

Category Theory and New Classes of Semi Bornological Group

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Abstract— Because, the new knowledge put every new structure in category theory. So, became a source of interest for many researchers, to put every new structure in category theory. This motivates us to put the new structures of semibornological group in category. Furthermore, new classes of semibornological group was constructed, which it is a semibornological group with respect to S -bounded maps and semibornological groups with respect to S^{**} -bounded maps. The main important results, every semi bornological group is S -semi bornological group and S^{**} -semi bornological group but the converse is not true. Also, we gave the certain condition for any codomain of S -semi bornological group to be S^{**} -semi bornological group. Additionally, every left (right) translation is S -bornological isomorphism and S^{**} -bornological isomorphism, every bornological isomorphism is S -bornological isomorphism and S^{**} -bornological isomorphism.

Keywords— Category theory, Bornological sets, Bornological groups, Semi bounded set, Semi bornological groups.

I. INTRODUCTION

The importance of bornology is to solve the problem of bounded of sets, functions and spaces by construct the structure of bornology on the set or the space. Researchers started to study abstract algebraic bornological structures, which it is bornological groups, bornological rings, bornological modules to solve the problem of bounded of the groups, rings, and modules. A bornological group is a group with bornology such that the group operation are bounded. See [2], [3], [4] and [5].

However, there are such kind of group cannot be bornological group because product map is not bounded, that means cannot solve the problem of bounded for this kind of group. In 2019 [1] this problem was solved by introducing a related structure to bornological group which is semibornological groups. A semibornological group is a group endowed with a bornology

such that, for each fixed $g \in G$, the translations $l_g, r_g : G \rightarrow G$, are bounded, so $l_g(x) = g \cdot x, r_g(x) = x \cdot g$ are bounded.

In this work, the category theory of semibornological groups is discussed. In addition, new classes of semi bornological group were constructed. The main goal is to restrict the condition of the group operations not condition to be bounded. We are going to assume that the group operation to be S -bounded or S^{**} -bounded maps. In the same way we define new classes of semi bornological groups called here S -semi bornological groups, S^{**} -semi bornological groups. Through the paper $(G, *, \beta)$ or simple G stands for a bornological group, X and Y stand for bornological sets.

II. THE CATEGORY THEORY

First, the effect of the category theory is to formalize mathematical structures and its concept in term of objects and a morphism. See [6-20].

Definition (2.1): [7]

A *category theory* consist of three pieces:

- 1) Collection of objects $\text{obj}(\square)$;
- 2) For any objects A_1, A_2 in the collection of objects $\text{obj}(\square)$, we have a set of morphisms $(\text{Hom}(A_1, A_2))$;
- 3) There is a composition $\circ : \text{Hom}(A_1, A_2) \times \text{Hom}(A_2, A_3) \rightarrow \text{Hom}(A_1, A_3)$ for A_1, A_2, A_3 in $\text{obj}(\square)$.



A functor is a map between categories.

Definition (2.2): [8]

A *functor F* from the category \square to the \mathcal{D} is a map that:

- i) Associates to each object X in \square an object in \mathcal{D} ;
- ii) Associates to each morphism in \square a morphism in \mathcal{D} .

Definition (2.3): [9]

Let \square be a category. Let X, Y, Z be three objects of \square . Assume that two morphisms $f: X \rightarrow Z$ and $g: Y \rightarrow Z$ are given. Then the **fiber product (pull back)** $X \times_Z Y$ is defined as an object $W = \{(x, y) \in X \times Y : f(x) = g(y)\}$ together with morphisms $p: W \rightarrow X$ and $q: W \rightarrow Y$ such that $f \circ p = g \circ q$ which is universal (for any $W_1 \in \text{ob}(\square)$ together with morphisms $q_1: W_1 \rightarrow Y$ and $p_1: W_1 \rightarrow X$ exist a unique morphism $h: W_1 \rightarrow W$ St. $p \circ h = p_1$ and $q \circ h = q_1$ hold).

The dual of fiber product is **push out (Equalizer)**. Let X and Y are objects, while f and g are morphism from X to Y , the equalizer consist of an object E and morphism $eq: E \rightarrow X$ satisfy $f \circ eq = g \circ eq$ and such that given any object O and morphism $m: O \rightarrow X$ if $f \circ m = g \circ m$. Then, there is $u: O \rightarrow E$ S.T. $eq \circ u = m$ morphism $m: O \rightarrow X$. Is said to equalize f and g if $f \circ m = g \circ m$.



This means, equalizer is the limit of a diagram consisting $f: X \rightarrow Y$ $g: X \rightarrow Y$ with common domain and codomain such satisfy universal property.

A *coequalizer* can be clarified by the next diagram.



III. CATEGORY THEORY OF SEMI BORNOLICAL GROUP

The category theory of semibornological group, the objects of this category are semibornological groups and the morphisms are bounded groups homomorphism's. We denote the category by (*SBorn*).

Product (direct product)

The *product* in semibornological group is the set theoretical product with the product bornology.

Definition (3.1):

Let (G_i, β_i) be a family of semibornological group, then the direct product $\prod_{i \in I} (G_i, \beta_i)$ is defined to be the group product, which is the Cartesian product of sets and coordinate wise operation, with the bornology generated by the family $\{B : B \prod_{i \in I} B_i, B_i \in \beta_i\}$.

Example (3.2) :

The product of semibornological groups (G_i, β_i) and (G_j, β_j) is the semi bornological group $(G_i, \beta_i) \times (G_j, \beta_j)$ with two bounded homomorphisms T_1, T_2 , when T_1, T_2 the canonical coordinate projection maps. Such that: $T_1: (G_i, \beta_i) \times (G_j, \beta_j) \rightarrow (G_i, \beta_i), T_2: (G_i, \beta_i) \times (G_j, \beta_j) \rightarrow (G_j, \beta_j)$.

Let (G_k, β_k) be any semibornological group and let $f: (G_k, \beta_k) \rightarrow (G_i, \beta_i), g: (G_k, \beta_k) \rightarrow (G_j, \beta_j)$ be any function. We construct the function $h: (G_k, \beta_k) \rightarrow (G_i, \beta_i) \times (G_j, \beta_j)$ such that: $h(G_k, \beta_k) = (f(G_k, \beta_k), g(G_k, \beta_k))$, for every (G_k, β_k) in category of semibornological group.

$$(T_1 \circ h)(G_k, \beta_k) = T_1(h(G_k, \beta_k)) = T_1(f(G_k, \beta_k))$$

$$(T_2 \circ h)(G_k, \beta_k) = T_2(h(G_k, \beta_k)) = T_2(g(G_k, \beta_k)).$$

Remark (3.3) :

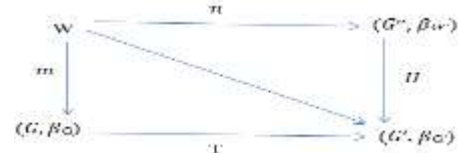
The concept of *coproduct* in semibornological group does not exist for a family of infinite semibornological groups. But for a family of finite semibornological commutative groups it is same as direct product.

Definition (3.4):

Let (G_i, β_i) be a family of semi bornological group, then the direct sum of the group $\bigoplus G_i$ equipped with the bornology generated by the family $\bigoplus_{i \in I} \beta_i = \{\bigoplus B_i, B_i \in \beta_i\}$.

Definition (3.5):

Let $T: (G, \beta_G) \rightarrow (G', \beta_{G'})$ and $U: (G'', \beta_{G''}) \rightarrow (G', \beta_{G'})$ be two bounded homomorphism's of semibornological group, the fiber product $(G, \beta_G) \times_{(G', \beta_{G'})} (G'', \beta_{G''})$ is defined as the **fiber product of groups** $G \times_{G'} G'' = \{(g, g'') \in G \times G'' : T(g) = U(g'')\}$. Equipped with the bornology generated by the family $\{B_1 \times G' B_2\}$ such that B_1 is bounded in (G, β_G) and B_2 is bounded in $(G'', \beta_{G''})$. Then the fiber product of bounded homomorphism's T and U consists of an object $W = \{(G, \beta_G) \times (G', \beta_{G'}) (G'', \beta_{G''})\}$ and two homomorphism's $n: W \rightarrow (G, \beta_G) m: W \rightarrow (G'', \beta_{G''})$ such that: $T \circ m = U \circ n$.



A **functor in semibornological group** is a map from category of semibornological group to semibornological group. A **forgetful functor in semi bornological group** is a functor from category of semi bornological group to category of group, forgets the semi bornological group structure remembering only the underlying group.

IV. S-SEMI BORNLOGICAL GROUPS AND S**-SEMI BORNLOGICAL GROUPS

In this section, two new classes of semi bornological group (S-semi bornological groups and S**-semi bornological groups) are studied. First of all, the idea of lessening the condition of bounded set in to semi bounded set by introducing semi bounded set in bornological set are studied in [1]. A subset S of a bornological set X is said to be a semi bounded set if there is a bounded subset B of X such that, $B \subseteq S \subseteq \overline{B}$, where $\overline{B} = \{ \text{all upper and lower bounds of } B \} \cup B$. Note that, every bounded set is semi bounded set, but the converse is not true. $SB(X)$ is that the collection of all semi bounded subsets of (X, β) . A map f from a bornological set (X, β) into a bornological set (Y, β) is said to be:

- **bounded map**, the image of every bounded set of (X, β) is bounded set in (Y, β) ;
 - **S*-bounded map**, the image of any semi bounded set in (X, β) is bounded set in (Y, β) .
- The identity map is clearly a semi bounded map. Evidently, a bounded map is S*-bounded map.

Definition (4.1):

A semi bornological group with respect to S-bounded translation is a group G endowed with a bornology such that, for each fixed $g \in G$, the translations $l_g, r_g : G \rightarrow G$, $l_g(a) = g \cdot a$, $r_g(a) = a \cdot g$ are S-bounded maps.

Theorem (4.2):

Let G be a semi bornological group with respect to S-bounded map and B is a nonempty bounded set of G , then for any fixed element $g \in G$ the set $g \cdot B$, $B \cdot g$ are semi bounded.

Proof: Since l_g, r_g are also semi bounded and bijection the result is obvious. \square

We prove that any product between the bounded set and semi bounded set in SBG is also semi bounded.

Theorem (4.3):

Assume that G is a S-semi bornological group with respect to S-bounded map. And B is a bounded set that is not empty of G , if A is any set of G then the set $A \cdot B$, $B \cdot A$ are semi bounded sets.

Proof: Let $B \cdot A$ as the union of semi bounded sets $B \cdot g$ for all $g \in A$ which are semi bounded set by Theorem 4.2. Since the union of finite semi bounded sets is semi bounded set [3]. Therefore, $B \cdot A$ is semi bounded set. Since the union of semi bounded sets is again semi bounded set. Similarly, $A \cdot B$ is semi bounded set by the same way. Since $A \cdot B$ can be written as the union of the sets $g \cdot B$ for all $g \in A$. \square

Definition (4.4):

A mapping $f: X \rightarrow Y$ is called S-semi bornological isomorphism if it is bijective, and f, f^{-1} are S-bounded maps.

However, it is time to give the notion of a semibornological group with respect to S**-bounded map.

Definition (4.5): A semibornological group with respect to S**-bounded map is $(G, *, \beta)$ such that $(G, *)$ is a group endowed with a bornology β also the left (right) translation are S**-bounded maps.

Proposition (4.6):

Every semibornological group is S-semibornological group and S**-semibornological group

Proof: Since a bounded map is also a S-bounded map and S**-bounded map. Therefore, every semibornological group up to concept of bounded map is also semi bornological group with respect to S-bounded map. But the inverse of this fact is not true in general.

Example (4.7):

Let $(\mathbb{R}, +)$ be the group of real number with the usual operation of addition and let β be a usual bornology induced by the norm (absolute value) and the sets $[c, \infty)$ where c in \mathbb{R} . In this case all translations are S**-bounded maps but not bounded.

An equivalent concept are given know.

Definition (4.8):

Let the map $f : G \times G \rightarrow G$ that maps (g_1, g_2) to $g_1 * g_2$. This map is S**-bounded map in g_1 when, for each A_1 semi bounded set contains g_1 there exist A semi bounded set contains $g_1 * g_2$ such that $A_1 g_2 \subset A$.

Example (4.9):

Let $(\mathbb{R}, +)$ be the group of real number with the usual operation of addition and let β be a usual bornology induced by the norm (absolute value) and the sets $[c, \infty)$ where c in \mathbb{R} . In this case all translations are S**-bounded maps but not bounded.

Theorem (4.10):

Let $(G, *, \beta)$ be a semi bornological group with respect to S**-bounded map then each left (right) translation $l_g: G \rightarrow G$, $r_g: G \rightarrow G$ is a S**-bornological isomorphism.

Proof The statement will be proved for r_g , the other translation follows in similar manner. First, every left (right) translation is clearly a bijection. Now to prove that r_g and it is inverse are S**-bounded maps. Since G is a semi bornological group with respect to S**-bounded map. That has a S**-bounded group operation in each variable separately. We know that for every semi bounded set A_1, A_2 contains g_1, g_2 respectively, there exist a semi bounded set A contains g_1, g_2 with $A_1 \cdot A_2 \subset A$. The map r_g fixed an element $g \in G$. So we get for every A_1 contains g_1 there exist A contains $g_1 \cdot g$ with $A_1 g \subset A$. Which make r_g is S**-bounded map. We have that $r_g^{-1}(g_1)$ which maps $g_1 g$ to g_1 , this is equivalent to the

map from g_1 to $g_1 g^{-1}$. So $r_g^{-1}(g_1) = r_{g^{-1}}(g_1)$. We have that $r_g^{-1}(g_1)$ is S^{**} -bounded map by the same argument as above. So r_g is a S^{**} -bornological isomorphism. Similarly with l_g is a S^{**} -bornological isomorphism.

Corollary (4.11):

Let G be a semi bornological group with respect to S^{**} -bounded map and $g_1 g_2 \in G$. There exist a S^{**} -bornological isomorphism f such that $f(g_1) = g_2$.

Proof: From Theorem 4.9 we now that r_g is a S^{**} -bornological isomorphism from all $g \in G$. By letting $f = r_{g^{-1}} g$ we get $f(g_1) = f(g_2)$.

Since $f(g_1) = g_1 \cdot g_1^{-1} g_2 = g_2 = f(g_2)$. As required.

Corollary (4.12):

Let G be a semi bornological group with respect to S^{**} -bounded map and A is a semi bounded set of G and if g is any point of G , then the set $g \cdot A, A \cdot g$ are semi bounded sets.

Proof: Since l_g, r_g are also S^{**} -bounded maps and bijection maps the result is obvious. \square

Corollary (4.13):

Consider G be a semi bornological group with respect to S^{**} -bounded map and A semi bounded set in G and A_1 any subset of G , then $A A_1, A_1 A$ are semi bounded sets.

Proof: Note that $A A_1$ can be written as the union of semi bounded sets $A \cdot g$ for all $g \in A_1$ which are semi bounded set by Theorem 4.2. Since the union of finite semi bounded sets is semi bounded set [3]. Therefore, $A A_1$ is semi bounded set. Since the union of semi bounded sets is again semi bounded set. Similarly, $A_1 A$ is semi bounded set by the same way. Since $A A_1$ can be written as the union of the sets $g \cdot A$ for all $g \in A_1$. \square

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