t_i}^n=X_{t_{i+1}}^n+\Psi(t,X_i^n(t),\Gamma_i^n(t))\pi+\Phi(t,X_i^n(t),\Gamma_i^n(t))\Delta W(t_{i+1})-\Gamma_i^n(t)\, \Delta B(t_{i+1}) \quad (5)$$

with $X(T) = \xi(T)$ on $0 \leq t \leq T$, so we consider a class of BFSDEs as follows:

$$X_{t_i}^n=\xi+\int_{t_0}^{T} [\Psi(s,X_i^n(s),\Gamma_i^n(s))]ds+\int_{t_0}^{T} [\Phi(s,X_i^n(s),\Gamma_i^n(s))]dWs$$
$$-\int_0^T \Gamma_i^n(s)dB(s) \quad (6)$$

Lemma (4.1). Assume the Lipschitz conditions are fulfilled, then it holds that

$$E[\sup_{0\leq t\leq T}|X(t)|^2+\int_0^T|\Gamma(s)|^2\, ds]\leq M \quad (7)$$

Proof. From the first part of inequality (7), we get that

$$|X(t)|^2=|X(t)+\int_0^T [\Psi(s, X(s), \Gamma(s))]\, ds + \int_0^T [\Phi(s, X(s), \Gamma(s))]dW(s) - \int_0^T \Gamma(s)dB(s)|^2$$

By elementary inequality $|a + b + c + d|^2 \leq 4(|a|^2 + |b|^2 + |c|^2 + |d|^2)$, we have that

$$|X(t)|^2\leq4|X(t)|^2+4\left|\int_0^T [\Psi(s, X(s),\Gamma(s))]ds\right|^2 + 4\left|\int_0^T [\Phi(s, X(s), \Gamma(s)\,]\, dW(s)\right|^2 +4\left|\int_0^T \Gamma(s)dB(s)\right|^2.$$

Taking the expectation and supremum, we obtain that

$$E\left[\sup_{0\leq t\leq T}|X(t)|^2\right]\leq4E\sup_{0\leq t\leq T}|X(t)|^2+ 4(T)E\sup_{0\leq t\leq T}\int_0^T \left[|\Psi(s, X(s), \Gamma(s))|^2\right] ds$$
$$+4E\left(\sup_{0\leq t\leq T}\left|\int_0^T [\Phi(s, X(s),\Gamma(s))]dW(s)\right|^2\right) +4E\left(\sup_{0\leq t\leq T}\left|\int_0^T \Gamma(s)\, dB(s)\right|^2\right).$$

By using theorem (2.1), we get that

$$E\left[\sup_{0\leq t\leq T}|X(t)|^2\right]\leq4E|X(t)|^2+4(T)\int_0^T E|\Psi(s, X(s),\Gamma(s)|^2\, ds$$
$$+4\left(\frac{2^3}{2(2-1)}\right)^{\frac{2}{2}} T^{\frac{2-2}{2}}E\int_0^T |\Phi(s, X(s),\Gamma(s))|^2ds$$
$$+4\left(\frac{2^3}{2(2-1)}\right)^{\frac{2}{2}} T^{\frac{2-2}{2}}E\int_0^T |\Gamma(s)|^2ds\,.$$

Using Lipschitz condition H_2 we have that

$$E\left[\sup_{0\leq t\leq T}|X(t)|^2\right]\leq4E|X(T)|^2+4(T)E\int_0^T K(1+|X|^2+|\Gamma|^2)\, ds +16E\int_0^T K(1+|X|^2+|\Gamma|^2)\, ds+16\int_0^T E|\Gamma(s)|^2\, ds,$$

where $K_1 = 4E|X(T)|^2$, we get that

$$E\left[\sup_{0\leq t\leq T}|X(t)|^2\right]\leq K_1+4K(T)(1+E|X|^2+E|\Gamma|^2)(T) +16E|\Gamma(t)|^2(T) \quad (8)$$

Now, we take the second part of inequality (7), we have that

$$E\int_0^T |\Gamma(s)|^2\, ds = \int_0^T E|\Gamma(s)|^2 ds\,.$$

Then, we obtain that

$$E\int_0^T |\Gamma(s)|^2 ds = E|\Gamma(t)|^2 \int_0^T ds \qquad (9)$$

From equations (8) and (9) we obtain that

$$E\left[\sup_{0\le t\le T}|X(t)|^2+\int_0^T|\Gamma(s)|^2ds\right]\le K_1+4K(T)(1+E|X|^2+E|\Gamma|^2)(T)$$
$$+17E|\Gamma(t)|^2(T).$$

Let $K_2 = 4K(T)(1+E|X|^2+E|\Gamma|^2)(T)$ and $K_3 = 17E|\Gamma(t)|^2(T)$, we have that

$$E\left[\sup_{0\le t\le T}|X(t)|^2+\int_0^T|\Gamma(s)|^2ds\right]\le K_1+K_2+K_3.$$

Choosing $M=K_1+K_2+K_3$, we get that

$$E\left[\sup_{0\le t\le T}|X(t)|^2+\int_0^T|\Gamma(s)|^2ds\right]\le M$$

Lemma (4.2). Under Lipschitz conditions, for all $p\ge 2$, there exists a positive constant M_1 and M_2 such that

$$E\left(\sup_{0\le t\le T}|X(t)|^p\right)\le M_1$$

And

$$E\left(\sup_{0\le t\le T}|\Gamma(t)|^p\right)\le M_2$$

Proof.

$$|X(t)|^p = |X(t)+\int_0^T\Psi(s, X(s), \Gamma(s))ds$$
$$+\int_0^T\Phi(s, X(s),\Gamma(s))dW(s)$$
$$+\int_0^T\Gamma(s)dB(s)|^p$$

By using elementary inequality $|a+b+c+d|^p \le 4^{p-1}(|a|^p+|b|^p+|c|^p+|d|^p)$, we have that

$$|X(t)|^p\le 4^{p-1}|X(t)|^p+4^{p-1}\left|\int_0^T\Psi(s,X(s),\Gamma(s))ds\right|^p$$
$$+4^{p-1}\left|\int_0^T\Phi(s,X(s),\Gamma(s))dW(s)\right|^p$$
$$+4^{p-1}\left|\int_0^T\Gamma(s)dB(s)\right|^p.$$

Taking the expectation and supremum of the inequality above, we obtain that

$$E\left[\sup_{0\le t\le T}|X(t)|^p\right]\le 4^{p-1}E\sup_{0\le t\le T}|X(t)|^p$$
$$+[4(T)]^{p-1}E\sup_{0\le t\le T}\int_0^T|\Psi(s, X(s),\Gamma(s))|^pds$$
$$+4^{p-1}E\left(\sup_{0\le t\le T}\left|\int_0^T\Phi(s, X(s),\Gamma(s))dW(s)\right|^p\right)$$
$$+4^{p-1}E\left(\sup_{0\le t\le T}\left|\int_0^T\Gamma(s)dB(s)\right|^p\right).$$

Using theorem (2.1), we have that

$$E\left[\sup_{0\le t\le T}|X(t)|^p\right]\le 4^{p-1}E|X(T)|^p+[4(T)]^{p-1}E\int_0^T|\Psi(s, X(s),\Gamma(s))|^pds$$
$$+4^{p-1}\left(\frac{p^3}{2(p-1)}\right)^{\frac{p}{2}}(T)^{\frac{p-2}{2}}E\int_0^T\left|\Phi(s,X(s),\Gamma(s))\right|^pds$$
$$+4^{p-1}\left(\frac{p^3}{2(p-1)}\right)^{\frac{p}{2}}(T)^{\frac{p-2}{2}}E\int_0^T|\Gamma(s)|^pds\,.$$

By using the Lipschitz condition H_2, we get that

$$E\left[\sup_{0\le t\le T}|X(t)|^p\right]\le 4^{p-1}E|X(T)|^p+[4(T)]^{p-1}E\int_0^TK(1+|X|^p+|\Gamma|^p)ds$$
$$+4^{p-1}\left(\frac{p^3}{2(p-1)}\right)^{\frac{p}{2}}(T)^{\frac{p-2}{2}}E\int_0^TK(1+|X|^p+|\Gamma|^p)ds$$
$$+4^{p-1}\left(\frac{p^3}{2(p-1)}\right)^{\frac{p}{2}}(T)^{\frac{p-2}{2}}E\int_0^T|\Gamma(s)|^pds\,,$$

Then

$$E\left[\sup_{0\le t\le T}|X(t)|^p\right]\le 4^{p-1}E|X(T)|^p+4^{p-1}K\left[(T)^{p-1}+\left(\frac{p^3}{2(p-1)}\right)^{\frac{p}{2}}(T)^{\frac{p-2}{2}}\right]$$
$$\int_0^TE(1+|X|^p+|\Gamma|^p)ds+4^{p-1}\left(\frac{p^3}{2(p-1)}\right)^{\frac{p}{2}}(T)^{\frac{p-2}{2}}E\int_0^T|\Gamma(s)|^pds\,,$$

Let $C_1=4^{p-1}E|X(T)|^p$, we have that

$$E\left[\sup_{0\le t\le T}|X(t)|^p\right]\le C_1+4^{p-1}K\left[(T)^{p-1}+\left(\frac{p^3}{2(p-1)}\right)^{\frac{p}{2}}(T)^{\frac{p-2}{2}}\right]$$
$$[1+E|X|^p+E|\Gamma|^p](T)+4^{p-1}\left(\frac{p^3}{2(p-1)}\right)^{\frac{p}{2}}(T)^{\frac{p-2}{2}}E|\Gamma(s)|^p(T).$$

Let $C_2 = 4^{p-1}K\left[(T)^{p-1}+\left(\frac{p^3}{2(p-1)}\right)^{\frac{p}{2}}(T)^{\frac{p-2}{2}}\right][1+E|X|^p+E|\Gamma|^p](T)$

and $C_3 = 4^{p-1}\left(\frac{p^3}{2(p-1)}\right)^{\frac{p}{2}}(T)^{\frac{p-2}{2}}E|\Gamma(s)|^p(T)$, we obtain that

$$E\left[\sup_{0\le t\le T}|X(t)|^p\right]\le C_1+C_2+C_3.$$

Choosing $M_1=C_1+C_2+C_3$, we obtain that

$$E\left[\sup_{0\le t\le T}|X(t)|^p\right]\le M_1.$$

And consequently, we have that

$$E\left(\sup_{0\le t\le T}|\Gamma(t)|^p\right)\le M_2$$

5-Main Result

The purpose of this section is to discuss approximation solutions to the backward fuzzy doubly stochastic differential equations

Theorem (5.1): Assume that the conditions $(H_1 - H_3)$ are fulfilled. Let $\pi = 0 = t_0 < \cdots < t_n = T$ be any partition of [0, T] with $|\pi|=\max_{0\le t\le T}|t_i - t_{i-1}|$, then there exists a constant A depending only on T and K, such that

$$E\left[\max_{1\le i\le n}|X_i(t)-X_i^n(t)|^2+\sum_{i=1}^n\int_{t_{i-1}}^{t_i}|\Gamma_i(s)-\Gamma_i^n(s)|^2ds\right]\le A|\pi| \qquad (10)$$

Proof. For all $t\in[0,T]$, from the first part of inequality (10), we obtain that

$$|X_i(t)-X_i^n(t)|^2=\Big|X_i(t)-X_i^n(t)$$
$$+\int_{t_{i-1}}^{t_i}\left[\Psi(s,X_i(s),\Gamma_i(s))-\Psi(s,X_i^n(s),\Gamma_i^n(s))\right]ds$$
$$+\int_{t_{i-1}}^{t_i}\left[\Phi(s,X_i(s),\Gamma_i(s))-\Phi(s,X_i^n(s),\Gamma_i^n(s))\right]dW(s)$$
$$-\int_{t_{i-1}}^{t_i}[\Gamma_i(s)-\Gamma_i^n(s)]\,dB(s)\Big|^2.$$

By using elementary inequality$|a+b+c+d|^2\le 4(|a|^2+|b|^2+|c|^2+|d|^2)$, we have that

$$|X_i(t)-X_i^n(t)|^2\le 4|X_i(t)-X_i^n(t)|^2$$
$$+4\left|\int_{t_{i-1}}^{t_i}\left[\Psi(s,X_i(s),\Gamma_i(s))-\Psi(s,X_i^n(s),\Gamma_i^n(s))\right]ds\right|^2$$
$$+4\left|\int_{t_{i-1}}^{t_i}\left[\Phi(s,X_i(s),\Gamma_i(s))-\Phi(s,X_i^n(s),\Gamma_i^n(s))\right]dW(s)\right|^2$$
$$+4\left|\int_{t_{i-1}}^{t_i}[\Gamma_i(s)-\Gamma_i^n(s)]\,dB(s)\right|^2.$$

Using Lipschitz condition H_1, we have that

$$|X_i(t)-X_i^n(t)|^2\le 4|X_i(t)-X_i^n(t)|^2+4(t_i-t_{i-1})$$
$$\int_{t_{i-1}}^{t_i}K[|X_i(s)-X_i^n(s)|^2+|\Gamma_i(s)-\Gamma_i^n(s)|^2]ds$$
$$+4\int_{t_{i-1}}^{t_i}K[|X_i(s)-X_i^n(s)|^2+|\Gamma_i(s)-\Gamma_i^n(s)|^2]|dWs|^2$$
$$+4\int_{t_{i-1}}^{t_i}\left[|\Gamma_i(s)-\Gamma_i^n(s)|^2\right]|dB(s)|^2.$$

Taking the expectation and maximum inequality above, we obtain that

$$E[\max_{1\le i\le n}|X_i(t)-X_i^n(t)|^2]\le 4E\max_{1\le i\le n}|X_i(t)-X_i^n(t)|^2$$
$$+4K\max_{1\le i\le n}(t_i-t_{i-1})\int_{t_{i-1}}^{t_i}E[|X_i(s)-X_i^n(s)|^2+|\Gamma_i(s)-\Gamma_i^n(s)|^2]ds$$
$$+4K\max_{1\le i\le n}\int_{t_{i-1}}^{t_i}E\left[|X_i(s)-X_i^n(s)|^2+|\Gamma_i(s)-\Gamma_i^n(s)|^2\right]ds$$
$$+4\max_{1\le i\le n}\int_{t_{i-1}}^{t_i}E[|\Gamma_i(s)-\Gamma_i^n(s)|^2]\,ds.$$

Using Lipschitz condition H_3, we have

$$E[\max_{1\le i\le n}|X_i(t)-X_i^n(t)|^2]\le 4K(T)+4K(T+1)$$
$$\int_0^T E[|X_i(s)-X_i^n(s)|^2+|\Gamma_i(s)-\Gamma_i^n(s)|^2]ds$$
$$+4\int_0^T E\left[|\Gamma_i(s)-\Gamma_i^n(s)|^2\right]|ds,$$

then

$$E[\max_{1\le i\le n}|X_i(t)-X_i^n(t)|^2]\le 4K(T)+4K(T+1)$$
$$E|X_i(s)-X_i^n(s)|^2+E|\Gamma_i(s)-\Gamma_i^n(s)|^2(T)$$
$$+4\;E\left[|\Gamma_i(s)-\Gamma_i^n(s)|^2\right](T).$$

Where $K_1 = 4K$, $K_2 = 4K(T+1)E|X_i(s)-X_i^n(s)|^2+E|\Gamma_i(s)-\Gamma_i^n(s)|^2$ and $K_3 = 4\;E[|\Gamma_i(s)-\Gamma_i^n(s)|^2]$

And $|\pi|=\max_{0\le t\le T}|t_i-t_{i-1}|=T$, we obtain that

$$E[\max_{1\le i\le n}|X_i(t)-X_i^n(t)|^2]\le K_1|\pi|+K_2|\pi|+K_3|\pi|. \qquad (11)$$

Now, we prove the second part of the inequality (10), and we get that

$$\sum_{i=1}^n\int_{t_{i-1}}^{t_i}|\Gamma_i(s)-\Gamma_i^n(s)|^2ds=\int_0^T|\Gamma_i(s)-\Gamma_i^n(s)|^2\,ds$$

Taking expectations, we have that

$$E\left[\sum_{i=1}^n\int_{t_{i-1}}^{t_i}|\Gamma_i(s)-\Gamma_i^n(s)|^2ds\right]=\int_0^T E|\Gamma_i(s)-\Gamma_i^n(s)|^2\,ds.$$

Then

$$E\left[\sum_{i=1}^n\int_{t_{i-1}}^{t_i}|\Gamma_i(s)-\Gamma_i^n(s)|^2ds\right]=E|\Gamma_i(s)-\Gamma_i^n(s)|^2(T),$$

Where $K_4 = E|\Gamma_i(s)-\Gamma_i^n(s)|^2$ And $|\pi|=\max_{0\le t\le T}|t_i-t_{i-1}|=T$, we obtain that

$$E\left[\sum_{i=1}^n\int_{t_{i-1}}^{t_i}|\Gamma_i(s)-\Gamma_i^n(s)|^2ds\right]=K_4|\pi|, \qquad (12)$$

from (11) and (12), we get that

$$E\left[\max_{1\le i\le n}|X_i(t)-X_i^n(t)|^2+\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}|\Gamma_i(s)-\Gamma_i^n(s)|^2ds\right]\le K_1|\pi|$$

$$K_2|\pi|+K_3|\pi|+K_4|\pi|$$

then

$$E\left[\max_{1\le i\le n}|X_i(t)-X_i^n(t)|^2+\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}|\Gamma_i(s)-\Gamma_i^n(s)|^2ds\right]\le [K_1+K_2$$

$$+K_3+K_4]|\pi|$$

Let constant A=$K_1+K_2+K_3+K_4$, we have that

$$E\left[\max_{1\le i\le n}|X_i(t)-X_i^n(t)|^2+\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}|\Gamma_i(s)-\Gamma_i^n(s)|^2ds\right]\le A\,|\pi|$$

Theorem (5.2): consider that {X (t), Γ(t)} be a solution of equation (1) then the approximated solution $\{X^n(t),\Gamma^n(t)\}$ converges to {X(t),Γ(t)} in the sense that for all t∈ $[0,T]$

$$\lim_{n\to\infty}E|X(t)-X^n(t)|^2=0$$

And

$$\lim_{n\to\infty}E\int_0^T|\Gamma(s)-\Gamma^n(s)|^2ds=0$$

Proof. First, we prove that

$$\lim_{n\to\infty}E|X(t)-X^n(t)|^2=0$$

$$|X(t)-X^n(t)|^2=\Big|\xi-\xi^n+\int_0^T[\Psi(s,X(s),\Gamma(s))$$
$$-\Psi(s,X^n(s),\Gamma^n(s))]ds+\int_0^T[\Phi(s,X(s),\Gamma(s))$$
$$-\Phi(s,X^n(s),\Gamma^n(s))]dW(s)-\int_0^T[\Gamma(s)-\Gamma^n(s)]\,dB(s)\Big|^2$$

.

By using elementary inequality$|a+b+c+d|^2\le 4(|a|^2+|b|^2+|c|^2+|d|^2)$, we have that

$$|X(t)-X^n(t)|^2\le 4|\xi-\xi^n|^2+4\Big|\int_0^T[\Psi(s,X(s),\Gamma(s))$$
$$-\Psi(s,X^n(s),\Gamma^n(s))ds\Big|^2$$
$$+4\Big|\int_0^T[\Phi(s,X(s),\Gamma(s))-\Phi(s,X^n(s),\Gamma^n(s))]dW(s)\Big|^2$$
$$+4\Big|\int_0^T[\Gamma(s)-\Gamma^n(s)]\,dB(s)\Big|^2,$$

Then

$$|X(t)-X^n(t)|^2\le 4|X(T)-X^n(T)|^2+4\Big|\int_0^T[\Psi(s,X(s),\Gamma(s))$$
$$-\Psi(s,X^n(s),\Gamma^n(s))ds\Big|^2$$
$$+4\Big|\int_0^T[\Phi(s,X(s),\Gamma(s))-\Phi(s,X^n(s),\Gamma^n(s))]dW(s)\Big|^2$$
$$+4\Big|\int_0^T[\Gamma(s)-\Gamma^n(s)]\,dB(s)\Big|^2.$$

Taking expectations and using Lipschitz condition H_1, we have that

$$E|X(t)-X^n(t)|^2\le 4E|X(T)-X^n(T)|^2+4(T)$$

$$E\int_0^T K[|X(s)-X^n(s)|^2+|\Gamma^n(s)-\Gamma(s)|^2]ds$$

$$+4E\int_0^T K[|X(s)-X^n(s)|^2+|\Gamma^n(s)-\Gamma(s)|^2]ds$$

$$+4E\int_0^T\left[|\Gamma(s)-\Gamma^n(s)|^2\right]ds,$$

using condition H_3, we have that

$$E|X(t)-X^n(t)|^2\le 4K(T)+4K(T+1)$$

$$\int_0^T E[|X(s)-X^n(s)|^2+|\Gamma^n(s)-\Gamma(s)|^2]ds$$

$$+4E\int_0^T\left[|\Gamma(s)-\Gamma^n(s)|^2\right]ds.$$

Let $C_1=4K(T)$, and $C_2=4K(T+1)$, we obtain that

$$E|X(t)-X^n(t)|^2\le C_1+C_2\int_0^T E[|X(s)-X^n(s)|^2$$

$$+|\Gamma(s)-\Gamma^n(s)|^2]ds+4E\int_0^T\left[|\Gamma(s)-\Gamma^n(s)|^2\right]ds.$$

And then

$$E|X(t)-X^n(t)|^2\le C_1+C_2\int_0^T E[|X(s)-X^n(s)|^2\,ds,$$

for all $t\in[0,T]$ using theorem (2.2), we get that

$$\lim_{n\to\infty}E|X(t)-X^n(t)|^2=0\,.$$

And so, we obtain the following

$$\lim_{n\to\infty}E\int_0^T|\Gamma(s)-\Gamma^n(s)|^2ds=0$$

Theorem (5.3): Suppose that the assumptions (H_1-H_3) are fulfilled, then there exists a unique solution of the following equation

$$X(t)=\xi+\int_0^T \Psi(s,X(s),\Gamma(s))ds+\int_0^T \Phi(s,X(s),\Gamma(s))dW(s)$$
$$\int_0^T (\Gamma(s))dB(s)$$

Proof. Existence.

By using theorem (5.1), then there exist

$X \in H_T^2(R^\lambda)$ and $\Gamma \in S_T^2(R^{\lambda\times\delta})$

Such that

$$\lim_{n\to\infty} (X^n,\Gamma^n)=(X,\Gamma).$$

Then the (lemma (4.2)) and (theorem (5.2)) shows that

$$\lim_{n\to\infty} E|X^n(t)-X(t)|^2=0 \qquad 0\le t\le T$$

Then by using a lemma (4.1), a lemma (4.2), and a theorem (5.1), we have the result.

Uniqueness.

Let pair (X_i,Γ_i), where i=1, 2 is the solution of BFDSDEs by theorem (5.2), we obtain that

$$\lim_{n\to\infty} E|X(t)-X_1^n(t)|^2=0$$

And

$$\lim_{n\to\infty} E|X(t)-X_2^n(t)|^2=0$$

Now, we prove that

$$\operatorname{Lim}_{n\to\infty} E|X_1^n(t)-X_2^n(t)|^2=0$$

$$|X_1^n(t)-X_2^n(t)|^2=|\xi_1^n-\xi_2^n+\int_0^T [\Psi(s,X_1^n(s),\Gamma_1^n(s)) -\Psi(s,X_2^n(s),\Gamma_2^n(s))]ds$$
$$+\int_0^T [\Phi(s,X_1^n(s),\Gamma_1^n(s)) -\Phi(s,X_2^n(s),\Gamma_2^n(s))]dWs +\int_0^T [\Gamma_1^n(s)-\Gamma_2^n(s)]\,dB(s)|^2.$$

By elementary inequality $|a+b+c+d|^2\le 4(|a|^2+|b|^2+|c|^2+|d|^2)$, we have that

$$|X_1^n(t)-X_2^n(t)|^2\le 4|\xi_1^n-\xi_2^n|^2+4|\int_0^T [\Psi(s,X_1^n(s),\Gamma_1^n(s)) - \Psi(s,X_2^n(s),\Gamma_2^n(s))]ds|^2+4|\int_0^T [\Phi(s,X_1^n(s),\Gamma_1^n(s))$$
$$-\Phi(s,X_2^n(s),\Gamma_2^n(s))]dWs|^2+4\left|\int_0^T [\Gamma_1^n(s)-\Gamma_2^n(s)]\,dB(s)\right|^2.$$

Taking the expectation and using the condition H_1, $\xi = X(T)$, we have that

$$E|X_1^n(t)-X_2^n(t)|^2\le 4E|X_1^n(T)-X_2^n(T)|^2+4(T)E\int_0^T K[|X_1^n(s)-X_2^n(s)|^2+|\Gamma_1^n(s)-\Gamma_2^n(s)|^2]ds$$
$$+4E\int_0^T K[|X_1^n(s)-X_2^n(s)|^2+|\Gamma_1^n(s)-\Gamma_2^n(s)|^2]|dW(s)|^2$$
$$+4E\int_0^T |\Gamma_1^n(s)-\Gamma_2^n(s)|^2\,|dB(s)|^2.$$

Using condition H_3, we have that

$$E|X_1^n(t)-X_2^n(t)|^2\le 4K(T)+4K(T)\int_0^T E[|X_1^n(s)-X_2^n(s)|^2 +|\Gamma_1^n(s)-\Gamma_2^n(s)|^2]ds$$
$$+4K\int_0^T E[|X_1^n(s)-X_2^n(s)|^2+|\Gamma_1^n(s)-\Gamma_2^n(s)|^2]ds$$
$$4\int_0^T E|\Gamma_1^n(s)-\Gamma_2^n(s)|^2\,ds,$$

then

$$E|X_1^n(t)-X_2^n(t)|^2\le 4K(T)+4K(T+1)\int_0^T E\,[|X_1^n(s)-X_2^n(s)|^2 +|\Gamma_1^n(s)-\Gamma_2^n(s)|^2]ds$$
$$+4\int_0^T E|\Gamma_1^n(s)-\Gamma_2^n(s)|^2\,ds.$$

Let $C_1 = 4K(T)$ and $C_2 = 4K(T+1)$, we get that

$$E|X_1^n-X_2^n|^2\le C_1+C_2\int_0^T E[|X_1^n(s)-X_2^n(s)|^2+|\Gamma_1^n(s)-\Gamma_2^n(s)|^2]ds$$
$$+4\int_0^T E|\Gamma_1^n(s)-\Gamma_2^n(s)|^2\,ds.$$

Then $E|X_1^n-X_2^n|^2\le C_1+C_2\int_0^T E|X_1^n(s)-X_2^n(s)|^2ds$,

for all $t\in[0,T]$. Using theorem (2.2), we obtain that

$$\lim_{n\to\infty} E|X_1^n(t)-X_2^n(t)|^2=0.$$

Hence, we get that $X_1^n(t)=X_2^n(t)$

And consequently, we obtain that

$$\lim_{n\to\infty} E|\Gamma_1^n(t)-\Gamma_2^n(t)|^2=0$$

CONCLUSION

In this work, we have introduced the BDFSDEs and studied that the approximate solution of backward double fuzzy stochastic differential equations converges to the exact solution under Lipschitz conditions using mean squared error. We also discuss the existence and uniqueness of the approximate solution of the BDFSDE.

ACKNOWLEDGMENT

We appreciate the referee's very constructive and detailed comments to improve the presentation of this paper

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