Distributive Semimodules: Theory, Properties, and Extensions

Ali Hasan Abdoul Khaleq
Department of Mathematics, College of Education for Pure Science, University of Babylon
Babylon, Iraq
ali.kufa94@gmail.com
Orcid.org/0009-0002-3872-7910

Asaad M. A. Alhossaini
Department of Mathematics, College of Education for Pure Science, University of Babylon
Babylon, Iraq
asaad_hosain@itnet.uobabylon.edu.iq
Orcid.org/0000-0002-4569-3352

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Abstract— *This article explores the properties and theorems pertaining to subtractive and distributive semimodules. The focus is on understanding the relations between these characteristics and their implications within the context of semimodules. The investigated propositions establish conditions for distributivity in semimodules. It is shown that under certain circumstances, the preimages of subsemimodules under homomorphisms exhibit specific additive properties. Furthermore, the study introduced definitions and propositions related to supplements, principally supplemented semimodules and H-supplemented semimodules. The findings reveal the importance of supplements in understanding the structure and properties of semimodules, particularly in relation to their distributive nature.*

Keywords—distributive semimodule, subtractive, principally supplemented semimodule, H-supplemented semimodule.

I. INTRODUCTION

A semiring is a nonempty set R on which operations of addition and multiplication have been defined such that the following conditions are satisfied: $(R,+)$ is a commutative monoid with identity element 0; $(R,\cdot)$ is a monoid with identity element $1 \neq 0\,(1 = 1_R)$; multiplication distributes over addition, i.e. $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for all $a, b, c \in R$; the element 0 is the absorbing element of the multiplication, i.e. $r \cdot 0 = 0 \cdot r = 0$ for all $r \in R$. The semiring R is said to be commutative if its multiplication is commutative [1, p. 1].

Let R be a commutative semiring with identity. An R-semimodule M is said to be distributive if, for all subsemimodules $A, B,$ and C of M, the following equality holds: $A \cap (B + C) = (A \cap B) + (A \cap C)$ [2].The notion of distributive semimodules has been studied and developed as a generalization independently in [2] and [3]. As for the module, in the last six decades much research and results on the structure of the modules with a distributive lattice of submodules (see for example [4], [5], [6], [7], and [8]). In this study, our focus is on investigating the properties of distributive semimodules. We aim to delve deeper into understanding the nature of the distribution property and its implications within this research. In section 2, some definitions and facts that are important in our study will be given. In section 3, some properties in homomorphisms of distributive semimodule. In section 4, the concept of supplemented extending (principally⊕-supplemented, principally supplemented, and H-supplemented) semimodules and a relationship between distributive semimodule and these concepts are studied.

II. PRELIMINARIES

In this section, we will present some definitions and remarks that are needed in the next section.

Definition 2.1. Let R be a semiring, a **left R-semimodule** is a commutative monoid $(M,+)$ with additive identity 0 for which we have a function $R \times M \to M$ defined by $(r,m) \mapsto rx$ (scalar multiplication), which satisfies the following conditions for all elements $r, r' \in R$ and all elements $m, m' \in M$:

1. $(rr')x = r(r'm)$;
2. $r(m + m') = rm + rm'$;
3. $(r + r')m = rm + r'm$;
4. $r0_M = 0_M = 0_R m.$

If the condition$1_R m = m$ for all $m \in M$ hold then the semimodule M is said to be unitary. [1, p. 149]

Definition 2.2. A non-empty subset N of a left R-semimodule M is called a **subsemimodule** of M if N is closed under addition and scalar multiplication that is N is an R-semimodule itself (denoted by $N \leq M$). The set of all subsemimodules of M is denoted by L(M) [1, p. 150]

Definition 2.3. An element m of an R −semimodule M is called **cancellable** if for all $m, m' \in M$ with $m + m' = m + m''$ implies that $m' = m''$. The semimodule M is cancellative if and only if every element of M is cancellable. [1, p. 49]

Definition 2.4. Let M be an R-semimodule. A **subtractive** subsemimodule (or k −subsemimodule) N is a subsemimodule of M such that if $z, z + w \in N$, then $w \in N$, and the semimodule M is **subtractive** if each subsemimodule is subtractive. [9]

Definition 2.5. A left R -semimodule M is called **Noetherian**, if M satisfies the ACC (Ascending chain condition) on its R - subsemimodules, i.e., if there does not exist an infinite strictly ascending chain $M_1 \subset M_2 \subset M_3 \subset \ldots$ of subsemimodules of M. [10]

Definition 2.6. An R -semimodule M is called **uniserial** if for any two subsemimodules H and L of M, either $H \subseteq L$ or $L \subseteq H$. [11]

Definition 2.7. If R is a semiring. The R-semimodule M is **semisubtractive** if each $x, y \in M$, there exists $z \in M$ where $x + z = y$ or $y + z = x$. [12]

Definition 2.8. Let M and D be R - semimodules the map f: $M \rightarrow D$ is called a homomorphism if $\forall\, e, n \in M, s \in R$

1- $f(e+n) = f(e) + f(n)$.
2- $f(s\,n) = sf(n)$. [1, p. 156]

Definition 2.9. [11] For a homomorphism of R - semimodules $\text{ɥ}: \mathcal{B} \rightarrow \mathcal{D}$ we define

1- $\text{ɥ}(\mathcal{B}) = \{\text{ɥ}(e) \mid e \in \mathcal{B}\}$.
2- $\ker(\text{ɥ}) = \{ e \in \mathcal{B} \mid \text{ɥ}(e) = 0\}$.
3- ɥ is $\boldsymbol{k}$**− regular**, if $\text{ɥ}(e) = \text{ɥ}(\acute{e})$ implies $e + h = \acute{e} + \acute{h}$ for some $h, \acute{h} \in \ker(\text{ɥ})$.
4- $Hom(\mathcal{B}, \mathcal{D}) = \{\text{ɥ}: \mathcal{B} \rightarrow \mathcal{D} \mid \text{ɥ is a homomorphism}\}$.

Definition 2.10.
A subsemimodule U of M is called a **fully invariant** subsemimodule if $f(U) \subseteq U$ for every f $\in End(M)$. [12]

Definition 2.11. [12]A semimodule U is said to be **duo** if each subsemimodule of U is fully invariant.

Definition 2.12. Let $\mathcal{D}$ be an $R-$semimodule, and $\mathcal{L}, \mathcal{L}_* \in L(\mathcal{D})$. $\mathcal{D}$ is called a **direct sum** of $\mathcal{L}$and $\mathcal{L}_*$, if each $u \in \mathcal{D}$ can be represented uniquely as $u = k + h$, where $k \in \mathcal{L}$ and $h \in \mathcal{L}_*$, then we can say that $\mathcal{L}$ (similarly $\mathcal{L}_*$) is a **direct summand** of $\mathcal{D}$ and denoted by $\mathcal{D} = \mathcal{L} \oplus \mathcal{L}_*$. [1, p. 184]

Definition 2.13. Let $\mathcal{D}$ be an $R-$semimodule, and $\mathcal{J}, \mathcal{S} \in L(\mathcal{D})$. The subsemimodule $\mathcal{J}$ is called **supplemented** of $\mathcal{S}$ in $\mathcal{D}$ if $\mathcal{J} + \mathcal{S} = \mathcal{D}$, and $\mathcal{J}$ is minimal with this property. That is, if $\mathcal{F} + \mathcal{S} = \mathcal{D}$ and $\mathcal{F} \subseteq \mathcal{J}$, then $\mathcal{F} = \mathcal{J}$. A semimodule $\mathcal{D}$ is called supplemented if each $\mathcal{S} \in L(\mathcal{D})$ has a supplemented in $\mathcal{D}$. [13]

Definition 2.14. A subsemimodule $\mathcal{K}$ of semimodule $\mathcal{D}$ is called **closed** if $\mathcal{K}$ has no proper essential extensions in $\mathcal{D}$. [12]

III. ON HOMOMORPHISMS OF DISTRIBUTIVE SEMIMODULE

Definition 3.1. [2] Let R be a semiring. A left R-semimodule $\mathcal{D}$ is called a distributive semimodule if, for all subsemimodules $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ of $\mathcal{D}$, the following equality holds: $\mathcal{A} \cap (\mathcal{B} + \mathcal{C}) = (\mathcal{A} \cap \mathcal{B}) + (\mathcal{A} \cap \mathcal{C})$.

Lemma 3.2. If $\mu \in Hom(U, A)$, where U and A are $R-$ semimodules. If B is a subtractive subsemimodule of A, then so is $\mu^{-1}(B)$.

Proof: The same as in the semiring (see [1, p. 116]) **.**

Lemma 3.3.
Let U and A be $R-$ semimodules and $\mu \in Hom(U, A)$, and $M \in L(A)$, then $\mu\left(\mu^{-1}(M)\right) = M \cap \mu(U)$.

Proof. The same as in the modules (see [14, p. 44]**).**

Remark 3.4: If $\mu \in Hom(U, A)$, where U and A are $R-$ semimodules, then:
i) For $W, N \in L(U)$, $\mu(W + N) = \mu(W) + \mu(N)$,
ii) for $Q, S \in L(A)$, $\mu^{-1}(Q \cap S) = \mu^{-1}(Q) \cap \mu^{-1}(S)$,
iii) $\mu^{-1}(\mu(U)) = U$,

Proof: it's clear.

The following Proposition mentioned in [2],will be proved under different conditions.

Proposition 3.5. Let U and A be an R-semimodules, μ is k-regular homomorphism from U to A

i. If A is distributive, W and N subsemimodules of A with $W + N$ subtractive, then $\mu^{-1}(W + N) = \mu^{-1}(W) + \mu^{-1}(N)$.
ii. If U is distributive, T and S subsemimodules of U, then $\mu(T \cap S) = \mu(T) \cap \mu(S)$.

Proof.

(i) It is clear that $\mu^{-1}(W) + \mu^{-1}(N) \subseteq \mu^{-1}(W + N)$.

Let $x \in \mu^{-1}(W + N)$, then $\mu(x) \in W + N$. So $\mu(x) \in (W + N) \cap \mu(U)$.

Since A is distributive, $\mu(x) \in [W \cap \mu(U) + N \cap \mu(U)]$, by Lemma 3.3 then $\mu(x) \in [\mu\left(\mu^{-1}(W)\right) + \mu(\mu^{-1}(N))]$ this mean $\mu(x) = \mu(x_1) + \mu(x_2)$ with $x_1 \in \mu^{-1}(W)$, $x_2 \in \mu^{-1}(N)$ by Remark 3.4 $\mu(x) = \mu(x_1 + x_2)$.

By hypothesis, μ a k-regular, hence $x + k_1 = x_1 + x_2 + k_2$ for some $k_1, k_2 \in ker\mu$. Now, $x_1 + x_2 + k_2 \in \mu^{-1}(W) + \mu^{-1}(N)$ (since $er\mu \leq \mu^{-1}(N)$), so $x + k_1 \in \mu^{-1}(W) + \mu^{-1}(N)$. But $k_1 \in ker\mu \subseteq \mu^{-1}(W) + \mu^{-1}(N)$, then $x \in \mu^{-1}(W) + \mu^{-1}(N)$(by Lemma 3. 3 and subtractive property).

(ii) It is clear that $\mu(T \cap S) \subseteq \mu(T) \cap \mu(S)$. Let $y \in \mu(T) \cap \mu(S)$, then $y = \mu(\mathrm{t}) = \mu(\mathrm{s})$, $\mathrm{t} \in T$, $s \in S$. Since , μ is k-regular, $t + k_1 = s + k_2$ for some $k_1, k_2 \in ker\mu$. Then $t + k_1 \in (T + ker\mu) \cap (S + ker\mu)$, since U is distributive, $t + k_1 \in T \cap S + T \cap ker\mu + T \cap ker\mu + ker\mu$ so, $t + k_1 \in T \cap S + ker\mu$ hence $t + k_1 = x + k_3$ where $x \in T \cap S$, $k_3 \in ker\mu$. Then $\mu(\mathrm{t}) = \mu(x)$implies $y = \mu(x) \in \mu(T \cap S)$. Therefore $\mu(T \cap S) = \mu(T) \cap \mu(S)$

Proposition 3.6. Let $\mathcal{D}$ and $\mathcal{M}$ be R-semimodules, and $\mathcal{N}, \mathcal{H} \in L(\mathcal{D})$. Consider the following statements:
i. $\mathcal{D}$ is a distributive semimodule.
ii. For any $f \in Hom(\mathcal{D}, \mathcal{M})$, $f(\mathcal{N} \cap \mathcal{H}) = f(\mathcal{N}) \cap f(\mathcal{H})$.
iii. For any $\in Hom(\mathcal{M}, \mathcal{D})$, then $g^{-1}(\mathcal{N} + \mathcal{H}) = g^{-1}(\mathcal{N}) + g^{-1}(\mathcal{H})$).
iv. $\mathcal{D}$ and $\mathcal{M}$ are modules and $\mathcal{D}$ is distributive.
Then the following implications: iii → i, ii → i, iv → ii, and iv → ii hold.

Proof. (iii→i)
If $\mathcal{B}, \mathcal{N}, \mathcal{H} \in L(\mathcal{D})$ and $\sigma: \mathcal{B} \to \mathcal{D}$ be the inclusion map, so $\mathcal{B} \cap (\mathcal{N} + \mathcal{H}) = \sigma^{-1}(\mathcal{N} + \mathcal{H}) = \sigma^{-1}(\mathcal{N}) + \sigma^{-1}(\mathcal{H}) = \mathcal{B} \cap \mathcal{N} + \mathcal{B} \cap \mathcal{H}$.
(ii→i)
Let $\mathcal{B}, \mathcal{N}, \mathcal{H} \in L(\mathcal{D})$ and let $\pi: \mathcal{D} \to \mathcal{D}/\mathcal{N}$ be the natural epimorphism
Since $\pi(\mathcal{B} \cap \mathcal{H}) = \pi(\mathcal{B}) \cap \pi(\mathcal{H})$, we have:
$\pi^{-1}\big(\pi(\mathcal{B} \cap \mathcal{H})\big) = \mathcal{B} \cap \mathcal{H} + ker\pi = \mathcal{B} \cap \mathcal{H} + \mathcal{N}$ ….(1)
Also,
$\pi^{-1}\big(\pi(\mathcal{B} \cap \mathcal{H})\big) = \pi^{-1}\big(\pi(\mathcal{B}) \cap \pi(\mathcal{H})\big) = \pi^{-1}(\pi(\mathcal{B})) \cap \pi^{-1}(\pi(\mathcal{H})$
$= (\mathcal{B} + ker\pi) + (\mathcal{H} + ker\pi) = (\mathcal{B} + \mathcal{N}) \cap (\mathcal{H} + \mathcal{N})$ ….(2)
Therefore, the distributivity of $\mathcal{D}$ follows from the equivalence of the two expressions (1) and (2).
(iv → ii)
Let $\in Hom(\mathcal{D}, \mathcal{M})$, such that $f(m) = f(n)$
Since $\mathcal{D}$ and $\mathcal{M}$ are modules we have,
$f(m) - f(n) = 0$ which implies that
$f(m - n) = 0$. Therefore, $m - n \in kerf$
Let $k = m - n$, so f is k-regular. By the Proposition 3.5, we have
$f(\mathcal{N} \cap \mathcal{H}) = f(\mathcal{N}) \cap f(\mathcal{H})$ for any $\mathcal{N}, \mathcal{H} \in L(\mathcal{D})$.
(iv → iii)

Now, since g is k-regular, Proposition 3.5 implies that
$g^{-1}(\mathcal{N} + \mathcal{H}) = g^{-1}(\mathcal{N}) + g^{-1}(\mathcal{H})$.

The following Proposition mentioned in [3],will be proved under different conditions.

Proposition 3.7. Let $\mathcal{D}$ and M be R – semimodule, and $q \in Hom\,(\mathcal{D}, M)$.If $\mathcal{D}$ is a distributive semimodule and q is k-regular, then $q(\mathcal{D})$ is a distributive semimodule.

Proof. Let A, B, and $C \in L(q(\mathcal{D}))$, since

$$A \cap (B + C) = A \cap q(D) \cap [B \cap q(D) + C \cap q(D)]$$

$$= q\big(q^{-1}(A)\big) \cap \big[q\big(q^{-1}(B)\big) + q\big(q^{-1}(C)\big)\big] \text{ (by Lemma 3.3)}$$

by Proposition 3.5

$$A \cap (B + C) = q[q^{-1}(A) \cap \big(q^{-1}(B) + q^{-1}(C)\big)]$$

$$= q[q^{-1}(A) \cap q^{-1}(B) + q^{-1}(A) \cap q^{-1}(C)] \text{(since } \mathcal{D} \text{ is a distributive)}$$

$$= q[q^{-1}(A \cap B) + q^{-1}(A \cap C)] \text{(by Remmark 3.4)}$$

$$= q(q^{-1}(A \cap B)) + q(q^{-1}(A \cap C))$$

$$= (A \cap B) \cap q(\mathcal{D}) + (A \cap C) \cap q(\mathcal{D}) \text{(by Lemma 3.3)}$$

$$= (A \cap B) + (A \cap C).$$

Lemma 3.8. Let C be a fully invariant subsemimodule of R-semimodule M, and let A, B be subsemimodules of M such that $A \cap B = 0$. If each idempotent endomorphism of every 2-generated subsemimodule of M is extended to an endomorphism of M. Then $C \cap (A + B) = C \cap A + C \cap B$.

Proof: Assume that $H = (Re_1 \oplus Re_2)$ where $e_1 \in A$, $e_2 \in B$. Let $f_i: H \to Re_i$ be natural projections. Idempotent endomorphisms f_i of H are extended to endomorphism $g_i \in End(M)$. Hence $g_1(e_1) = e_1$, $g_2(e_2) = e_2$ and $g_1(e_2) = g_2(e_1) = 0$. If $h = e_1 + e_2 \in C \cap (A + B)$, then $e_i = g_i(e_1 + e_2) = g_i(h)$. By hypostasis $g_i(h) \in C$, therefore $h = g_1(e_1) + g_2(e_2) \in (C \cap A + C \cap B)$.

Proposition 3.9. Assume that each idempotent endomorphism of any 2-generated left ideal of any arbitrary factor semiring B of a commutative semiring R is extended to an endomorphism of B (as a semimodule over itself). If R is subtractive, then it is distributive.

Proof. It is clear that a commutative semiring R is duo as (a left or right) semimodule over itself, for if $f \in End({}_RR)$ and I is a left ideal of R, then for each $x \in I$, $f(x) = xf(1) = f(1)x \in I$, that is $f(I) \subseteq I$. This property hold for any factor of ${}_RR$. Let F, G, and H be left ideals of R, and let $h: R \to R/(G \cap H)$ be a natural epimorphism.
Not that, $h(\text{G})$ and $h(\text{H})$ are subsemimodules of the factor semiring $R/(G \cap H)$, with $h(\text{F}) \cap h(\text{G}) = 0$.
Applying Lemma 3.8 to $R/(G \cap H)$, we have

$$h(\text{F}) \cap \big(h(\text{G}) + h(\text{H})\big) = h(\text{F}) \cap h(\text{G}) + h(\text{F}) \cap h(\text{H})$$

the left-hand side

$$F/_{G \cap H} \cap \big(G/_{G \cap H} + H/_{G \cap H}\big) = F \cap (G + H)/_{(G \cap H)}$$

the right-hand side

$$\big(F/_{G \cap H} \cap G/_{G \cap H}\big) + \big(G/_{G \cap H} \cap H/_{G \cap H}\big) = (F \cap G + F \cap H)/_{(G \cap H)}$$

Therefore $F \cap (G + H) = (F \cap G + F \cap H)$, then is, ${}_RR$ is distributive.

Proposition 3.10. [3] Let $\mathcal{D}$ and M be subtractive R −semimodules with M isomorphic to $\mathcal{D}$, then $\mathcal{D}$ is distributive if and only if M is distributive

Recall that the semiring R is called uniserial if for any two ideal I and J of R, either $I \subseteq J$ or $J \subseteq I$. [15]

Proposition 3.11. If M is a subtractive cyclic semimodule over a uniserial semiring R, then M is a distributive semimodule.

Proof. Assume that $M = Rm$, for some m in M, and let $\gamma: R \to M$ be a homomorphism map such that $\gamma(r) = rm$, then γ is an epimorphism with $\ker(\gamma) = l(m)$, hence $M \cong R/_{l(m)}$. By hypothesis, R-semimodule R is a distributive semimodule. So, $R/_{l(m)}$ is a distributive semimodule, by Proposition 3.10 M is a distributive semimodule.

Lemma 3.12. [16]Let B be a semisubtractive cancellative semimodule, and $\{B_i\}_{i\in I}$ is a family of subsemimodules of B, such that $\sum_{i\in J} B_i$ is subtractive for each $J \subseteq I$. Then $\sum_{i\in I} B_i = \oplus_{i\in I} B_i$ if and only if $B_j \cap \sum_{i\in I} B_i = 0$ for each $i \in J$.

Lemma 3.13. If $A, B \in L(M)$, $A \cap B = 0$, $A + B$ is **cancellative** and **semisubtractive**, then there is a bijection between the complements of B in $A \oplus B$ and the split of maps the projection of $A \oplus B$ onto A.

proof. Let $p: A\oplus B \rightarrow A$ be a natural projection map then p splits, there exists $q: A \rightarrow A\oplus B$ such that $pq = 1_A$, hence $A\oplus B = Imq \oplus kerp = Imq \oplus B$. It is clear that Imq is a complement of B in $A\oplus B$. Now if $q \neq \acute{q}$ are distinct split maps of p then $q(a) \neq \acute{q}(a)$ for some $a \in A$.

Claim: $q(a) \notin Im\acute{q}$, and then $Im\, q \neq Im\, \acute{q}$.

For, if $q(a) \in Im\acute{q}$, then $q(a) = q'(\acute{a})$ for some $\acute{a} \in A$. Hence $p\big(q(a)\big) = p\big(q'(\acute{a})\big)$, so, $a = \acute{a}$ (since $pq = pq' = 1_A$) that is, $q(a) = q(\acute{a}) = \acute{q}(\acute{a})$ $C!$, therefore $Im\, q \neq Im\, \acute{q}$.

Hence there is a bijective between the split maps of p and the complements of B in $A\oplus B$.

Lemma 3.14. If $A, B \in L(M)$, $A \cap B = 0$, $A + B$ is **cancellative** and **semisubtractive**, then there is a bijection between $Hom\,(A, B)$ and the split maps of the projection p of A⊕B onto A .

proof. Let α ∈ Hom (A, B) and $p: A\oplus B \rightarrow A$ be the natural projection

Define $q: A \rightarrow A\oplus B$ by $a \mapsto a + \alpha(a)$, then

$$(pq)(a) = p(a + \alpha(a)) = a$$

q is a right inverse of p, that is, a split map of p

It is clear that if $\alpha \neq \alpha' \in Hom(A, B)$, then for some $a \in A$.

$$\alpha(a) \neq \acute{\alpha}(a), \qquad q(a) = a + \alpha(a) \neq a + \acute{\alpha}(a) = \acute{q}\,(a).$$

On the other hand, if q is a split map of p, then it is 1-1, and it induces an $\alpha \in Hom\,(A, B)$, by $\alpha = \pi_B\, q$ where π_B is the natural projection of $A\oplus B$ onto B. Therefore, there is a bijection between $Hom\,(A, B)$ and the split maps p.

Corollary 3.15. If $A, B \in L(M)$,$A \cap B = 0$, $A + B$ is cancellative and semisubtractive, then there is a bijective between $Hom\,(A, B)$ and the set of complements of B in $A\oplus B$.

Proposition 3.16. A Lattice L is distributive if and only if for each interval I of L any two elements of I which are related in I are equal. (See [17, p. 68])

Proposition 3.17. Suppose that M is a cancellative and semisubtractive semimodule. $A, B \in L(M)$ such that $A \cap B = 0$. Then M is a distributive semimodule if and only if $Hom\,(A, B) = 0$.

proof. Since $A \cap B = 0$, by Corollary 3.15, there is a bijection between $Hom\,(A, B)$ and the set of complements of B in $A\oplus B$, and by Proposition 3.16 a lattice is distributive if and only if relative complements are unique. Now, we get that $Hom\,(A, B) = 0$ if and only if M is a distributive semimodule.

Corollary 3.18. Suppose that M is a cancellative and semisubtractive semimodule. $A, B \in L(M)$, such that $A \cap B$ subtractive subsemimodule. Then $M/(A \cap B)$ is a distributive semimodule if and only if $Hom\left(A/(A \cap B), B/(A \cap B)\right) = 0$.

Corollary 3.19. If M is a cancellative, semisubtractive, and distributive semimodule, such that $M = A + B$, where A, B are subtractive subsemimodules of M,then $Hom(M/A, M/B) = 0$.

Definition 3.20. Let M be R - semimodule, and $m \in M$ we define: $l(m) = \{r \in R: rm = 0\}$. [1]

Corollary 3.21. Let M be a distributive, cancellative, and semisubtractive semimodule over cancellative, semisubtractive semiring R. If $l(m) = 0, m \in M$, then Rm is an essential subsemimodule of M.

Proof. Let f: R → Rm be epimorphism such that $f(r) = rm$. Since $kerf = l(m) = 0$, then f is a monomorphism, hence R ≅ Rm. If Rm ∩ $A = 0$, for some $A \in L(M)$, if $0 \neq a \in A$, then there exist $g: R \rightarrow A$ defined by $g(r) = ra$. But we have $h: Rm \rightarrow R$ (inverse of f) which is not zero hence $0 \neq gh \in Hom(\text{Rm}, A)$ C!, hence $A = 0$.

Proposition 3.22. [2] Let U be a subtractive distributive R - semimodule and B be an R –semimodule, and $\omega, g \in Hom\,(U, B)$ such that $\omega + g$ is k- regular. Then, for any $C \in L(B)$, $C = (C + \omega(g^{-1}(C)) \cap \big(C + g(\omega^{-1}(C))\big)$.

Theorem 3.23. Let M be subtractive and distributive semimodule, N a noetherain semimodule, $f, g \in Hom\,(M, N)$ such that f $+ g$ is k-regular, if $g\,(M) \subseteq f\,(M)$ then ker $f \subseteq kerg$.
Proof. Suppose that $g\,(M) \subseteq f\,(M)$ but ker $f \nsubseteq kerg$. Let $A = \cap \{B \in L(N): f^{-1}(B) \subseteq g^{-1}(B)\}$
note that $f^{-1}(N) = g^{-1}(N) = M$, that is, the above set is not empty. Clearly, A is the smallest subsemimodule of N such that $f^{-1}(A) \subseteq g^{-1}(A)$. In particular, we see that $A \subseteq g\,(M) \subseteq f\,(M)$…** [since $f^{-1}(g\,(M)) \subseteq g^{-1}\big(g\,(M)\big)$].
Since ker $f \nsubseteq kerg$, $A \neq 0$ and so, by hypothesis, we can find a subsemimodule X such that $0 \leq X \leq A$ and A/X is simple.
But $g\,(f^{-1}(X)) \subseteq g\,(f^{-1}(A)) \subseteq$ A, and so $A = X + g\,(f^{-1}(X))$..*[since $\nsubseteq g^{-1}(X)$ and X is maximal in A].
By proposition 3.22

$$X = (X + g\,(f^{-1}(X))) \cap (X + f(g^{-1}(X))$$

and so

$$X = A \cap (X + f(g^{-1}(X))$$

leads to

$$f^{-1}(X) = f^{-1}(A) \cap [f^{-1}(X) + f^{-1}\big(f(g^{-1}(X)\big)]$$
$$f^{-1}(X) = f^{-1}(A) \cap [f^{-1}(X) + g^{-1}(X)]$$

[from * $g^{-1}(A) = g^{-1}(X) + f^{-1}(X)$]

$$f^{-1}(X) = f^{-1}(A) \cap g^{-1}(A) = f^{-1}(A)$$
$$X \cap f(M) = A \cap f(A)$$

, but $X \subseteq A \subseteq f\,(M)$ by **
so $X = A$ contradiction, hence $A = 0$, that is $\ker f = f^{-1}(A) \subseteq g^{-1}\,(A) = kerg$.

Proposition 3.24. Let $\mathcal{D}$, and N be modules over semiring R. Where $\mathcal{D}$ be distributive , N is a noetherain and A, B are submodules of $\mathcal{D}$. If $\mathcal{D}/_A \cong \mathcal{D}/_B$, then $A = B$.

Proof. Assume that $A, B \in L(\mathcal{D})$, such that $\mathcal{D}/_A \cong \mathcal{D}/_B$ by α. Let $\beta_C: \mathcal{D} \to \mathcal{D}/_C$ be the natural epimorphism for $C \in L(\mathcal{D})$. Then $\alpha\beta_A: \mathcal{D} \to \mathcal{D}/_B$ is an epimorphism and $\alpha\beta_A + \beta_B$ is k-regular (since $\mathcal{D}, N$ are modules). Thus $\alpha\beta_A(\mathcal{D}) = \beta_B(\mathcal{D})$, so by using Proposition 3.23 $\ker(\alpha\beta_A) = \ker(\beta_B)$, hence $A = B$.

IV. On Supplement Property in Distributive Semimodules

In this section, we aim to define the concepts originally introduced in the context of modules to semimodules, with a focus on generalizing their associated properties related to distributive laws. The definitions we will be exploring and expanding upon include supplemented extending [18], principally ⊕-supplemented [19], principally supplemented [20], and H-supplemented [21].

Definition 4.1. An R− semimodule $\mathcal{D}$ is called supplemented extending if every closed subsemimodule is supplemented.

Lemma 4.2. Suppose that $M = M_1 \oplus M_2$ where M, M_1, M_2 are R− semimodules and M distributive , if $I \in L(M)$ is closed in M, then $I \cap M_i$ closed in M_i $(i = 1{,}2)$.

Proof. Assume that $I \cap M_i \leq^e {M_i}^* \leq M_i$ where $= 1{,}2$. Then $I \cap M_1 \oplus I \cap M_2 \leq^e {M_1}^* \oplus {M_2}^* \leq M_1 \oplus M_2 = M$, thus $I \leq^e {M_1}^* \oplus {M_2}^*$ since I closed in M, then $I = {M_1}^* \oplus {M_2}^*$. Hence $I \cap M_1 \oplus I \cap M_2 = {M_1}^* \oplus {M_2}^*$ implies that $I \cap M_1 = {M_1}^*$ and $\cap M_2 = {M_2}^*$, so $I \cap M_i$ closed in $M_i (i = 1{,}2)$.

Lemma 4.3. [3] Let $\mathcal{D}$ be a distributive R− semimodule, and $\mathcal{T}, \mathcal{L} \in L(\mathcal{D})$. Then $\mathcal{T}$ is a supplemented of $\mathcal{L}$ if and only if $\mathcal{D} = \mathcal{L} + \mathcal{T}$ and $\mathcal{T} \cap \mathcal{L}$ is small in $\mathcal{T}$.

Proposition 4.4. Let $M = M_1 \oplus M_2$ where M, M_1, M_2 are R− semimodules and M distributive , then M is supplemented extending, if M_1, M_2 are supplemented extending.

Proof. Suppose that M_1, M_2 are supplemented extending, and L is closed subsemimodule in M. By Lemma 4.2 $L \cap M_i$ is closed subsemimodule in $M_i (i = 1{,}2)$ but M_i is supplemented extending , then $L \cap M_i$ is supplemented in M_i, then by Lemma 4.3 there exist a subsemimodule K_i of M_i such that $K_i + (L \cap M_i) = M_i$ and $K_i \cap L \cap M_i = K_i \cap L \ll L \cap M_i \ll L$. Now, $(K_1 + K_2) \cap L = (K_1 \cap L) + (K_2 \cap L) \ll L$ and $M = [K_1 + (L \cap M_1)] + [K_2 + (L \cap M_2)] \subseteq (K_1 + K_2) + L$, by Lemma 4.3 L is supplemented.

Following Mohammad in [22] we use a concept T-direct summand in the module. We use the definition in semimodule. Let $A, B,$ and C be subsemimodules of R-semimodule M. A is called the T-direct sum of $B,$ and C (denoted by $A = B \oplus_T C$) if $A = B + C$ and $B \cap C \in L(T)$. In this case, each of $B,$ and C is called a T-direct summand of A.

Proposition 4.5. Let A, B and C be subsemimodules of cancellative, semisubtractive, subtractive, and distributive semimodule M, such that $M = B \oplus_A C$. Then $M/_A = (B + A)/_A \oplus (C + A)/_A$.

Proof. It's clear $M/_A = (B \oplus_A C)/_A = (B + A)/_A + (C + A)/_A$

and $(B + A)/_A \cap (C + A)/_A = [(B + A) \cap (C + A)]/_A$
$= [B \cap (C + A) + A \cap (C + A)]/_A =$
$[B \cap C) + (B \cap A) + (A \cap C) + A)]/_A = A/_A = 0$ hence by Lemma 3.12 $M/_A = (B + A)/_A \oplus (C + A)/_A$.

Proposition 4.6. Let $M = M_1 \oplus M_2 = K + N$ be a semimodule and $K \leq M_1$. If M is distributive and $K \cap N$ is small in N, then $K \cap N$ is small in $M_1 \cap$ N.

Proof. Let $M_1 \cap N = (K \cap N) + L$, Where L is a subsemimodule of $M_1 \cap N$. Since M is distributive, $N = (M_1 \cap N) \oplus (M_2 \cap N)$. We have $M = K + N = K + (M_1 \cap N) + (M_2 \cap N) = K + L + (M_2 \cap N)$ and $N = (K \cap N) + L + (M_2 \cap N)$. Since $K \cap N$ is small in N, we have $N = L \oplus (M_2 \cap N)$. Then $N = (N \cap M_1) \oplus (N \cap M_2)$ and $L \leq M_1 \cap N$ imply $L = M_1 \cap N$. Hence $K \cap N$ is small in $M_1 \cap N$.

Definition 4.7. Let M be an R-semimodule, $m \in M$, and L a direct summand of M, the subsemimodule L is called principally ⊕-supplemented of Rm in M if Rm and L satisfy $M = Rm + L$ and $Rm \cap L$ is small in L, and the subsemimodule M is called principally ⊕-supplemented if every cyclic subsemimodule of M has a principally $\oplus$-supplemented in M.

Proposition 4.8. Let M be a semisubtractive, cancellative, and distributive principally $\oplus$-supplemented semimodule. Then every homomorphic image of M is principally $\oplus$-supplemented.

Proof. Let L be a subsemimodule of M and $(Rm + L)/L$ a cyclic subsemimodule of M/L. Then there exists a direct summand A of M such that $M = A \oplus B = Rm + A$ and $Rm \cap A$ is small in A. We prove $(A + L)/L$ is a principally $\oplus$-supplemented of $(Rm + L)/L$. Now $M/L = (Rm + L)/L + (A + L)/L$ and, since M is distributive, $(Rm + L) \cap (A + L) = L + (Rm \cap A)$. So $(Rm + L)/L \cap (A + L)/L$ is small in $(A + L)/L$. Again by distributivity and $A \cap B = 0$, we have $(A + L) \cap (B + L) = L$. Hence $(A + L)/L$ is a direct summand of M/L.

Recall that a semimodule M is said to be regular if every cyclic subsemimodule is a direct summand of M (refer Definition3.3 in [23]). In the context of modules, this concept is known as principally semisimple [24].

Lemma 4.9. Let M be a semisubtractive, cancellative semimodule with $Rad(M) = 0$. Then M is principally $\oplus$-supplemented if and only if M is regular.

Proof. Let M be a principally $\oplus$-supplemented module and $m \in M$. Then there exists a direct summand A of M such that $M = A + Rm$ and $A \cap Rm$ is small in A. Since $A \cap Rm$ is also small in M and $Rad(M) = 0$, then $A \cap Rm = 0$ so, Rm is a direct summand of M. Therefore M is regular. The rest is clear.

Proposition 4.10. Let M be a semisubtractive, cancellative, and distributive principally $\oplus$-supplemented semimodule. Then $M/Rad(M)$ is regular.

Proof. By Proposition 4.8, $M/Rad(M)$ is principally $\oplus$-supplemented. Since $Rad(M/Rad(M)) = 0, M/Rad(M)$is regular from Lemma 4.8.

Definition 4.11. Let N be a cyclic subsemimodule of M. A subsemimodule L is called a principally supplemented of N in M if N and L satisfy the condition $M = N + L$ and $N \cap L$ is small in L. The semimodule M is called principally supplemented if every cyclic subsemimodule of M has supplemented in M.

Proposition 4.12. Let M be a principally supplemented distributive semimodule. Then every direct summand of M is a principally supplemented semimodule.

Proof. Let $M = M_1 \oplus M_2$ and $m \in M_1$. There exists a subsemimodule A of M such that $M = mR + A$ and $(mR) \cap A$ is small in A. Then $M_1 = (mR) + (M_1 \cap A)$. By Proposition 4.6, $(mR) \cap A$ is small in $M_1 \cap A$.

Proposition 4.13. Let M be a principally supplemented distributive semimodule. Then $M/Rad(M)$ is a principally semisimple semimodule.

Proof. Let $m \in M$. There exists a subsemimodule M_1 such that $M = mR + M_1$ and $(mR) \cap M_1$ is small in M_1. Then $M/Rad(M) = [(mR + Rad(M))/Rad(M)] + [(M1 + Rad(M))/Rad(M)]$. Now we prove that $(mR + Rad(M)) \cap (M_1 + Rad(M)) = Rad(M)$. The distributivity of M implies $(mR + Rad(M)) \cap (M_1 + Rad(M)) = (mR) \cap M_1 + Rad(M)$. Since $(mR) \cap M_1$ is small in M_1, therefore small in M, $(mR) \cap M_1 \leq Rad(M)$. Hence $M/Rad(M) = [(mR + Rad(M))/Rad(M)] \oplus [(M_1 + Rad(M))/Rad(M)]$ and so every principal submodule of $M/Rad(M)$ is a direct summand.

Definition 4.14. Let M be an R-semimodule. M is called H-supplemented if, given any subsemimodule A of M, there exists a direct summand D of M such that $M = A + X$ holds if and only if $M = D + X$.

Lemma 4.15. Let M be an H-supplemented semimodule and X a subsemimodule of M. If for every direct summand K of M, $(X + K)/X$ is a direct summand of M/X, then M/X is H-supplemented.

Proof. Let $N/X \leq M/X$. Since M is H-supplemented, there exists a direct summand D of M such that $M = N + Y$ if and only if $M = D + Y$. By hypothesis, $(D + X)/X$ is a direct summand of M/X. Then $M/X = N/X + L/X$ if and only if $M/X = (D + X)/X + L/X$ for every $L/X \leq M/X$.

Proposition 4.16. Let M be a semisubtractive, cancellative, and H-supplemented distributive semimodule**.** then M/X is H-supplemented for every subtractive subsemimodule X of M.

Proof. Let D be a direct summand of M. Then $M = D \oplus D'$ for some subsemimodule D' of M. Now $M/X = [(D + X)/X] + [(D' + X)/X]$ and $X = X + (D \cap D') = (X + D) \cap (X + D')$. So $M/X = [(D + X)/X] \oplus [(D' + X)/X]$. By Lemma 4.15, M/X is H-supplemented.

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