Shadowing in Fuzzy Dynamical Systems

Safa Ezghair Ali
Department of Mathematics, Collage of Education for Pure Sciences
University of Babylon
Iraq

Safa.ali.pure376@student.uobabylon.edu.iq
Orcid.org/0000-0003-1687-176X

May Alaa.A.Al-Yaseen
Department of Mathematics, Collage of Education for Pure Sciences
University of Babylon
Iraq

pure.may.alaa@uobabylon.edu.iq
Orcid.org/0009-0007-6056-7649

DOI: http://dx.doi.org/10.31642/JoKMC/2018/100224

Received May. 26, 2023. Accepted for publication Jul.10, 2023

Abstract**—***In this paper we introduce the notions (L-shadowing property, two – sided limit shadowing, negative limit shadowing property, simple two – sided limit shadowing) for fuzzy dynamical systems and prove that properties (fuzzy shadowing property, fuzzy L-shadowing property, fuzzy two – sided limit shadowing) are invariant under fuzzy topological conjugation.*

Keywords—* *Fuzzy metric space, fuzzy dynamical system, fuzzy L-shadowing property, the fuzzy two – sided limit shadowing property, fuzzy negative limit shadowing property , fuzzy h-shadow.

I. INTRODUCTION

Qualitative theory of *dynamical systems* is the study of the long-term behavior of evolving systems under perturbations ([2])[4].The evolution of a particular state of a dynamical system is related to its *orbits*. For this purpose we need to know the structure of these orbits by solving a *system* of differential equations. If we cannot solve a system of differential equations analytically, then *pseudo orbits* are suggested as replacements for the *orbits* [4].

The notion *pseudo − orbit tracing* property, also known as *shadowing*, are among the most important concepts in *topological dynamics* [1]. Also, the rest of the shadowing properties can be studied in the fuzzy dynamic

In this paper we are going to use of *fuzzy* distances to approximate the pseudo orbits and their *shadowing* properties [4].

In section 2 we introduce the notions (fuzzy *L − shadowing property*, fuzzy two – sided limit *shadowing*, fuzzy negative limit *hadowing property*) for fuzzy dynamical system. We prove that *these properties* (fuzzy shadowing property, fuzzy L-shadowing property ,fuzzy two–sided limit shadowing) are invariant under fuzzy topological conjugacy. Also, the rest of the shadowing properties can be studied in the fuzzy dynamical system.

II. PRELIMINARIES

Definition 2.1[6]

A binary operation $*: [0,1] \times [0,1] \rightarrow [0,1]$ is called *a continuous triangular norm* $(t-norm)$ if $*$ satisfies the following conditions:

1. $*$ is *a ssociative and commutative*;

2. $*$ is *continuous*;

3. $a * 1 = a$ for all $a \in (0,1]$;

4. $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$;

5. If $a * b = a * c$ *then* $b = c$.

Properties 4 and 5 of *a continuous* $t-norm$ imply that if

$a * b \leq a * c$ then $b \leq c$.

Definition 2.2[5]

A fuzzy *metric space* is a triple n $(X, M,*)$ where X is a non-empty set, $*$ is *a continuous* $t-norm$ and $M : X \times X \times (0,\infty) \longrightarrow [0,1]$ is a mapping which has the following properties :

For every $x, y, z \in X$ *and* $t, s > 0$:

1. $M(x,y,t) > 0$;

2. $M(x,y,t) = 1$ if and only if $x = y$;

3. $M(x,y,t) = M(y,x,t)$;

4. $M(x,z,t+s) \geq M(x,y,t) * M(y,z,s)$;

5. $M(x, y, \cdot) : (0, \infty) \longrightarrow [0, 1]$ is a continuous map.

Definition 2.3 [5]

Let $(X, M, *)$ be a fuzzy metric space. We define *open ball* $B(x, r, t)$ with center $x \in X$ and radius, $0 < r < 1$ as

$$B(x, r, t) = \{ y \in X;\ M(x, y, t) > 1 - r \}.$$

Result 2.4[5]

Let $(X, M, *)$ be a fuzzy *metric space*. Define

$\tau = \{A \subset X: x \in A \Leftrightarrow there\ exist\ t > 0\ and\ r \in (0,1)\ s.t\ B(x, r, t) \subset A\}$.

Then τ is a topology on X.

Result 2.5[7]

Let (X, d) be a metric space. Then $M(x, y, t) = M_d(x, y, t) = \frac{t}{t + d(x,y)}$ with the $t - norm$ $a * b = ab$ is a fuzzy metric defined on X (called standard *fuzzy* metric space) and the $topology\ \tau_d$ induced by the metric d and the $topology\ \tau_M$ are the same.

Definition 2.6 [4]

A $fuzzy\ map\ T: X \to X$ is $fuzzy\ continuous\ in\ x_0$, for each $0 < \varepsilon < 1\ and\ t > 0$ there is $0 < \delta < 1$ so that for each $x \in X$ with $M(x_0, y, t) > 1 - \delta$, we deduce $M(T(x_0), T(y), t) > 1 - \varepsilon$.

Definition 2.7[4]

We say that two $fuzzy\ maps$ $T : X_1 \to X_1\ and\ g : X_2 \to X_2$ on fuzzy $metric\ spaces$ $(X_1, M_1, *_1)$ and $(X_2, M_2, *_2)$ are $topologically\ conjugate$ if there is a fuzzy $homeomorphism\ h : X_1 \to X_2 (a\ fuzzy\ continuous\ bijection$ map with fuzzy continuous inverse).

Definition 2.8[4]

Let $(X, M, *)$ be a $fuzzy\ metric\ space$. Let $T : X \to X$ be a $fuzzy\ homeomorphism$ on X.We say that T has the $fuzzy\ pseudo - orbit\ tracing$ property on X, if for each $\varepsilon \in (0,1)\ and\ t > 0, there\ exists\ \delta \in (0,1)$ so that for a given sequence $(x_k)_{k \in \mathbb{Z}} \subset X$ with $M(T(x_k), x_{k+1}, t) \leq 1 - \delta$ for every $k \in \mathbb{Z}$ (called $\delta - pseudo\ orbit$) there is a point $p \in X$ $such\ that\ M(T^k(p), x_k, t) \leq 1 - \varepsilon$ for every $k \in \mathbb{Z}$.

Definition 2.9 [4]

Let $(X, M, *)$ be a $fuzzy$ metric space. Let $T : X \to X$ be a $fuzzy\ homeomorphism$ on X.We say that T has the $fuzzy\ limit\ shadowing\ property$ on X, if for each $t > 0$ and each $sequence (x_k)_{k \in \mathbb{Z}} \subset X$ with $\lim_{k \to \infty} M(T(x_k), x_{k+1}, t) = 1$ $there\ exists\ a\ point\ p \in X$ so that $\lim_{k \to \infty} M(T^k(p), x_k, t) = 1$.

Definition 2.10

Let $(X, M, *)$ be a fuzzy $compact\ metric\ space$ .Let $T : X \to X$ a fuzzy $homeomorphism$. We say that T has fuzzy $L - shadowing\ property$ if for every $0 < \varepsilon < 1$, there exists $0 < \delta < 1$ such that for every sequence $(x_k)_{k \in \mathbb{Z}}$ satisfying $M(T(x_k), x_{k+1}, t) \leq 1 - \delta$ for every $k \in \mathbb{Z}$,and $\lim_{k \to \infty} M(T(x_k), x_{k+1}, t) = 1$, there is $p \in X$ satisfying $M(T^k(p), x_k, t) \leq 1 - \varepsilon\ for\ every\ k \in \mathbb{Z}$,$\lim_{k \to \infty} M(T^k(p), x_k, t) = 1$.

Definition 2.11

Let $(X, M, *)$ be a $fuzzy\ compact\ metric\ space$ .Let $T: X \longrightarrow X\ a\ homeomorphism$. We say that T has the fuzzy two-sided limit $shadowing\ property$ if every fuzzy $t$wo-sided limit $pseudo - orbit\ is\ fuzzy\ two - sided$ limit $shadow$.

A sequence $(x_k)_{k \in \mathbb{Z}}$ of point in X is $fuzzy$ two-sided limit pseudo-orbit if it satisfies $\lim_{k \to \infty} M(T(x_k), x_{k+1}, t) = 1$

A fuzzy $two - sided\ limit\ pseudo - orbit$ is fuzzy $two - sided\ limit - shadowed$ if there is a point $p \in X$ such that $\lim_{k \to \infty} M(T^k(p), x_k, t) = 1$.

Definition 2.12

Let $(X, M, *)$ be a fuzzy $compact\ metric\ space$ and $T: X \to X$ a homeomorphism .The fuzzy limit shadowing property for T^{-1} means the following: for every sequence $(x_k)_{k \in \mathbb{N}_0} \subset X$ satisfying

$$\lim_{k \to \infty} M(T^{-1}(x_k), (x_{k+1}), t) = 1$$

There exists $p \in X$ satisfying

$$\lim_{k \to \infty} M(T^{-k}(p), (x_k), t) = 1$$

During the text we will need the following property: for every sequence$(x_k)_{k \in -\mathbb{N}_0} \subset X$ ($-\mathbb{N}_0$denotes the set of non-positive integers) satisfying

$$\lim_{k \to -\infty} M(T^{-1}(x_k), (x_{k-1}), t) = 1$$

There exists $p \in X$

$$\lim_{k \to -\infty} M(T^k(p), (x_k), t) = 1.$$

Definition 2.13

This property will be called fuzzy negative limit shadowing and the sequence $(x_k)_{k \in -\mathbb{N}_0}$ will be called a fuzzy negative limit pseudo-orbit for T. We will say that p fuzzy limit shadows $(x_k)_{k \in -\mathbb{N}_0}$in the past and that $(x_k)_{k \in -\mathbb{N}_0}$ is fuzzy limit shadowed in the past by p.

Definition 2.14

For $p, z \in X$ and we consider the sequence$(x_k)_{k\in\mathbb{Z}}$ defined by

$$x_k = \begin{cases} T^k(z), & k \geq 0; \\ T^k(p), & k < 0, \end{cases}$$

This sequence consists of the past $orbit\ of\ p$ and the future orbit of z. Sequence of this type will be called T fuzzy simple $two-sided\ limit\ pseudo-orbits$. We say that T has the fuzzy simple two-sided limit $shadowing\ property$ when every fuzzy simple two-sided limit $pseudo-orbit$ is fuzzy two-sided limit $shadowed$.

III. MAIN RESULTE

To prove the propositions, we first need the following $lemma$.

Lemma 3.1 [1]

Let $(X_1, M_1, *_1)$ and $(X_2, M_2, *_2)$ be two fuzzy compact metric space and $h: X_1 \to X_2$ be $fuzzy\ homeomorphism$ .For each $0 < \varepsilon < 1$ there exists $0 < \delta <1$with the property $M_1(x, y, t) > 1 - \delta$, then

$M_2(h(x), h(y), t) > 1 - \varepsilon$.

Proposition 3.2

If $T: X \to X$ is fuzzy $homeomorphism$ then T has fuzzy $limit\ shadow$ if and only if so does T^{-1}.

Proof:

Let T fuzzy limit shadow. Let the sequence $\{x_k\}_{k\in\mathbb{Z}} \subset X$ satisfies

$$\lim_{k\to\infty} M(T(x_k), x_{k+1}, t) = 1 \quad \dots (1)$$

There exists $p \in X$

$$\lim_{k\to\infty} M(T^k(p), x_k, t) = 1 \quad \dots (2)$$

By mma 3.1 , for every $\varepsilon \in (0,1)$ there is $\varepsilon_1 \in (0,1)$ $and\ t > 0$for h

Let $g = T^{-1}$ and the sequence $\{y_k\}_{k\in\mathbb{Z}} \subset X$ .To put $\{x_k: x_k = y_{-k}\}_{k_0\geq k}$

There is $k_0 \in \mathbb{N}$ such that $M(g(y_k), y_{k+1}, t) > 1 - \varepsilon$ for $k \geq k_0$

Then

$$\lim_{k\to\infty} M(g(y_k), y_{k+1}, t) = \lim_{k\to\infty} M(y_k, T(y_{k+1}), t)$$
$$= \lim_{k\to\infty} M(T(g(y_k), T(y_{k+1}), t)$$
$$= 1$$

So there exists $p \in X$

$$\lim_{k\to\infty} M(g(p), y_k, t) = \lim_{k\to\infty} M(T^{-k}(p), x_{-k}, t) = 1$$

Hence T^{-1} has the fuzzy $limit\ shadowing\ property$.

Conversely, let T^{-1} fuzzy $limit\ shadow$ .Let the sequence

$\{y_k\}_{k\in\mathbb{Z}} \subset X$ satisfies

$$\lim_{k\to\infty} M(T^{-1}(y_k), y_{k+1}, t) = 1 \quad \dots (1)$$

There exists $q \in X$

$$\lim_{k\to\infty} M(T^{-k}(q), y_k, t) = 1 \quad \dots (2)$$

By $Lemma$ 3.1, for every $\varepsilon_1 \in (0,1)$ there is $\varepsilon \in (0,1)$ $and\ t > 0$ for h^{-1} .

Let $g = T$ and the sequence $\{x_k\}_{k\in\mathbb{Z}} \subset X$ .To put $\{y_k: y_k = x_{-k}\}_{k_0\geq k}$

There is $k_0 \in \mathbb{N}$ such that $M(g(x_k), x_{k+1}, t) > 1 - \varepsilon_1$ for $k \geq k_0$

Then

$$\lim_{k\to\infty} M(g(x_k), x_{k+1}, t) = \lim_{k\to\infty} M(x_k, T^{-1}(x_{k+1}), t)$$
$$= \lim_{k\to\infty} M(T^{-1}(g(x_k), (T^{-1}(x_{k+1}), t) = 1$$

So there exists $q \in X$

$$\lim_{k\to\infty} M(g^k(q), x_k, t) = \lim_{k\to\infty} M(T^k(q), y_{-k}, t) = 1$$

Hence T has the $uzzy\ limit\ shadowing\ property$. □

Proposition 3.3

A fuzzy homeomorphism T $on\ a\ fuzzy$ compact metric space X has the fuzzy $negative\ limit\ shadowing\ property$ if and only if T^{-1} has the fuzzy $limit\ shadowing\ property$.

Proof:

Suppose that T has the fuzzy negative limit shadowing property.

Let $(x_k)_{k\in-\mathbb{N}_0}$ a fuzzy $negative\ limit\ pseudo-orbit$ for T satisfying

$$\lim_{k\to-\infty} M(T^{-1}(x_k), x_{k-1}, t) = 1$$

There exists $p \in X$

$$\lim_{k\to-\infty} M(T^k(p), x_k, t) = 1$$

The sequence $(y_k)_{k\in\mathbb{N}_0}$ is a fuzzy $limit\ pseudo-orbit$ for T^{-1} , $(x_k) = y_{-k}$

$$\lim_{k\to-\infty} M(T^{-1}(x_k), x_{k-1}, t) = \lim_{k\to-\infty} M(T^{-1}(y_{-k}), y_{-k+1}, t)$$
$$= 1$$

Which implies that

$$\lim_{k\to\infty} M(T^{-k}(p), y_k, t) = 1$$

It follows that $(y_k)_{k\in\mathbb{N}_0}$is fuzzy $limit\ shadowed$ for T^{-1} and T^{-1} that has the fuzzy $limit\ shadowing\ property$.

Conversely, let T^{-1} has the fuzzy limit $shadowing\ property$. Let $(y_k)_{k\in-\mathbb{N}_0}$ a fuzzy negative limit $pseudo-orbit$ for T satisfying

$$\lim_{k\to\infty} M(T^{-1}(y_k), y_{k+1}, t) = 1$$

There exists $p \in X$

$$\lim_{k\to\infty} M(T^{-k}(p), y_k, t) = 1$$

The sequence $(x_k)_{k\in\mathbb{N}_0}$ is a fuzzy $limit\ pseudo-orbit$ for T , $y_k = x_{-k}$

$$\lim_{k\to\infty} M(T^{-1}(y_k), y_{k+1}, t) = \lim_{k\to\infty} M(T^{-1}(x_{-k}), x_{-k-1}, t) = 1$$

Which implies that

$$\lim_{k\to-\infty} M(T^k(p), x_k, t) = 1$$

Hence T has the $zzy\ limit\ shadowing\ property$. □

Proposition 3.4

Let $(X_1, M_1,*_1)$ and $(X_2, M_2,*_2)$ be two fuzzy compact metric spaces. Let $T : X_1 \to X_1$ and $g : X_2 \to X_2$ be two fuzzy $homeomorphisms$ and $h : X_1 \to X_2$ be a conjugation between T and g.Then T has fuzzy $shadowing\ property$ if and only if g is too.

Proof:

$Let\ T$ has fuzzy $shadowing\ property$ .By $Lemma$ 3.1 , for every $\varepsilon \in (0,1)$ there is $\varepsilon_1 \in (0,1)$ for h .From the fuzzy $shadowing\ property$ there is $\delta_1 \in (0,1)$ with property that every δ_1-$fuzzy\ pseudo\ orbit$ $(x_k)_{k\in\mathbb{Z}}$ of T can be ε_1-fuzzy $traced$ by some point in X_1.

$M_1(T(x_k)), x_{k+1}, t) \geq 1 - \delta_1$.

Let $0 < \delta < 1$ satisfies $Lemma$ 3.1 for δ_1 and h^{-1}

Given $\{y_k\}_{k\in\mathbb{Z}} \subset X_2$ .To do this put $x_k = h^{-1}(y_k)$ for $k \in \mathbb{Z}$

We have

$M_1(T(x_k)), x_{k+1}, t) = M_1(h^{-1}(g(y_k)), h^{-1}(y_{k+1}), t)$

$= M_2(g(y_k)), y_{k+1}, t) \geq 1 - \delta$ $for\ k \in \mathbb{Z}$,

(x_k) is a δ_1-$fuzzy\ pseudo\ orbit$ for T and for $k \in \mathbb{Z}$.

$$M_1(T^k(y), x_{k+1}, t) \geq 1 - \varepsilon_1.$$

For some $p \in X_1$ .therefor, for $k \in \mathbb{Z}$ it follows that

$$M_1(T^k(p), x_{k+1}, t) = M_2(h(T^k(p)), h(x_{+1}), t)$$

$$= M_2(g^k(h(p)), y_k, t) \geq 1 - \varepsilon.$$

Hence g has the fuzzy $shadowing\ property$.

Conversely, let g has the $zzy\ shadowing\ property$. By $Lemma$ 3.1, for every $\varepsilon_1 \in (0,1)$ there is $\varepsilon \in (0,1)$ for h^{-1}.From the fuzzy s$hadowing\ property$ there is $\delta \in (0,1)$ $with\ property\ that\ every\ \delta$ $-fuzzy\ pseudo\ orbit$ $(y_k)_{k\in\mathbb{Z}}$ of g can be ε-$fuzzy\ traced$ by some point in X_2.

$M_2(g(y_k), y_{k+1}, t) \geq 1 - \delta$ for $k \in \mathbb{Z}$

Let $0 < \delta_1 < 1$ satisfies $Lemma$ 3.1 for and h.

Given $\{x_k\}_{k\in\mathbb{Z}} \subset X_1$ .To do this put $y_k = h(x_k)$ for $k \in \mathbb{Z}$

We have

$$M_2(g(k), y_{k+1}, t) = M_2(h(T(x_k)), h(x_{k+1}), t)$$

$$M_1(T(x_k), x_{k+1}, t) \geq 1 - \delta_1$$

(g_k) is a δ- $fuzzy\ pseudo\ orbit$ for g and so for $k \in \mathbb{Z}$.

$$M_2(g^k(q), y_k, t) \geq 1 - \varepsilon\ .$$

For some $q \in X_2$, therefore

$$M_2(g^k(q), y_{k+1}, t) = M_1(h^{-1}(g^k(q)), h^{-1}(x_k), t)$$

$$= M_1(T^k(h^{-1}(q)), x_k, t) \geq 1 - \varepsilon_1.$$

Hence T has the $fuzzy\ shadowing\ property$. □

Proposition 3.5

Let $(X_1, M_1,*_1)$ and $(X_2, M_2,*_2)$ be two fuzzy $compact$ metric spaces. Let $T : X_1 \to X_1$ and $g : X_2 \to X_2$ be two $fuzzy\ homeomorphisms$ and $h : X_1 \to X_2$ be $a\ conjugation$ between T and g.Then T is fuzzy L-shadowing property if and only if g is too.

Proof:

Let T has the $fuzzy\ L - shadowing\ property$. By $Lemma$ 3.1, for every $0 < \varepsilon < 1$ and $t > 0$ there is $0 < \varepsilon_1 < 1$ for h. From the $fuzzy\ L - shadowing\ property$ if every $0 < \varepsilon_1 < 1$ there is $0 < \delta_1 < 1$ such that for every sequence$\{x_k\}_{k\in\mathbb{Z}} \subset X_1$

$$M_1(T(x_k), x_{k+1}, t) \geq 1 - \delta_1$$

And

$$\lim_{k\to\infty} M_1(T(x_k), x_{k+1}, t) = 1$$

There exists $p \in X_1$

$$M_1(T^k(p), x_k, t) \geq 1 - \varepsilon_1$$

And

$$\lim_{k\to\infty} M_1(T^k(p), x_k, t) = 1$$

Holds

Let $0 < \delta < 1$ satisfies $Lemma$ 3.1 for δ_1 and h^{-1}

Given $\{y_k\}_{k\in\mathbb{Z}} \subset X_2$ .To do this put $x_k = h^{-1}(y_k)$ for $k \in \mathbb{Z}$

We have

$$M_1(T(x_k), x_{k+1}, t) = M_1(h^{-1}(g(y_k)), h^{-1}(y_{k+1}), t) \geq 1 - \delta_1$$

Thus $\{x_k\}$ satisfies the relation

$M_1(T(x_k), x_{k+1}, t) \geq 1 - \delta_1$

Hence

$$M_2(g(y_k), y_{k+1},) \geq 1 - \delta$$

And

$$\lim_{k\to\infty} M_2\ (g(y_k), y_{k+1}, t) = 1$$

For some $p \in X_1$

$$M_1(T^k(p), x_k, t) \geq 1 - \varepsilon_1$$

Therefore it follows that

$$M_1(T^k(p), x_k, t) = M_2(h(T^k(p)), h(x_k), t)$$

$$= M_2(g^k(h(p), y_k,) \geq 1 - \varepsilon$$

And

$$\lim_{k\to\infty} M_2\ (g^k(q), y_k, t) = 1$$

Hence g has the fuzzy L-$shadowing\ property$.

Conversely, let g has the fuzzy L-$shadowing\ property$. By mma3.1 , for every $0 < \varepsilon_1 < 1$ and $t > 0$ $there\ is$ $0 < \varepsilon < 1$ $for\ h^{-1}$. From the $fuzzy\ L - shadowing\ property$ if every $0 < \delta < 1$ there is $0 < \varepsilon < 1$ such that for every sequence$\{y_k\}_{k\in\mathbb{Z}} \subset X_2$

$$M_2(g(y_k), y_{k+1}, t) \geq 1 - \delta$$

And

$$\lim_{k\to\infty} M_2\,(g(y_k), y_{k+1}, t) = 1$$

For some $q \in X_2$

$$M_2(g^k(q), y_k, t) \ge 1 - \varepsilon$$

And

$$\lim_{k\to\infty} M_2\,(g^k(q), y_k, t) = 1$$

Holds

Let $0 < \delta_1 < 1$ satisfies $Lemma\ 3.1$ for and h.

Given $\{x_k\}_{k\in\mathbb{Z}} \subset X_1$ .To do this put $y_k = h(x_k)$ for $k \in \mathbb{Z}$

We have

$$M_2(g(y_k), y_{k+1}, t) = M_1\big(h\big(T(x_k)\big), h(x_{k+1}), t\big) \ge 1 - \delta.$$

Thus $\{y_k\}$ satisfies the relation

$$M_2(g(y_k), y_{k+1}, t) \ge 1 - \delta$$

Hence

$$M_1(T(x_k), x_{k+1}, t) \ge 1 - \delta_1$$

And

$$\lim_{k\to\infty} M_1(T(x_k), x_{k+1}, t) = 1$$

For some $q \in X_2$

$$M_2(g^k(q), y_k, t) \ge 1 - \varepsilon$$

Therefore it follows that

$$M_2(g^k(q), y_k, t) = M_1\big(h^{-1}\big(g^k(q)\big), h^{-1}(x_k), t\big)$$
$$= M_1(T^k(h^{-1}(q), x_k, t) \ge 1 - \varepsilon_1$$

And

$$\lim_{k\to\infty} M_1(T^k(p), x_k, t) = 1$$

Hence T has $fuzzy\ L - shadowing\ property$. □

Proposition 3.6

Let$(X_1, M_1, *_1)$and$(X_2, M_2, *_2)$ be two fuzzy compact metric spaces. Let $T : X_1 \to X_1\ and\ g : X_2 \to X_2$ be two fuzzy $homeomorphisms$ and $h : X_1 \to X_2$ be $a\ conjugation$ between $T\ and\ g$.Then T is fuzzy two – sided limit $shadowing\ property$ if and only if g is too.

Proof:

Let T has fuzzy two –sided limit $shadowing\ property$. By $Lemma$ 3.1, for every $0 < \varepsilon < 1$ there is $0 < \varepsilon_1 < 1$ for h. Let $0 < \delta < 1$ satisfies $Lemma\ 3.1$ fore δ_1 and h^{-1}.To do this put $x_k = h^{-1}(y_k)$ for $k \ge k_0$

$$\lim_{k\to\infty} M_1(T(x_k), x_{k+1}, t) = 1$$

There exists a point $p \in X_1$

$$\lim_{k\to\infty} M_1(T^k(p), x_{k+1}, t) = 1$$

The sequence $(x_k)_{k\in\mathbb{Z}} \subset X_1$

We have

$$M_1(T(x_k), x_{k+1}, t) = M_1\big(h^{-1}\big(g(y_k)\big), h^{-1}(y_{k+1}), t\big)$$
$$= M_2(g(y_k), y_{k+1}, t) \ge 1 - \delta, \quad k \ge k_0$$

Hence there is $k_0 \in \mathbb{N}$

Thus

$$\lim_{k\to\infty} M_2(g(y_k), y_{k+1}, t) = 1$$

Therefore there is $k_1 \in \mathbb{N}$ so that for

$M_1(T^k(p), x_{k+1}, t) \ \ge\ 1 - \varepsilon_1 \quad k \ \ge\ k_1.$

Thus

$$M_1(T^k(p), x_{k+1}, t) = M_2\big(h\big(T^k(p)\big), h(x_{k+1}), t\big)$$
$$= M_2(g^k(h(p)), y_k, t) \ge 1 - \varepsilon \ \text{ for } k \ \ge\ k_1$$

Hence

$$\lim_{k\to\infty} M_2(g^k(h(p)), y_k, t) = 1$$

Hence g has fuzzy two –sided limit $shadowing\ property$.

Conversely, let g has fuzzy two –sided limit shadowing property. By $Lemma\ 3.1$, for every $0 < \varepsilon_1 < 1$ there is $0 < \varepsilon < 1$ for h^{-1}. Let $0 < \delta_1 < 1$ satisfies Lemma 3.1 for δ and h .To do this put

$y_k = h(x_k)$ for $k \ge k_0$, $\lim_{k\to\infty} M_2(g(y_k), y_{k+1}, t) = 1$

There exists $q \in X_2$

$$\lim_{k\to\infty} M_2(g^k(q), y_k, t) = 1$$

The sequence $(y_k)_{k\in\mathbb{Z}}$

We have

$$M_2(g(y_k), y_{k+1}, t) = M_2(h(T(x_k), h(x_{k+1}), t)$$
$$= M_1(T(x_k), x_{k+1}, t) \ge 1 - \delta_1, k \ \ge\ k_0$$

There is $k_0 \in \mathbb{N}$

Thus

$$\lim_{k\to\infty} M_1(T(x_k), x_{k+1}, t) = 1$$

Therefore there is $k_1 \in \mathbb{N}$ so that for

$$M_2(g^k(q), y_{k+1}, t) \ \ge 1 - \varepsilon, \qquad k \ \ge\ k_1$$

Thus

$$M_2(g^k(q), y_k, t) = M_1\big(h^{-1}\big(g^k(q)\big), h^{-1}(x_k), t\big)$$
$$= M_1(T^k(h^{-1}(q)), x_k, t) \ge 1 - \varepsilon_1 \ \ for\ k \ \ge\ k_1$$

Hence

$$\lim_{k\to\infty} M_1(T^k(h^{-1}(q), x_k, t) = 1$$

Hence T has fuzzy two –sided limit shadowing property. □

Proposition 3.7

A $fuzzy\ homeomorphism$ on a fuzzy $compact\ metric$ space has the fuzzy $two - sided$ limit $shadowing\ property$

if and only if it has the fuzzy limit $shadowing\ property$, the fuzzy negative $limit\ shadowing\ property$ and the fuzzy simple $two - sided\ limit\ shadowing\ property$.

Proof:

$It\ is\ obvious$ that the fuzzy $two - sided$ limit shadowing property implies the fuzzy limit $shadowing\ property$, the fuzzy negative limit shadowing property and the fuzzy simple two-sided limit $shadowing\ property$. It suffices to prove the converse statement. The fuzzy $limit\ shadowing\ property$ and the fuzzy $negative\ limit\ shadowing\ property$ assure the existence of points $p_1, p_2 \in X$, $(x_k)_{k\in\mathbb{Z}}$ satisfying

$$\lim_{k\to\infty} M(T^k\,(p_1), x_k, t) = 1\ and$$

$$\lim_{k\to\infty} M(T^k(p_2), x_k, t) = 1$$

Thus the sequence

$$y_k = \begin{cases} T^k(p_2), k \geq 0; \\ T^k(p_1), k < 0, \end{cases}$$

Is a fuzzy simple two-sided limit $pseudo - orbit$.The fuzzy simple two-sided limit $shadowing\ property$ the existence confims of a point $p \in X$ that $fuzzy$ two- sided limit shadows$(y_k)_{k\in\mathbb{Z}}$ .This point also fuzzy two- sided limit $shadowing\ (x_k)_{k\in\mathbb{Z}}$. □

Corollary 3.8

If a fuzzy homeomorphism has the fuzzy limit $shadowing$ and the $fuzzy\ negative\ limit\ shadowing\ properties$ then it has the fuzzy two-sided limit $shadowing\ property$ if and only if it has the fuzzy simple two-sided limit $shadowing\ property$.

Conclusion

In this paper, we have succeeded in defining the concepts (L-shadowing property**,** two – sided limit shadowing**,** negative limit shadowing property, simple two – sided limit shadowing) in fuzzy dynamical system. We have proven that the properties (fuzzy shadowing property, fuzzy L-shadowing property, fuzzy two–sided limit shadowing) are invariant under fuzzy topological conjugation**.**

REFERENCES

[1] Badilla, Leonel, et al. "Topological stability for fuzzy expansive maps." FuzzySets and Systems 425 (2021): 34-47.

[2] Molaei, Mohammad Reza. "The Concept of Synchronization from the Observer's Viewpoint." Cankaya University Journal of Science and Engineering 8.2 (2011).

[3] Pilyugin, Shadowing in Dynamical Systems, Lecture Notes in Mathematics 1706, Springer, Berlin 1999.

[4] Ahmadi, Seyyed A., and Mohammad R. Molaei. "Stochastic stability in fuzzy dynamical systems." Cankaya University Journal of Science and Engineering 9.1 (2012).

[5] George, A., and P. Veeramani. "On some results in fuzzy metric spaces." Fuzzy sets and systems 64.3 (1994): 395-399.

[6] Schweizer, Berthold, and Abe Sklar. "Statistical metric spaces." Pacific J. Math 10.1 (1960): 313-334.

[7] George, A., and P. Veeramani. "On some results of analysis for fuzzy metric spaces." Fuzzy sets and systems 90.3 (1997): 365-368.