On Best Multiplier Approximation of K −Monotone Unbounded Functions by Spline Polynomials in L_{P,λ_n} −Space

Saheb K. Al-Saidy
Department of Communication, College of Engineering
Uruk University
Baghdad, Iraq
sahebalsaidy@gmail.com
Orcid.org/0009-0004-6150-4330

Hasan W. Maktoof
Department of Mathematics, College of Science
Mustansiriyah University
Baghdad, Iraq
hasan2017@uomustansiriyah.edu.iq
Orcid.org/0000-0002-6759-0629

Abdul khaleq owaid mazeel
Department of Mathematics, College of Science
Mustansiriyah University
Baghdad, Iraq
khaleqmazeel@uomustansiriyah.edu.iq
Orcid.org/0000-0003-4830-767X

DOI: http://dx.doi.org/10.31642/JoKMC/2018/100212

Received Jun. 10, 2023. Accepted for publication Jul. 11, 2023

Abstract—*The main purpose of this research is to study the degree of the best multiplier approximation of monotone unbounded functions in $L_{P,\lambda_n}(X)$ −space, where $X=(a,b)$ by spline polynomials in terms of averaged multiplier modulus smoothness $\tau(f,\delta)_{P,\lambda_n}$ using some definitions and theorems necessary for this.*

Keywords—Multiplier Integral, Multiplier Averaged Modulus of Smoothness, Multiplier Norm, Spline Polynomial.

I. INTRODUCTION

Many specialists and researchers in this field approximation theory have studied the approximation of monotone function by polynomials, such as; in 1995 [1], Hu, Y. K., obtained some results about approximation of bounded function in $L_P[0,1]$ −space by spline polynomials, in 1995 [2], Kopotun, Kirill A., introduced a paper on k −monotone polynomial and spline approximation $L_P, 0<P<\infty$ quasi norm. in 1998 [3], Hu, Y. K, Kopotun K. and Yu, X. M, once again gave us a study on approximation by polynomial. In 2014 [4], S. K. Jassim and Zaboon had studied and analyzed the approximation of unbounded functions by the trigonometric polynomials in locally-global space in $L_{P,\sigma,w}$ and they got important results. In 2016 [11], S. A. Abdul-Hameed and Murtada J. Mohammed, obtained some results about approximation by using the method of Korovik-type statistical approximation. In 2017 [12], Ghazi Abdullah, introduced a paper about $\forall n$ nonnegative integer number, we can find a monotone function depending on n. In this paper, the spline polynomial will be utilized to study and analyze the degree of the best multiplier approximation of monotone unbounded functions f in $L_{P,\lambda_n}(X)$ −space, $X=(a,b)$ in terms of averaged multiplier modulus smoothness $\tau(f,\delta)_{P,\lambda_n}$. in this work, we take the span (basis) of spline function be a B −spline, $(B_1^k,\ldots,B_{n+1+k}^k)$ [9].

We give a definition for basis spline function which are usually denoted by B −spline and we will also record various properties of the B −spline.

Let $t=\{t_i\}$ be a non-decreasing sequence (which may be finite or infinite). The ith normalized B −spline of order k for the knot sequence t is denoted by $B_{i,k,t}$ and is defined by the rule:

$$B_{i,k,t}(x)=(t_{i+k}-t_i)[t_i,t_{i+1},\ldots,t_{i+k}](-x)_+^{k-1}, \forall x\in\mathbb{R},$$

where $[t_i,t_{i+1},\ldots,t_j]$ is the divided deference of order $j-i$ of f at the points $t_i,t_{i+1},\ldots,t_j$ for all i,j are integers.

As for the properties of B −splines:

We notice that it is right a way that $B_{i,k,t}$ has small support, i.e.:

$$B_{i,k,t}(x)=0, \text{for all } x\notin[t_i,t_{i+k}]$$

We have: $\sum_i B_{i,k,t}(x)=1\ \forall\ t_r<x<t_s$,

$B_{i,k,t}(x) > 0,\ t_i < x < t_{i+k}$.

Note that the space of spline with simple knots $(x_1, x_2, \cdots, x_k)$ denoted by $S_N(x_1, x_2, \cdots, x_k)$, consider $a = x_0 < x_1, \cdots, < x_k < x_{k+1} = b$ a partition on the interval, for $k = 1,2, \cdots$.
Let $S_k = S_N(x_1^k, x_2^k, \cdots, x_k^k)$, for some k knot.
Let $m_k = max_{i=0,1,\cdots,k}(x_{i+1}^k - x_i^k)$ is called the mesh length. And the formula for the spline polynomial is:

$$S_k(x) = \sum_{i=1}^{n+k+1} f(t_i^k)B_i^k(x), \forall\, j = 0,1, \cdots k, x \in [x_j^k, x_{j+1}^k],$$

where $B_i^k(x)$ is B −spline.

II. DEFINITIONS AND CONCEPTS

Definition 1. [7]

We denote by $L_\infty(X), X = (a,b)$, the space of all bounded measurable function of f on X with the norm:

$$\|f\|_\infty = sup\{|f(x)|, x \in X\} < \infty$$

Definition 2. [6]

Let $f \in L_P[a,b]$, where $1 \le P \le \infty$, be the space of all bounded measurable function f on X with the norm:

$$\|f\|_P = \left[\int_a^b |f(x)|^p\, dx\right]^{1/p} < \infty, 1 \le P < \infty$$

Definition 3. [8]

A series $\sum_{n=0}^{\infty} a_n$ is called a multiplier convergent series if there is a convergent sequence of real number $\{\lambda_n\}_{n=0}^{\infty}$ such that $\sum_{n=0}^{\infty} \alpha_n \lambda_n \le \infty$ and $\{\lambda_n\}_{n=0}^{\infty}$ is called multiplier for the convergence.

Definition 4.

For any real valued function $f \in L_{P,\lambda_n}(X)$, where $X = (a,b)$, if there is a sequence $\{\lambda_n\}_{n=0}^{\infty}$, such that:

$$\int_a^b f(x)\, \lambda_n dx < \infty,$$

then f is called a multiplier integrable function, λ_n is called a multiplier integrable sequence.

Definition 5.

Let $f \in L_{P,\lambda_n}(X)$, where $X = (a,b)$, then: $\|f\|_{P,\lambda_n}$, is given by the below definite multiplier integral norm:

$$\|f\|_{P,\lambda_n} = \left[\int_a^b \left|(f\lambda_n)_{(x)}\right|^p dx\right]^{1/p}$$

This represents the space of all unbounded functions defined on X with the norm $\|f\|_{P,\lambda_n}$, where $1 \le P < \infty$.

Definition 6. [7]

Let $f \in C[a,b]$, then $\omega(f,h) = max_{|x-y|<h}|f(x) - f(y)|$ is called the modulus of continuity of the function f with size (distance).

Definition 7.

Let $f \in L_{P,\lambda_n}(X)$, where $X = (a,b), 1 \le P < \infty$, then the multiplier continuity modulus of the function f with step size (distance) is known as the following:

$$\omega(f,h)_{P,\lambda_n} = \max_{|x-y|<h} |\lambda_n(f(x) - f(y))|$$

Definition 8. [7]

Let $f \in L_P[a,b]$, where $1 \le P \le \infty$, then the integral modulus (L_P −modulus or P −modulus) of order k of function f is the following function of $\delta \in [0,(b-a)/k]$:

$$\omega_k(f;\delta)_{L_P} = sup_{0\le h\le\delta}\left\{\int_a^{b-kh} \left|\Delta_h^k f(x)\right|^p dx\right\}^{1/p}$$

Definition 9.

Let $f \in L_{P,\lambda_n}(X)$, where $X = (a,b), 1 \le P < \infty$, then: the multiplier integral modulus of smoothness of order k of the function f, where $0 \le \delta \le b - ak$, is defined by:

$$\omega_k(f,\delta)_{P,\lambda_n} = sup_{h\in[0,\delta]}\left(\int_a^{b-kh} \left|\Delta_h^k(\lambda_n f)_{(x)}\right|^p dx\right)^{1/p}$$

Where:

$$\Delta_h^k(\lambda_n f)_{(x)} = \sum\nolimits_{m=i}^{k} (-1)^{m+k}\binom{k}{m}(\lambda_n f)(x+mh); \binom{k}{m} = \frac{k!}{m!\,(k-m)!}$$

Definition 10. [7]

Let $f \in L_P(X)$; where $X = [a,b]$ and $1 \le P \le \infty$. The local modulus of smoothness of the function f of order k at a point $x \in [a,b]$ is the following function of $\delta \in [0,(b-a)/k]$:

$$\omega_k(f,x;\delta) = sup\left\{\left|\Delta_h^k f(t)\right|; t, t+kh \in \left[x - \frac{k\delta}{2}, x + \frac{k\delta}{2}\right] \cap [a,b]\right\}$$

Definition 11.

If $f \in L_{P,\lambda_n}(X)$, where $X = (a,b), 1 \le P < \infty$, then the multiplier local modulus of smoothness of a function f of order k at a point $x \in [a,b], 0 \le \delta \le \frac{b-a}{k}$ is defined by:

$$\omega_k(f,x,\delta)_{P,\lambda_n} = sup_{h\in[0,\delta]}\left\{\Delta_h^k(\lambda_n f)(t); t, t+kh \in \left[x-\frac{k\delta}{2}, x+\frac{k\delta}{2}\right]\cap[a,b]\right\}.$$

Definition 12. [7]

The averaged modulus of smoothness of order k (or τ −modulus) of the function $f \in M[a,b]$ is the following function of $\delta \in [0,(b-a)/k]$:

$$\tau_k(f;\delta)_P = \|\omega_k(f,.,;\delta)\|_{LP} = \left[\int_a^b (\omega_k(f,x,\delta))^p dx\right]^{1/p}.$$

Definition 13.

If $f \in L_{P,\lambda_n}(x)$, where $X=(a,b), 1 \le P < \infty$, then the multiplier averaged modulus of smoothness of order k of $f \in L_{P,\lambda_n}(X)$, is defined by:

$$\tau_k(f,\delta)_{P,\lambda_n} = \|\omega_k(f,.,\delta)\|_{P,\lambda_n} = \left(\int_a^b (\omega_k(\lambda_n f,x,\delta))^p dx\right)^{1/p}.$$

Definition 14. [7]

If $f \in L_P(X), X=[a,b]$, then:

$$E_n(f)_P = inf\{\|f-P_n\|_P : P_n \in P\},$$

where $E_n(f)_P$ is called the degree of the best monotone multiplier approximation of f by polynomial P_n.

Definition 15.

If $f \in L_{P,\lambda_n}(X), X=(a,b)$, then:

$$E_n(f)_{P,\lambda_n} = inf\{\|f-S_n\|_{P,\lambda_n} : S_n \in P\}$$

Such that $E_n(f)_{P,\lambda_n}$ is called the degree of the best monotone multiplier approximation of f by polynomial S_n.

Definition 16. [9]

Let X be a Banach space with real or complex scalar. Let $x_1, x_2, \cdots, x_n$ be given vectors in X.

Consider the polynomials of the form $y=\sum_{i=1}^n \alpha_i x_i$, where α_i are scalars, $x_i \in X$, then the degree of approximation of $x_i \in X$ is $E_n(x) = inf\|x-y\|$.

If the infimum is attained for some $y=y_0$, this y_0 is called a linear combination of the best approximation or a polynomial of the best approximation to x.

III. MAIN RESULTS

In this work, we prove theorems to find degree of the best multiplier approximation of monotone unbounded functions by spline polynomials in $L_{P,\lambda_n}(X)$ −space in terms of averaged multiplier modulus of smoothness $\tau(f,\delta)_{P,\lambda_n}(X), X=(a,b)$.

Theorem 1. [5]

For each $f \in C[a,b]$, there exist $s_k \in S_k$ such that:

$$\lim_{k\to\infty}\|f-s_k\|_\infty = 0 \text{ if and only if } \lim_{k\to\infty} m_k = 0$$

Now, we want to find some results, but for all unbounded function in the space L_{P,λ_n} −space, $1 \le P < \infty$.

Theorem 2.

Let f be unbounded function, $f \in L_{P,\lambda_n}(a,b), s_k \in S_k$, then:

$$E_n(f)_{P,\lambda_n} \le \|f-s_k\|_{P,\lambda_n} \le C_P\tau(f,\frac{1}{n})_{P,\lambda_n}$$

Proof:

Let $f \in L_{P,\lambda_n}(X)$, be unbounded function on (a,b), we divide the interval (a,b) to n subintervals $a \le x_0 < x_1 < x_2 < x_3 < \cdots < x_n \le 1$, such that $|x_i - x_{i+1}| = \frac{1}{n}, n\to\infty$. Let $\omega(f,\delta)_{P,\lambda_n} = \max_{|x-y|\le\delta}\{|\lambda_n(f(x)-f(y))|\}, 0 \le \delta < \frac{1}{n}$, $\forall\delta > 0$. Let $t_i^k \in B_i^k \cap (a,b), i=1,\cdots n+k+1$.

Let s_k be quasi operation denoted by:

$$s_k(x) = \sum_{i=1}^{n+k+1} f(t_i^k)\, B_i^k(x), \forall j = 0,1,2,\cdots k, x \in \left[x_j^k, x_{j+1}^k\right]$$

Then:

$$E_n(f)_{P,\lambda_n} \le \|f-s_k\|_{P,\lambda_n} = \left\{\int_a^b |\lambda_n(f(x)-s_k(x))|^p\, dx\right\}^{1/p}$$

$$= \left\{\int_a^b \left|\lambda_n(f(x) - \sum_{i=1}^{n+k+1} f(t_i^k)\, B_i^k(x))\right|^p dx\right\}^{1/p} \quad (1)$$

Now, we use new step, since:

$$\sum_{i=1}^{n+k+1} B_i^k(x) = 1 \quad (2)$$

We multiply both sides of equation (2) by $f(x)$, we get:

$$\sum_{i=1}^{n+k+1} B_i^k(x)\, f(x) = f(x) \quad (3)$$

We substitute equation (3) in equation (1), we get by using Jensen's inequality [10] which are: for $P \ge 1$ and $\alpha_i > 0$, we have: $|\sum_{i=1}^n \alpha_i\beta_i|^p \le \sum_{i=1}^n \alpha_i\, |\beta_i|^p$, where $\sum_{i=1}^n \alpha_i = 1$.

$$E_n(f)_{P,\lambda_n} = \left\{\int_a^b \sum_{i=1}^{n+k+1} \left|\lambda_n(f(x)-f(t_i^k))B_{i,k}(x)\right|^p dx\right\}^{1/p}$$

$$E_n(f)_{P,\lambda_n} = \left\{\int_a^b \sum_{i=1}^{n+k+1} \left|\lambda_n(f(x) - f(t_i^k))\right|^p dx\right\}^{1/p}$$

Since $\sum_{i=1}^{n+k+1} B_{i,k}(x) = 1$.

$$E_n(f)_{P,\lambda_n} \leq \left\{\int_a^b \left|\Delta_{h,m}^k(\lambda_n f)_{(x)}\right|^p dx\right\}^{1/p}$$

$$E_n(f)_{P,\lambda_n} \leq \left\{\int_a^b \left|\omega(f; x; (n+2)_{m_k})\right|^p dx\right\}^{1/p}$$

$$E_n(f)_{P,\lambda_n} = C_P \left\|\omega(f; x; (n+2)_{m_k})\right\|_{P,\lambda_n}$$

$$E_n(f)_{P,\lambda_n} = C_P \tau(f, (n+2)_{m_k})_{P,\lambda_n}$$

Corollary 3.

Let $f \in L_{P,\lambda_n}(X), X = (a,b), f$ be unbounded function, $s_k \in S_k$, then:

$$\lim_{k\to\infty} \|f - s_k\|_{P,\lambda_n} = 0 \text{ if and only if } \lim_{k\to\infty} m_k = 0$$

Proof:

First, we assume that $\lim_{k\to\infty} m_k = 0$, such that $|x_{i+1} - x_i| \to 0$ by Theorem 2

$$\|f - s_k\|_{P,\lambda_n} \leq \tau_P(\delta, \frac{1}{n})_{P,\lambda_n}$$

Since $\lim_{k\to\infty} m_k = 0$, such that $|x_{i+1} - x_i| \to 0$

$$\tau(f,\delta)_{P,\lambda_n} = \|\omega(f, x, \delta)\|_{P,\lambda_n}$$

$$= sup_{|h|\leq\delta} \left\{\Delta_{h,m}^k(\lambda_n f)(t): t, t+kh \in \left[x - \frac{k\delta}{2}, x + \frac{k\delta}{2}\right] \cap \{a,b\}\right\}$$

Since $m_k \to 0$ implies that $\delta \to 0$.

Then $\left|\Delta_{h,m}^k(\lambda_n f)(x)\right|$ tend to zero ($\Delta_h^k \to 0$).

Therefore $\tau(f,\delta)_{P,\lambda_n}$ tend to zero ($\tau \to 0$).

Since $\|f - s_k\|_{P,\lambda_n} \leq \tau_P(\delta, \frac{1}{n})_{P,\lambda_n}$ by Theorem 2, we have that:

$\|f - s_k\|_{P,\lambda_n}$ tend to zero.

Hence, $\lim_{k\to\infty} \|f - s_k\|_{P,\lambda_n} = 0$ as $k \to 0$.

Converslely:

Suppose that $\lim_{k\to\infty} \|f - s_k\|_{P,\lambda_n} = 0$, if $\lim_{k\to\infty} m_k$ does not tend to zero, so there exist a subinterval, such that $d(x_k, x_{k+1})$ does not tend to zero.

That is big interval, so $s_k(x)$ does not converge to function f in this subinterval.

Then, in this subinterval, we get that:

$$\lim_{k\to\infty} \|f - s_k\|_{P,\lambda_n} \geq C$$

Where C is constant and this is contraducti

$$\lim_{k\to\infty} \|f - s_k\|_{P,\lambda_n} = 0, \text{hence } \lim_{k\to\infty} m_k = 0$$

CONCLOSION

the aim of this paper is to obtain the degree of the best multiplier approximation of monotone unbounded functions, $f \in L_{P,\lambda_n}(X)$ −space, $X = (a,b)$ by spline polynomials in terms of averaged multiplier modulus smoothness $\tau(f,\delta)_{P,\lambda_n}$ using some definitions and theorems necessary for this.

REFERENCES

[1] Y.-K. Hu, "Positive and copositive spline approximation in Lp [0, 1]," Computers & Mathematics with Applications, vol. 30, no. 3-6, pp. 137-146, 1995.

[2] K. A. Kopotun, Approximation in Lp, 0< p< oo (Quasi) norm (Approximation Theory Viii-Volume 1: Approximation And Interpolation). World Scientific Publishing Co Pte Ltd, 1995.

[3] Y. Hu, K. Kopotun, and X. Yu, "On Positive and Copositive Polynomial and Spline Approximation inLp [−1, 1], 0< p<∞," journal of approximation theory, vol. 86, no. 3, pp. 320-334, 1996.

[4] S. K. Al-Saidy and A. H. Zuboon, "Approximation of Unbounded Functions by Trigonometric Polynomials in Locally-Global Weighted " Al-Mustansiriya J. Sci., vol. 25, no. 3, pp. 97-112, 2014.

[5] A. Pinkus, "Density in approximation theory," arXiv preprint math/0501328, 2005.

[6] G. G. Lorentz, "Approximation of Functions," Scientific Research, Chelsea, New York, 1986.

[7] B. Sendov and V. A. Popov, The averaged moduli of smoothness: applications in numerical methods and approximation. John Wiley & Sons Incorporated, 1988.

[8] G. Hardy, "Divergent series," ed: Clavendan Press, 1949.

[9] C. De Boor and C. De Boor, A practical guide to splines. springer-verlag New York, 1978.

[10] G. I. Gaudry and I. R. Inglis, "Approximation of multipliers," Proceedings of the American Mathematical Society, vol. 44, no. 2, pp. 381-384, 1974.

[11] M. J. Mohammed, and S. A. Abdul-Hammed. "On modified approximation properties by q-analogue summation-integral type operators." Journal of Kufa for Mathematics and Computer, vol. 3, no. 1, pp. 7-12, 2016.

[12] Abdullah, G. "Negative Theorem for Lp, 0< P< 1 Monotone Approximation," Journal of Kufa for Mathematics and Computer vol. 4, no. 3, pp. 1-3, 2017.