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The Cayley Graph of Semi-Direct Product of finite Groups: Interrelationships and Construction

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Abstract— *In this paper, we study Cayley graph of the semi-direct product of two finite groups where* $\mathbf{p}$ *is an odd prime numbe. Specifically, we endeavor to establish a comprehensive understanding of the Cayley graph by investigating the interrelationships among the constituent elements of the group. Leveraging this acquired knowledge, we systematically construct the Cayley graph, there by contributing to the scholarly discourse on this subject matter.*

Keywords—: direct product of groups, automorpihism group, semi-direct product of groups, digraph.

1- INTRODUCTION AND PRELIMINARIES

The present investigation examines the characteristics of groups by analyzing their underlying graph structures. To accomplish this, we utilize the concept of identity within groups, which enables the definition of an associated identity graph for each group. By means of this graph, we can determine adjacency relationships among group elements, whereby two elements are deemed adjacent if they share a common identity element. Notably, the utilization of commutativity is not essential in this process. In the identity graph, the vertices correspond to the elements of the group, establishing a direct link between the group's order and the number of vertices in its associated identity graph. [1]

Cayley digraphs serve as a fundamental portrayal of finite groups, providing a potent mechanism for visually depicting group structures. Through the visualization of a group's multiplication table, Cayley digraphs have the capability to unveil crucial group properties, such as commutativity. Additionally, these digraphs play a vital role in identifying isomorphisms between groups, thereby bestowing them with significant value in the realm of group theory. It is worth noting that Cayley initially introduced the graph-theoretic representation of groups in 1878, which laid the groundwork for a more methodical examination of algebraic structures. Notably, Caucal's seminal work extensively characterizes the Cayley digraphs associated with diverse algebraic structures, encompassing semigroups, monoids, right-cancellative monoids, left-cancellative monoids, and groups. [2][3]

Despite the versatility of graph theory in modeling various problems, its adequacy may be limited, particularly when dealing with directed networks that prioritize the orientation of links. In such instances, a standard network graph may not sufficiently capture the underlying structure. However, a directed graph provides a more precise representation by assigning a direction to each edge. Notably, recent research has confirmed the conjecture proposed by Babai and Godsil [4] that Cayley graphs of abelian groups possess automorphism groups of minimal size. Additionally, Melissa Sherman-Bennett's thesis explores the intricate relationship between groups and the Cayley graphs they generate. In the realm of graph theory, a graph G is defined as an ordered triple $(V(G), E(G), \varphi_G)$, comprising a non-empty set of vertices $V(G)$, a set of edges $E(G)$ (disjoint from $V(G)$), and an incidence function φ_G that associates an unordered pair of vertices with each edge. When an edge e connects two vertices u and v, denoted as $\varphi_G(e) = uv$, u and v are referred to as the endpoints of e. [4-6]

The automorphism group of a given group G, denoted as $Aut(G)$, refers to a group comprising all the automorphisms of G. An automorphism is defined as a bijective function $\mu: G \rightarrow G$ that maintains the group structure, meaning that it preserves the operation of the group. In other words, for any elements g^1 and g^2 in G, the automorphism μ satisfies the property $\mu(g^1 * g^2) = \mu(g^1)\ o\ \mu(g_2)$, where $*$ represents the group operation. Within Aut(G), the group operation is defined as the composition of automorphisms. This implies that for any two automorphisms μ_1 and μ_2 in Aut(G), and any element g in G, their composition $(\mu_1 * \mu_2)(g)$ is obtained by first applying μ_2 to g and then μ_1 to the result, i.e., $(\mu_1 * \mu_2)(g) = \mu_1(\mu_2(g))$.

The automorphism group of a given group G is a subgroup of the group of all permutations of G, denoted as Sym(G), which represents the collection of all bijective functions mapping elements from G to G. The group structure of the automorphism group, denoted as $Aut(G)$, is derived from the underlying group structure of Sym(G), indicating that $Aut(G)$ is a subgroup of Sym(G) under the composition operation of functions.

In the case where G is a cyclic group with a total of m elements, $Aut(G)$ consists of m distinct elements, formally expressed as $|Aut(G)| = |G| = m$. Furthermore, each automorphism of G is uniquely characterized by its effect on a generator of G, given that every element in G can be represented as a power of such generator. As a result, the determination of $Aut(G)$ solely relies on counting the number of generators present in G, which is facilitated by Euler's totient function denoted as $\varphi(m)$.

Consider a scenario where we have a subgroup H within a larger group G, and K is a normal subgroup of G such that their intersection, denoted as $K \cap H$, only contains the trivial subgroup $\{e\}$. In this context, we can establish that G is the semi-direct product of K and H, symbolized as $G \cong K \rtimes H$, if each element g in G can be uniquely expressed as a product $g = kh$, where k belongs to K and h belongs to H. The multiplication operation for this semi-direct product is defined as follows: $(k_1h_1) * (k_2h_2) = k_1(h_1 * k_2{h_1}^{-1})h_2$, where k_1 and k_2 are elements of K, and h_1 and h_2 are elements of H. Here, $*$ represents the group operation in H, and ${h_1}^{-1}$ denotes the inverse of h_1 in H. Additionally, it should be noted that in this case, K is also a normal subgroup of G, and the semi-direct product can be treated as a direct product, resulting in $G \cong K \times H$ as groups.

It is important to emphasize that the semi-direct product of two groups, K and H, is not uniquely determined. To construct a semi-direct product, one must have knowledge of the automorphisms induced by conjugation of elements in K on the group H. Specifically, if $G = KH$ represents a semi-direct product, then each element k in K corresponds to an automorphism $\alpha_k\ in\ Aut(H)$, where α_k is the result of conjugation by k: $\alpha_k(h) = khk^{-1}$ for h in H. This correspondence from k to α_k can be understood as a homomorphism from K to $Aut(H)$.

On the contrary, in the context of arbitrary groups K and H, it can be established that given any homomorphism $\psi: K \to Aut(H)$, there exists a distinctive semi-direct product of K by H, where $\alpha_k = \psi(k)$ holds for all k in K. This semi-direct product is a specific instance of an extension of the group H by the group K, commonly referred to as a split extension. The notation used to represent the semi-direct product of K by H is often denoted as $H \rtimes K$ or $H{:}K$.

The term "internal" semi-direct product is used when considering subgroups K and H within a given group G. In contrast, the "external" semi-direct product of groups K and H involves a homomorphism $\psi: K \to Aut(H)$, and is represented by the Cartesian product $K \times H$. The multiplication operation in this product is defined as $(k_1, h_1)(k_2, h_2) = (k_1k_2, h_1^{\psi(k_2)}h_2)$, where the exponent signifies the action of the automorphism α_2 on h_1.

To provide further clarity to the definition, let's examine some examples in graph theory. Consider a graph $G_1 = (V(G_1), E(G_1), \varphi(G_1))$, where the vertex set $V(G_1) = \{v_1, v_2, v_3, v_4, v_5\}$ and the edge set $E(G_1) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$. The mapping $\varphi(G_1)$ is defined as $\varphi(G_1)(e_1) = v_1v_2, \varphi(G_1)(e_2) = v_1v_2, \varphi(G_1)(e_3) = v_1v_2, \varphi(G_1)(e_4) = v_1v_2, \varphi(G_1)(e_5) = v_1v_2, \varphi(G_1)(e_6) = v_1v_2, \varphi(G_1)(e_7) = v_1v_2, and\ \varphi(G_1)(e_8) = v_1v_2.$

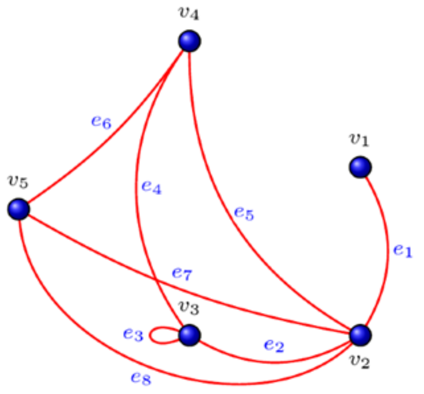

Fig.1. G_1

Let $G_2 = (V(G_2), E(G_2), \varphi(G_2))$ be a graph, where $V(G_2) = \{v_1, v_2, v_3, v_4, v_5\}$ denotes the set of vertices and $E(G_2) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$ represents the set of edges in G_2. The function $\varphi(G_2)$ defines the mapping of each edge to a pair of vertices. Specifically, $\varphi(G_2)(e_1) = v_1v_2, \varphi(G_2)(e_2) = v_1v_2, \varphi(G_2)(e_3) = v_1v_2, \varphi(G_2)(e_4) = v_1v_2, \varphi(G_2)(e_5) = v_1v_2, \varphi(G_2)(e_6) = v_1v_2, \varphi(G_2)(e_7) = v_1v_2$, and $\varphi(G_2)(e_8) = v_1v_2$.

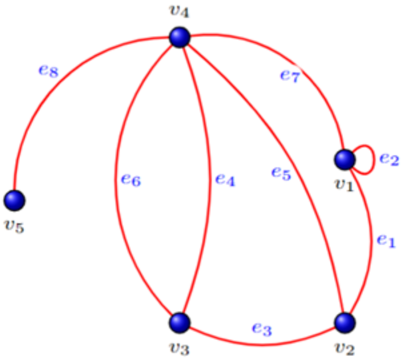

Fig.2. G_2

Directed graphs, also known as digraphs, are mathematical structures employed for modeling the interconnectedness between entities. These graphs incorporate edges that possess a defined direction, allowing for the representation of directed relationships. In such graphs, the in-degree of a vertex indicates the number of edges directed towards it, while the out-degree of a vertex denotes the number of edges emanating from it and connecting to other vertices within the graph. These fundamental concepts, commonly referred to as "in-degree" and "out-degree," are valuable tools for characterizing the dissemination of information or allocation of resources within a directed graph.

Furthermore, it is important to highlight that in the presence of a loop at a vertex, both the in-degree and out-degree contribute a value of 1 to their respective degrees. This observation has been established in previous research [7][8]. To illustrate this concept, we will employ the term "digraph" to exemplify a directed graph throughout the forthcoming discussions in this paper.

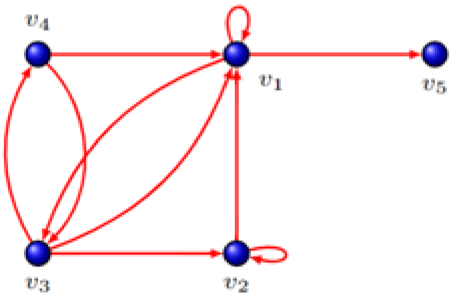

Fig.3. Digraph

In the provided digraph, the vertex denoted as v_4 has an incoming degree of 1 $(deg^-(v_4) = 1)$ and an outgoing degree of 2 $(deg^+(v_4) = 2)$. Similarly, the vertex v_3 does not have an incoming degree of 2 $(deg^-(v_3) = 2)$, but it has an outgoing degree of 3 $(deg^+(v_3) = 3)$. The vertex v_2 has an incoming degree of 2 $(deg^-(v_2) = 2)$ and an outgoing degree of 2 $(deg^+(v_2) = 2)$. Furthermore, the vertex v_1 has an incoming degree of 4 $(deg^-(v_1) = 4)$ and an outgoing degree of 3 $(deg^+(v_1) = 3)$. Lastly, the vertex v_5 has an incoming degree of 1 $(deg^-(v_5) = 1)$ and an outgoing degree of 0 $(deg^+(v_5) = 0)$. This graph comprises five vertices and ten directed edges. The presence of directed edges indicates that the graph is a directed graph.

Theorem (1.1): Let $G = (V, E)$ be a directed graph. Then, the number of edges, denoted as $|E|$, is equal to the sum of the out-degrees and the sum of the in-degrees of all vertices, expressed as:

$|E| = \sum_{v\in V} deg^-(v) = \sum_{v\in V} deg^+(v)$ [7][8]

In this context, the first sum is calculated by considering the count of outgoing edges for each vertex, while the second sum is obtained by considering the count of incoming edges for each vertex. It is essential to ensure that both sums are equal in order to obtain the correct result.

By applying this theorem to the provided example, the following result is obtained:

$$\sum_{v\in V} deg^-(v) = deg^-(v_4) + deg^-(v_3) + deg^-(v_2) + deg^-(v_1) + deg^-(v_5) = 1 + 2 + 2 + 4 + 1 = 10$$

$$\sum_{v\in V} deg^+(v) = deg^+(v_4) + deg^+(v_3) + deg^+(v_2) + deg^+(v_1) + deg^+(v_5) = 2 + 3 + 2 + 3 + 0 = 10$$

Therefore, based on the equality of these sums, it can be concluded that the graph depicted in Figure 2 is a directed graph.

Geometric representations of groups are commonly illustrated through Cayley graphs. These diagrams provide a visual tool to facilitate the understanding of abstract group actions in various contexts.

Definition (1.2): Let G be a group and S be a subset of G. The *Cayley graph* $Cay(G{:}S)$ is an undirected graph that incorporates the elements of G as its vertices and establishes connections between them through edges.

Each element g in G is connected to two other elements, namely sg and $s^{-1}g$, for every s in S. These connections are represented by directed edges originating from g and pointing towards sg and $s^{-1}g$, respectively. Notably, when the order of g is 2, the directed edges from g to sg and $s^{-1}g$ coincide, resulting in the existence of only one edge.

The Cayley graph $Cay(G{:}S)$ pertaining to a group G and its generating set S exhibits a property of connectedness, indicating the presence of a path connecting any two vertices within the graph. This property holds if and only if S serves as a generating set for G. Stated differently, if and only if the corresponding Cayley graph is connected, it implies that every element in G can be expressed as a composition of elements from S and their respective inverses. This finding proves to be valuable in comprehending the interplay between a group's generating set and the connectedness exhibited by its associated Cayley graph.

Definition (1.3): The directed *Cayley graph*, represented as $DiCay(G{:}S)$, refers to a directed graph derived from the *Cayley graph* of a given group G and its generating set S by incorporating directional properties to each edge.

More precisely, for every element g in G and each generator s in S, the directed Cayley graph exhibits a pair of edges with opposing directions: the first edge connects g to sg, while the second edge connects g to $s^{-1}g$. This approach is in accordance with reference [9].

Definition (1.4): Let G be a group and let S be a generating set of G. The *Cayley graph* $Cay(G, S)$ associated with G is a directed graph with the vertex set G, where each vertex represents an element of G. The edge set of $Cay(G, S)$ consists of directed edges (x, y) for every pair of elements x and y in G, such that there exists a generator s in S satisfying the equation $x * s = y$. To emphasize this connection, each edge (x, y) is adorned with the label s, indicating the generator that produced the edge.

Vertex Set: The vertices of $Cay(G, S)$ precisely correspond to the elements of the group G.

Edge Set: The directed edge (x, y) is created between two vertices x and y in G if and only if y is obtained from x by applying the group operation with the generator 1. In accordance with the customary conventions of Cayley graphs, we assign the label 1 to each edge in the graph, representing the generator responsible for its creation.

To illustrate, let's consider the following example:

Example (1.5): Within the confines of our research, we shall focus our attention on a group referred to as $G = (Z, +)$, where Z represents the set of integers and the operation $'+'$ denotes addition. To examine the inherent structural characteristics of G in a visual form, we employ the utilization of a Cayley graph, designated as $Cay(G, \{1\})$. The construction of the Cayley graph is facilitated by employing the generating set $\{1\}$ and is precisely defined as follows:

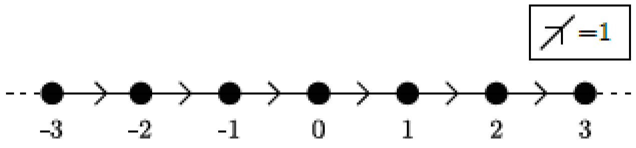

Fig.4. $Cay(Z, \{1\})$

In the context of group theory, let us consider the group G, which is defined as the set of integers Z equipped with a generating set S consisting of the elements 2 and 3. The associated Cayley graph, illustrating the group's structure and interactions, is depicted below.

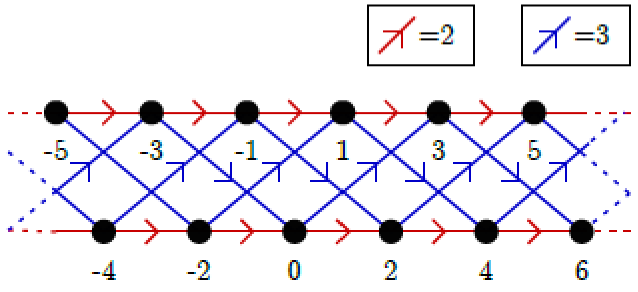

Fig.5. $Cay(Z, \{2,3\})$

We examine a mathematical group represented by integers as vertices in the context of the Cayley graph. This group is generated by two distinct elements, denoted as $S = \{2,3\}$, which establish the corresponding edges in the Cayley graph. Specifically, the generator 2 establishes connections between any pair of integers that differ by 2, while the generator 3 connects any pair of integers that differ by 3. It is important to highlight that this particular *Cayley graph* differs from the previously presented one. We observe that a group can exhibit multiple distinct Cayley graphs, with each graph being uniquely determined by the choice of generators [9].

Example (1.6) involves the analysis of the group $G = S_3$, where the generating set S is defined as $S = \{(12)), (123)\}$. Our objective is to ascertain the group elements obtained through the composition of these generators.

TABLE 1: edge and vertices of $Cay(S_3, \{(12), (123)\})$

	id	(12)	(13)	(23)	(123)	(132)
(12)	(12)	id	(123)	(132)	(13)	(23)
(123)	(123)	(23)	(12)	(13)	(132)	(123)

By employing the multiplication table of the group $G \cong S_3$, where the generating set $S = \{(12), (123)\}$ is given, it is possible to construct the associated *Cayley digraph*.

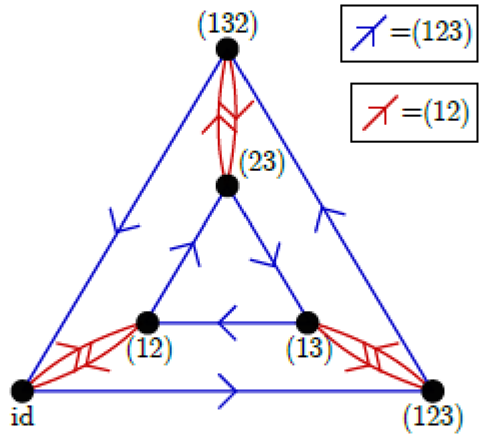

Fig.6. $Cay(S_3, \{(12), (123)\})$

The preceding examples demonstrate that, within a specific group, the construction of its Cayley digraph can yield distinct outcomes depending on the selection of generating sets S_1 and S_2.

2- MAIN RESULTS

In this section we will present some of results of Cayley graph of semi-direct product,

Theorem (2.1): Suppose that p is a prime number, the Semi-direct product of groups C_p and C_4 be isomorphic to $C_p \rtimes C_4$.

Proof. Before proof above theorem, we must be finding the relation between the elements belong to Semi-direct product of C_4 and C_p it is clear that the Cyclic groups are define by $C_p =< a | a^p = e >$ and $C_4 =< b | b^4 = e >$. In the first we must prove the group C_p is normal subgroup of $C_p \rtimes C_4 =< a, b | a^p = b^4 = e, b^3 a^i b = a^{p-i} >$

$$b^3 a^i b = a^{p-i}$$
$$b^3 a^i b^2 = a^{p-i} b$$
$$b^3 a^i b^3 = a^{p-i} b^2$$
$$b^3 a^i b^4 = a^{p-i} b^3$$

$$b^3a^ie = a^{p-i}b^3$$
$$b^3a^i = a^p\,a^{-i}b^3$$
$$b^3a^i = a^{-i}b^3$$

To proof the C_4 is not normal subgroup of $C_4 \rtimes C_p$

$$a^ib^jC_4(a^ib^j)^{-1} = a^ib^j < b^k > (a^ib^j)^{-1}$$
$$a^ib^jC_4(a^ib^j)^{-1} =< a^ib^jb^k(a^ib^j)^{-1} >$$
$$a^ib^jC_4(a^ib^j)^{-1} =< a^ib^jb^kb^{-j}a^{-i} >$$
$$a^ib^jC_4(a^ib^j)^{-1} =< a^ib^ka^{-i} >$$
$$a^ib^jC_4(a^ib^j)^{-1} =< b^ka^{-i}a^{-i} > = < b^ka^{-2i} >.$$

Since, for any element $a^tb^k \in C_p \rtimes C_4$, we can see,

$$a^tb^kC_p(a^tb^k)^{-1} = a^tb^k < a^i > (a^tb^k)^{-1}$$
$$a^tb^kC_p(a^tb^k)^{-1} =< a^tb^ka^i(a^tb^k)^{-1} >$$
$$a^tb^kC_p(a^tb^k)^{-1} =< a^tb^ka^ib^{-k}a^{-t} >$$
$$a^tb^kC_p(a^tb^k)^{-1} =< a^ta^ib^kb^{-k}a^{-t} >$$
$$a^tb^kC_p(a^tb^k)^{-1} =< a^ta^ia^{-t} > = < a^i >.$$

And $C_p \cap C_4 = \{e\}$, thus $\left|C_p \rtimes C_4\right| = \frac{|C_p||C_4|}{|C_p \cap C_4|} = 4p$.

the general formula useful us to find the relation between elements in $C_p \rtimes C_4$ group which helpful in direct product of groups such that

$$C_p \rtimes C_4 =< a, b | a^p = b^4 = e, b^3a^ib = a^{p-i} >$$
$$b^3a^i = a^{-i}b^3$$

Theorem (2.2): Let p be an odd prime, and let $G = C_p \rtimes C_4$ be the semidirect product of C_p and C_4, where $C_4 = \langle a\rangle$ and $C_p = \langle b\rangle$. In the Cayley digraph $Cay(G, \{a, b\})$, the indices h and k can be determined as follows:

If $(a^i, b^j) = (a^h, b^k) \rtimes (a, e)$, then $h = i + 1$ and $k = j$.

If $(a^i, b^j) = (a^h, b^k) \rtimes (e, b)$, then $h = i$ and $k = j - 1$.

Proof. Let (a^i, b^j) and (a^h, b^k) be elements of the Cayley graph $Cay(C_p \rtimes C_4, \{a, b\})$. We aim to find the values of h and k such that $(a^i, b^j) = (a^h, b^k) \rtimes (a, e)$ and $(a^i, b^j) = (a^h, b^k) \rtimes (e, b)$.

For the first case, we have:

$$(a^h, b^k) \rtimes (a, e) = (a^h\varphi_{b^k}(a)\ , b^ke)$$
$$= (a^ha^{-1}, b^ke) = (a^{h-1}, b^k)$$

and so $(a^i, b^j) = (a^{h-1}, b^k)$, implying that $a^i = a^{h-1}$ and $b^j = b^k$. Therefore, we have $h = i + 1$ and $k = j$.

For the second case, we have

$$(a^h, b^k) \rtimes (e, b) = (a^h\varphi_{b^k}(e), b^kb) = (a^he^{-1}, b^{k+1})$$
$$= (a^h, b^{k+1}$$

and so $(a^i, b^j) = (a^h, b^{k+1})$, implying that $a^i = a^h$ and $b^j = b^{k+1}$. Therefore, we have $h = i$ and $k = j - 1$

Hence, we have shown that the values of h and k for the two cases are $h = i + 1$ and $k = j$ for the first case, and $h = i$ and $k = j - 1$ for the second case.

Example (2.3): The group $G = C_p \rtimes C_4$ is a semidirect product of the cyclic group of order p and the cyclic group of order 4. Let a and b be generators of C_p and C_4 respectively. The *Cayley digraph* of $C_p \rtimes C_4$ with respect to the generating set $S = \{a, b\}$ is a directed graph with vertices labeled by the elements of the group and directed edges labeled by the generators in S.

TABLE 2: edge and vertices of $Cay(C_7 \rtimes C_4, \{a, b\})$

g	$g \times (a,e) = h$	$g \times (e,b) = h$	$g \rtimes (a,e) = h$	$g \rtimes (e,b) = h$
(e,e)	$(e,e) \times (a,e) = (a,e)$	$(e,e) \times (e,b) = (e,b)$	$(e,e) \rtimes (a,e) = (a^6,e)$	$(e,e) \rtimes (e,b) = (e,b)$
(a,e)	$(a,e) \times (a,e) = (a^2,e)$	$(a,e) \times (e,b) = (a,b)$	$(a,e) \rtimes (a,e) = (e,e)$	$(a,e) \rtimes (e,b) = (a,b)$
(a^2,e)	$(a^2,e) \times (a,e) = (a^3,e)$	$(a^2,e) \times (e,b) = (a^2,b)$	$(a^2,e) \rtimes (a,e) = (a,e)$	$(a^2,e) \rtimes (e,b) = (a^2,b)$
(a^3,e)	$(a^3,e) \times (a,e) = (a^4,e)$	$(a^3,e) \times (e,b) = (a^3,b)$	$(a^3,e) \rtimes (a,e) = (a^2,e)$	$(a^3,e) \rtimes (e,b) = (a^3,b)$
(a^4,e)	$(a^4,e) \times (a,e) = (a^5,e)$	$(a^4,e) \times (e,b) = (a^4,b)$	$(a^4,e) \rtimes (a,e) = (a^3,e)$	$(a^4,e) \rtimes (e,b) = (a^4,b)$
(a^5,e)	$(a^5,e) \times (a,e) = (a^6,e)$	$(a^5,e) \times (e,b) = (a^5,b)$	$(a^5,e) \rtimes (a,e) = (a^4,e)$	$(a^5,e) \rtimes (e,b) = (a^5,b)$
(a^6,e)	$(a^6,e) \times (a,e) = (e,e)$	$(a^6,e) \times (e,b) = (a^6,b)$	$(a^6,e) \rtimes (a,e) = (a^5,e)$	$(a^6,e) \rtimes (e,b) = (a^6,b)$
(e,b)	$(e,b) \times (a,e) = (a,b)$	$(e,b) \times (e,b) = (e,b^2)$	$(e,b) \rtimes (a,e) = (e,e)$	$(e,b) \rtimes (e,b) = (e,e)$
(e,b^2)	$(e,b^2) \times (a,e) = (a,b^2)$	$(e,b^2) \times (e,b) = (e,b^3)$	$(e,b^2) \rtimes (a,e) = (e,e)$	$(e,b^2) \rtimes (e,b) = (e,e)$
(e,b^3)	$(e,b^3) \times (a,e) = (a,b^3)$	$(e,b^3) \times (e,b) = (e,e)$	$(e,b^3) \rtimes (a,e) = (e,e)$	$(e,b^3) \rtimes (e,b) = (e,e)$
(a,b)	$(a,b) \times (a,e) = (a^2,b)$	$(a,b) \times (e,b) = (a,b^2)$	$(a,b) \rtimes (a,e) = (e,b)$	$(a,b) \rtimes (e,b) = (a,b^2)$
(a^2,b)	$(a^2,b) \times (a,e) = (a^3,b)$	$(a^2,b) \times (e,b) = (a^2,b^2)$	$(a^2,b) \rtimes (a,e) = (a,b)$	$(a^2,e) \rtimes (e,b) = (a^2,b^2)$
(a^3,b)	$(a^3,b) \times (a,e) = (a^4,b)$	$(a^3,b) \times (e,b) = (a^3,b^2)$	$(a^3,b) \rtimes (a,e) = (a^2,b)$	$(a^3,e) \rtimes (e,b) = (a^3,b^2)$
(a^4,b)	$(a^4,b) \times (a,e) = (a^5,b)$	$(a^4,b) \times (e,b) = (a^4,b^2)$	$(a^4,b) \rtimes (a,e) = (a^3,b)$	$(a^4,e) \rtimes (e,b) = (a^4,b^2)$
(a^5,b)	$(a^5,b) \times (a,e) = (a^6,b)$	$(a^5,b) \times (e,b) = (a^5,b^2)$	$(a^5,b) \rtimes (a,e) = (a^4,b)$	$(a^5,e) \rtimes (e,b) = (a^5,b^2)$
(a^6,b)	$(a^6,b) \times (a,e) = (e,b)$	$(a^6,b) \times (e,b) = (a^6,b^2)$	$(a^6,b) \rtimes (a,e) = (a^5,b)$	$(a^6,e) \rtimes (e,b) = (a^6,b^2)$
$(a,b^2$)	$(a,b^2) \times (a,e) = (a^2,b^2)$	$(a,b^2) \times (e,b) = (a,b^3)$	$(a,b^2) \rtimes (a,e) = (e,b^2)$	$(a,b^2) \rtimes (e,b) = (a,b^3)$
$(a^2,b^2$	$(a^2,b^2) \times (a,e) = (a^3,b^2)$	$(a^2,b^2) \times (e,b) = (a^2,b^3)$	$(a^2,b^2) \rtimes (a,e) = (a,b^2)$	$(a^2,b^2) \rtimes (e,b) = (a^2,b^3)$

$(a^3,b^2$	(a^3,b^2) $\times (a,e)$ $= (a^4,b^2)$	(a^3,b^2) $\times (e,b)$ $= (a^3,b^3)$	(a^3,b^2) $\rtimes (a,e)$ $= (a^2,b^2)$	(a^3,b^2) $\rtimes (e,b)$ $= (a^3,b^3)$
$(a^4,b^2$	(a^4,b^2) $\times (a,e)$ $= (a^5,b^2)$	(a^4,b^2) $\times (e,b)$ $= (a^4,b^3)$	(a^4,b^2) $\rtimes (a,e)$ $= (a^3,b^2)$	(a^4,b^2) $\rtimes (e,b)$ $= (a^4,b^3)$
$(a^5,b^2$	(a^5,b^2) $\times (a,e)$ $= (a^6,b^2)$	(a^5,b^2) $\times (e,b)$ $= (a^5,b^3)$	(a^5,b^2) $\rtimes (a,e)$ $= (a^4,b^2)$	(a^5,b^2) $\rtimes (e,b)$ $= (a^5,b^3)$
$(a^6,b^2$	(a^6,b^2) $\times (a,e)$ $= (e,b^2)$	(a^6,b^2) $\times (e,b)$ $= (a^6,b^3)$	(a^6,b^2) $\rtimes (a,e)$ $= (a^5,b^2)$	(a^6,b^2) $\rtimes (e,b)$ $= (a^6,b^3)$
(a,b^3)	(a,b^3) $\times (a,e)$ $= (a^2,b^3)$	(a,b^3) $\times (e,b)$ $= (a,e)$	(a,b^3) $\rtimes (a,e)$ $= (e,b^3)$	(a,b^3) $\rtimes (e,b)$ $= (e,e)$
$(a^2,b^3$	(a^2,b^3) $\times (a,e)$ $= (a^3,b^3)$	(a^2,b^3) $\times (e,b)$ $= (a^2,e)$	(a^2,b^3) $\rtimes (a,e)$ $= (a,b^3)$	(a^2,b^3) $\rtimes (e,b)$ $= (a^2,e)$
$(a^3,b^3$	(a^3,b^3) $\times (a,e)$ $= (a^4,b^3)$	(a^3,b^3) $\times (e,b)$ $= (a^3,e)$	(a^3,b^3) $\rtimes (a,e)$ $= (a^2,b^3)$	(a^3,b^3) $\rtimes (e,b)$ $= (a^3,e)$
$(a^4,b^3$	(a^4,b^3) $\times (a,e)$ $= (a^5,b^3)$	(a^4,b^3) $\times (e,b)$ $= (a^4,e)$	(a^4,b^3) $\rtimes (a,e)$ $= (a^3,b^3)$	(a^4,b^3) $\rtimes (e,b)$ $= (a^4,e)$
$(a^5,b^3$	(a^5,b^3) $\times (a,e)$ $= (a^6,b^3)$	(a^5,b^3) $\times (e,b)$ $= (a^5,e)$	(a^5,b^3) $\rtimes (a,e)$ $= (a^4,b^3)$	(a^5,b^3) $\rtimes (e,b)$ $= (a^5,e)$
$(a^6,b^3$	(a^6,b^3) $\times (a,e)$ $= (e,b^3)$	(a^6,b^3) $\times (e,b)$ $= (a^6,e)$	(a^6,b^3) $\rtimes (a,e)$ $= (a^5,b^3)$	(a^6,b^3) $\rtimes (e,b)$ $= (a^6,e)$

From above table we get these paths:

In $C_7 \times C_4$ group that mean identity automorphism function $\varphi_{b^k}(a) = a$:

1- $b^n \to ab^n \to a^2b^n \to a^3b^n \to a^4b^n \to a^5b^n \to a^6b^n \to b^n$ for $1 \le n \le 4$

2- $a^m \to a^mb \to a^mb^2 \to a^mb^3 \to a^m$ for $1 \le m \le 7$

In $C_7 \rtimes C_4$ group that mean inverse automorphism function $\varphi_{b^k}(a) = a^{-1}$:

1- $b^n \to a^6b^n \to a^5b^n \to a^4b^n \to a^3b^n \to a^2b^n \to ab^n \to b^n$ for $1 \le n \le 4$

2- $a^m \to a^mb \to a^mb^2 \to a^mb^3 \to a^m$ for $1 \le m \le 7$

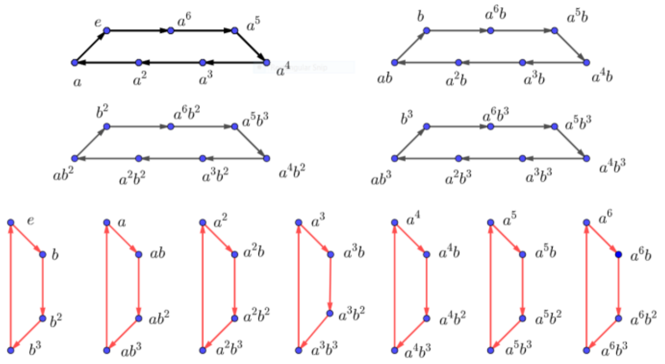

Fig.9. Paths generating by a and b

From paths above we get the right and left cosets of $C_7 \rtimes C_4$ group

1- Right cosets of $C_7 \rtimes C_4$ group

Let $H =< a >$ be a sub group of group $C_7 \rtimes C_4$ and isomorphic to cyclic group of order 7, thus the quotient group is given by

$\frac{C_7 \rtimes C_4}{H} = \{H, Hb, Hb^2, Hb^3\}$ be isomorphic to cyclic group of order 4.

$\frac{C_7 \rtimes C_4}{H} = \{H, Hb, Hb^2, Hb^3\} = \{Hb : o(Hb) = 4, b \in C_4\} \cong \{b : o(b) = 4\} = C_4$

2- Left cosets of $C_7 \rtimes C_4$ group

let $H =< b >$ be a subgroup of group $C_7 \rtimes C_4$ and isomorphic to cyclic group of order 4, thus, the quotient group is given by

$$\frac{C_7 \rtimes C_4}{H} = \{H, a^6H, a^5H, a^4H, a^3H, a^2H, aH\}$$

From left and right cosets above of $C_7 \rtimes C_4$ group we get the Cayley digraph of $C_7 \rtimes C_4$ group with respect to generator set $S = \{a, b\}$ As shown in the figure below

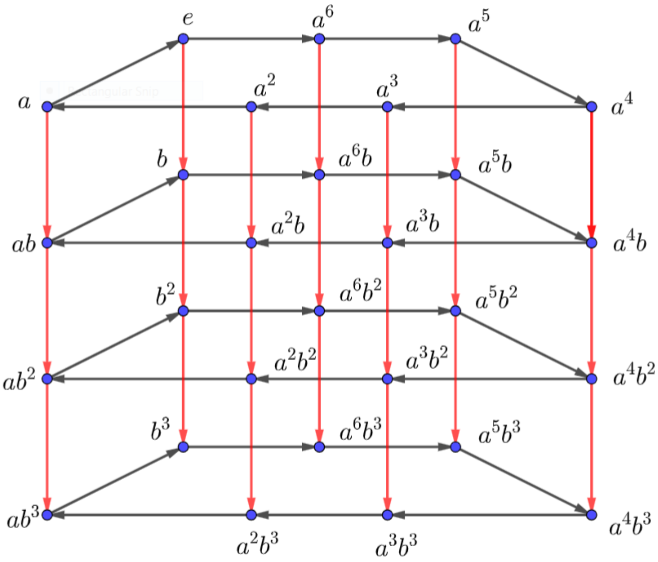

Fig.8. $Cay(C_7 \rtimes C_4, \{a,b\})$

The connectivity of the Cayley digraph associated with a specific group, denoted $Cay(C_7 \rtimes C_4, \{a^2, b^2\})$, is established for all cases except one. In this exceptional case, the connectivity is disrupted when the generator set S includes the element b^2.

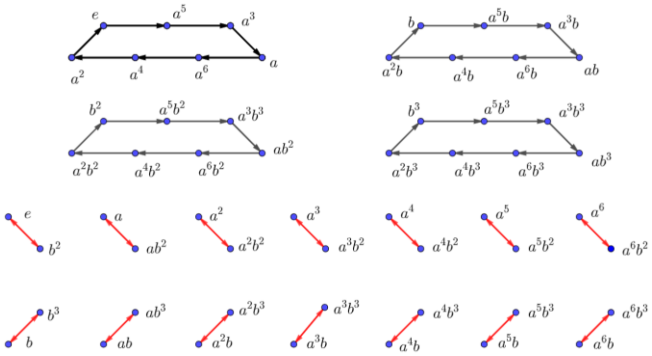

Fig.7. Paths generating by a^2 and b^2

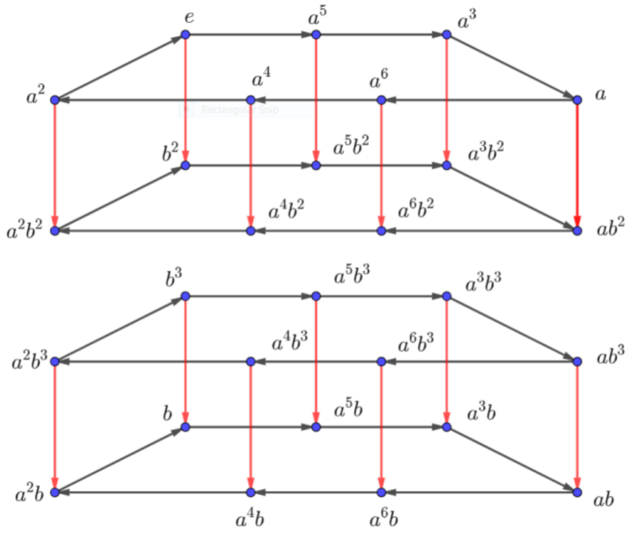

Fig.10. $Cay(C_7 \rtimes C_4, \{a^2, b^2\})$

CONCLUSION

By finding the relationship linking the elements (vertices) of a specific group with those of another group, the research seeks to determine the shape of the Cayley graph for a specific group and shows its importance to determining the structural structure of that group as well as its application to various fields.

A result of the study was:

TABLE 3: find the index of $C_7 \rtimes C_4$ and $C_7 \times C_4$

Group	**Generator Set**	**Condition of Cayley Digraph**	**Finding the index**	**Show that by proofing**
$C_p \times C_4$	$S = \{a,b\}$	$(a^i, b^j) = (a^h, b^k) \times (a, e)$	$h = i + 1, k = j$	$a^i b^j = a^h b^k a$ $a^i b^j = a^h a^{-1} b^k$ $a^i b^j = a^{h-1} b^k$ $h - 1 = i, k = j$ $h = i + 1, k = j$
		$(a^i, b^j) = (a^h, b^k) \times (e, b)$	$h = i, k = j - 1$	$a^i b^j = a^h b^k b$ $a^i b^j = a^h b^{k+1}$ $h = i, j = k + 1$ $h = i + 1, k = j - 1$
$C_p \rtimes C_4$	$S = \{a,b\}$	$(a^i, b^j) = (a\text{^}h, b^k) \rtimes (a, e)$	$h = i + 1, k = j$	$a^i b^j = a^h\ \varphi_{b^k}(a) b^k$ $a\text{^}i\ b\text{^}j = a^h a^{-1} b^k$ $a\text{^}i\ b\text{^}j = a^{h-1} b^k$ $h - 1 = i, k = j$ $h = i + 1, k = j$

		$(a^i, b^j) = (a^h, b^k) \rtimes (e, b)$	$h = i + 1, k = j - 1$	$a^i b^j = a^h \varphi_{b^k}(a) b^k b$ $a^\wedge i\ b^\wedge j = a^h a^{-1} b^{k+1}$ $a^\wedge i\ b^\wedge j = a^{h-1} b^{k+1}$ $h - 1 = i, k + 1 = j$ $h = i + 1, k = j - 1$

Despite the finding that the only cyclic group exists, more research is needed on other groups, including dihedral, symmetric, and quaternion groups. The semidirect product contains at least two automorphism identity functions and their inverses. Our study focuses on only one automorphism function. Additional research is warranted in regards to other forms of automorphism functions. Furthermore, a comprehensive analysis should be conducted on the characteristics of Cayley graphs pertaining to the groups $C_p \rtimes C_4$ and $C_p \times C_4$. As an illustration, determining the shortest path connecting any two vertices within the Cayley graph of these groups could prove to be advantageous.

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REFERENCES

[1] I. Grossman and W. Magnus, “Groups and their graphs,” vol. 14, pp. 44-51, New York: Random House, 1964.

[2] M. Krebs and A. Shaheen, “Expander families and Cayley graphs: a beginner's guide,” pp. 19-20, Oxford University Press, 2011.

[3] D. Caucal, “Cayley graphs of basic algebraic structures,” Discrete Mathematics & Theoretical Computer Science, vol. 21, pp. 2-3, 2020.

[4] E. Dobson, P. Spiga, and G. Verret, “Cayley graphs on abelian groups,” Combinatorica, vol. 36, no. 4, pp. 371-393, 2016.

[5] M. U. Sherman-Bennett, “On Groups and Their Graphs,” pp. 27-29, Bard College, MA, 2016.

[6] J. A. Bondy and U. S. R. Murty, “Graph theory with applications,” vol. 290, pp. 1-13, London: Macmillan, 1976.

[7] J. Bang-Jensen and G. Z. Gutin, “Digraphs: theory, algorithms and applications,” pp. 2-5, Springer Science & Business Media, 2008.

[8] G. Chartrand, L. Lesniak, and P. Zhang, “Graphs & digraphs,” vol. 22, pp. 47-49, London: Chapman & Hall, 1996.

[9] A. Tripi, “Cayley graphs of groups and their applications,” pp. 5-7, 2017.

[10] J. Meier, “Groups, graphs, and trees: an introduction to the geometry of infinite groups,” pp. 17-22, Cambridge University Press, 2008.

[11] J. A. Gallian, “Contemporary abstract algebra,” pp. 72-77, Chapman and Hall/CRC, 2021.