Representation of n- BiHom Γ-Lie algebra

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Abstract—The purpose of this paper, is to define and discuss a representation on an n-Bi-Hom Γ-Lie algebra and give some results about them.

Keywords—Algebra, Lie algebra, Representation.

1. INTRODUCTION

Kitoune – Makhlof – Silvestrov in [1] define n-BiHom Lie algebra. The representation notion of algebra is very essential since it defect some of its difficult structure invisible below. In [2,5] they study the theory of a representation of 3-BiHom Lie algebra inserted and construction of n-Lie algebras. Rezaei and Davvaz in [3] define Γ- algebra. A Γ-algebra is an algebraic structure composed of a vector-space V, a groupoid Γ jointly with function from V×Γ×V to V. Then, on every associative Γ-algebra V and for all x ∈ Γ they establish an x-Lie algebra. The purpose of this paper, is to introduced a new concept which is representation of an n-BiHom Γ-Lie algebra and equivalent two representation of an n-BiHom Γ-Lie algebra. It is mentioned the reader that, and field K of characteristic 0 and every the vector-spaces are on K. Every place here after, the notation s attened that s is except, for example, record f(s₁,...,s_i,...,s_n) for f(s₁,...,s_i,...,s_i,...,s_i,...,s_i,...,s_n).

Now, we will recall the followings concepts which are necessary in this paper.

Definition 1.1 [1]: An n- Bi-Hom Lie-algebra be a vector-space g, supplied with n-linear process [ , , , ] with 2-linear function  satisfying the following conditions: 1- for all  ,  ,  ∈ g .  ( [ , , , ] ) = [ , , , ] , , , .

2- For all  , , , ∈ g .  ( [ , , , ] ) =  [ , , , ]

3- BiHom - skew symmetry: for all  , , , ∈ g ,

[ , , , ] = − [ , , , ]

when  = 1, 2, ...,  − 2.

4- n- BiHom Jacobi-identity: for all  , , , , ∈ g ,

[ , , , , ] = 

\[\sum_{k=1}^{n}(-1)^{n-k}\left[ [ , , , , ] + [ , , , , ] + [ , , , , ] \right] \]

Note 1.2 [8]-[9]

1- when  = , we obtain n-Lie algebra .

2- when  = 2, the Bi-Hom Jacobi-identity be:

\[ [ , , ] + [ , , ] + [ , , ] = 0 \]

(1.1) and the identity (1.1) is equivalent to

\[ \sum_{k=1}^{n}(-1)^{n-k}\left[ [ , , ] + [ , , ] + [ , , ] \right] = 0 \]

3- For  = 3, the 3 Bi-Hom Jacobi-identity be:

\[ [ , , , ] + [ , , , ] + [ , , , ] = 0 \]

\[ \sum_{k=1}^{n}(-1)^{n-k}\left[ [ , , , ] + [ , , , ] + [ , , , ] \right] = 0 \]

Definition 1.3 [4] Suppose Γ be a groupoid and V is a vector-space on a field F. Then , V be invited a Γ- algebra on the field F if there exist function V×Γ×V→V ( the image be sαt for s,t ∈ V and α ∈ Γ ) such the following satisfy:

(1) (s+t)az = sad + cza, sa(z + t) = sat + cza,

(2) s(α + β)t = sat + sβt,

(3) (cα)t = c(sαt) = scαt,

(4) 0αt = tα0 = 0, for all s, y, z ∈ V, c ∈ F and s ∈ Γ.

Moreover, a Γ- algebra is called associative if

(5) (sα)bαz = sa(tβz).
Example 1.4 :- [4]
Suppose that $V$ is a vector-space and $\Gamma$ a groupoid. For all $s,t \in V$ and $\sigma \in \Gamma$ we define $s \cdot t = 0$. There after, $V$ is a $\Gamma$-algebra.

Definition 1.5 :- [3]
Suppose $V$ is an associative $\Gamma$-algebra on a field $F$. There after, for all $\lambda \in \Gamma$ one can construct an $\Lambda$-Lie-algebra $L_{\lambda}(V)$. As a vector-space, $L_{\lambda}(V)$ is the like with $V$ . The Lie $\Gamma$ bracket of 2-elements of $L_{\lambda}(V)$ be define to be their commutator in $V$, $[r,v]_{\lambda} = r \cdot \lambda v - \lambda r \cdot v$. Note that $[r,v]_{\lambda} = -[v,r]_{\lambda}$.

Theorem 1.6 :- [9]-[7]
Let $(g, […], …, g)_{g}$ be an n-Lie-algebra, and assume $D_{\Gamma} \sigma : g \rightarrow g$ is algebra morphism such $D_{\Gamma} \sigma = \sigma \circ D$. The algebra $(g, […], …, g)_{D_{\Gamma} \sigma}$, when $[, …, ]_{D_{\Gamma} \sigma}$ be define $s_{1}, …, s_{n-1} \rightarrow [D(s_{1}), …, D(s_{n-1}), s_{n}]_{g}$, is an Bi-Hom-Lie-algebra.

Definition 1.7 :- [9]
A homomorphism $f : (g, […], …, g, D, \sigma) \rightarrow (\hat{g}, […], …, g, \hat{D}, \hat{\sigma})$ form an n-Bi-Hom-Lie algebra is a linear-function $g \rightarrow g$ such $\hat{D} = D \circ f = f \circ D = \sigma \circ f$ and for any $s_{k} \in g$, $f(s_{1}, …, f(s_{n})$$_{g}$)

Definition 1.8 :- [9]-[6]
Let $(g, […], …, g, D, \sigma)$ be an n- Bi-Hom-Lie algebra. We define, for every $s = s_{1} \land … \land s_{n-1}$ , $t = t_{1} \land … \land t_{n-1}$ in $\Lambda^{n-1}g$ and $z \in g$ the nest : 1- the action fundamental subject over $g$.
2- The linear functions $D_{\Gamma}F, F : \Lambda^{n}(n-1)g \rightarrow \Lambda^{n}(n-1)g$.
3- $D_{\Gamma}F(s) = D(s_{1}) \land … \land D(s_{n-1})$ and $\sigma(s) = \sigma(s_{1}) \land … \land \sigma(s_{n-1})$.

II. MAIN RESULTS
In this section, we give a definition of n- Bi-Hom $\Gamma - Lie$-algebra, a representation on n- Bi-Hom $\Gamma - Lie$-algebra, two equivalent representation of an n-Bi-Hom $\Gamma$-Lie algebra.

Definition 2.1:-
A representation an n- Bi-Hom $\Gamma$-Lie algebra $(g, […], …, g, D, \sigma)$ over a vector space $V$ regarding endomorphisms $D_{\Gamma}F, F \in End(V)$ is a linear-function $\alpha : \Lambda^{n-1}g \rightarrow End(V)$ such for every $s = s_{1} \land … \land s_{n-1}$, $t = t_{1} \land … \land t_{n-1} \in \Lambda^{n-1}g$, $D(s) = D(s_{1}) \land … \land D(s_{n-1})$, $\sigma(s) = \sigma(s_{1}) \land … \land \sigma(s_{n-1})$, we have
1- $\rho(D(s)) \circ D_{\Gamma}F = D_{\Gamma}F \circ \rho(s)$;
2- $\rho(\sigma(s)) = \sigma_{\Gamma} \rho(s)$;
3- $\rho(D_{\Gamma}F(s)) = \rho(t) - \rho(\sigma(t)) \circ \rho(D(s))$;

$$= \sum_{i=1}^{n-1} \rho(\sigma(t_{i}), \sigma(\sigma_{i-1}), [\sigma(s_{i}), …, \sigma(s_{n-1}), t_{i}]_{D_{\Gamma}F})$$

$$\circ \sigma_{i+1}, \sigma(t_{n-1})$$

4- $\rho([\sigma(t_{1}), …, \sigma(s_{n-1}), t_{1}]_{D_{\Gamma}F}) = \sigma_{\Gamma}v$,

$$= \sum_{i=1}^{n} (-1)^{i-1} \rho(D_{\Gamma}F(s_{i}), …, D_{\Gamma}F(s_{i}), \sigma(\sigma_{i}), …, \sigma(\sigma_{n-1}), \sigma(t_{1}))$$

$$\circ \rho(t_{1}, …, t_{n-1}, D_{\Gamma}F(s_{1})) + \rho(D_{\Gamma}F(s_{1})) \circ \rho(t)$$

We symbolize a representation by $(v, \rho, D_{\Gamma}F, \sigma)$. 

Proposition 2.2:-
Let $(g, […] , …, g, D, \sigma)$ is an n-Bi-Hom $\Gamma - Lie$-algebra. Define ad : $\Lambda^{n-1}g \rightarrow End(g)$ by $ad_{\Gamma}F(s_{1}, …, s_{n-1}, t)_{D_{\Gamma}F}$, for every $s = s_{1} \land … \land s_{n-1} \in \Lambda^{n-1}g$, $t \in g$. Then $(g, ad_{\Gamma}F, \sigma)$ is a representation of the n-Bi-Hom $\Gamma$-Lie algebra $(g, […] , …, g, D, \sigma)$ on $g$, called adjoin representation.

Proof:-
Let n- Bi-Hom $\Gamma - Lie$-algebra $(g, […] , …, g, D, \sigma)$ over a vector space $V$ regarding endomorphisms $D_{\Gamma}F \in End(V)$ is a linear function $ad_{\Gamma}F : \Lambda^{n-1}g \rightarrow End(V)$ for all $s = s_{1} \land … \land s_{n-1} \in \Lambda^{n-1}g$, $t \in g$. Then $(g, ad_{\Gamma}F, \sigma)$ is a representation of the n-Bi-Hom $\Gamma$-Lie algebra $(g, […] , …, g, D, \sigma)$ on $g$, called adjoin representation.

Proposition 2.3:-
Let $(v, \rho, D_{\Gamma}F, \sigma_{\Gamma})$ be a representation of n-Bi-Hom $\Gamma$-Lie algebra $(g, […] , …, g, D, \sigma)$, suppose the maps $D_{\Gamma}$ and $\sigma_{\Gamma}$ be bijective. There after $(g\circ D_{\Gamma}F, […] , …, g\circ D_{\Gamma}F, \sigma_{\Gamma})$ is an n-Bi-Hom $\Gamma$-Lie algebra. When $D_{\Gamma}F \circ D_{\Gamma}F , D_{\Gamma}F \circ D_{\Gamma}F$ be defined $(D\circ D_{\Gamma}F)(s + t) = D(s) + D_{\Gamma}F(t)$ and $(\sigma \circ D_{\Gamma}F)(s + t) = \sigma(s) + \sigma_{\Gamma}(t)$ and the bracket operation $[,] : \Lambda^{n}(g\circ D_{\Gamma}F) \rightarrow g\circ D_{\Gamma}F$ be define by $[s_{1} + t_{1}, …, s_{n} + t_{n}]_{\Gamma} = [s_{1}, …, s_{n}]_{\Gamma} + \rho([s_{1}, …, s_{n}])_{\Gamma} + \rho([s_{1}, …, s_{n}])_{\Gamma} \circ \rho(t)$, for all $s_{1} \in g, t_{1} \in V$. We symbolize this semi direct n- Bi-Hom $\Gamma$-Lie algebra simply by $g \circ \alpha v$.

Proof:-
It is clear that
\((\mathcal{D} + \mathcal{D}_v) \circ (\mathcal{D} \circ \mathcal{D}_v) = (\mathcal{D} + \mathcal{D}_v) \circ (\mathcal{D} + \mathcal{D}_v)\)
from the fact \(\mathcal{D} \circ \mathcal{D} = \mathcal{D} \circ \mathcal{D}_v = \mathcal{D} \circ \mathcal{D}_v\).
Now, we clear that \(\mathcal{D} + \mathcal{D}_v\) be an algebra morphism. On the other hand, we have
\[(\mathcal{D} + \mathcal{D}_v)(s_1 + f_1, \ldots, s_n + f_n)_{\lambda^g} =
(\mathcal{D} + \mathcal{D}_v) \left(\sum_{i=1}^n (-1)^{n-i} \rho(s_1, \ldots, s_{i-1}, \mathcal{D}^\ast \sigma(s_n)(\mathcal{D}_v \mathcal{D}_v^{-1}(f_i))\right)
\]
so that \(\mathcal{D} + \mathcal{D}_v\) is an algebra morphism regarding the bracket \([\ldots, \ldots]_{\lambda^g}\). The same way, we get \(\mathcal{D} + \mathcal{D}_v\) an algebra morphism regarding the bracket \([\ldots, \ldots]_{\lambda^g}\).

follow the fact that the bracket \([\ldots, \ldots]_{\lambda^g}\) satisfying Bi-Hom \(\Gamma\)-Lie algebra skew-symmetry, for every \(s_1 \in g, f_1 \in V\)
\[\mathcal{D}_v \mathcal{D}_v^{-1}((\mathcal{D} + \mathcal{D}_v)(s_1 + f_1, \ldots, s_n + f_n)_{\lambda^g}) + (\mathcal{D} + \mathcal{D}_v)(s_1 + f_1, \ldots, s_n + f_n)_{\lambda^g}\]
\[= ([\mathcal{D} + \mathcal{D}_v](s_1 + f_1, \ldots, s_n + f_n)_{\lambda^g}) - [\mathcal{D}_v \mathcal{D}_v^{-1}((\mathcal{D} + \mathcal{D}_v)(s_1 + f_1, \ldots, s_n + f_n)_{\lambda^g})]d_{\lambda^g}\]
\[= \sum_{k=1}^{n} (-1)^{n-k} \left[ (\sigma + \omega_r)^2 (t_1 + v_1), ..., (\sigma + \omega_r)^2 (t_k - 1) + v_{k-1}, (\sigma + \omega_r)^2 (t_{k+1} + v_{k+1}), ..., (\sigma + \omega_r)^2 (t_n + v_n), (\sigma + \omega_r)(s_{n+1} - \epsilon_{n-1}), (\sigma + \omega_r)(s_1 - \epsilon_1), (\sigma + \omega_r)(s_2 - \epsilon_2) \right] \]

Then \( g \propto \nu := (g \circ \nu, ..., \nu, (\sigma + \omega_r, \nu, \omega_r)) \) is an n-Bi-Hom \( \Gamma \)-Lie algebra.

**Definition 2.4:-**

Assume \((\nu_1, \rho_1, D_1, \sigma_1)\) and \((\nu_2, \rho_2, D_2, \sigma_2)\) is a 2-representation from n-Bi-Hom \( \Gamma \)-Lie-algebra \((g, \ldots, \nu, \rho)\). They are said to be equivalent if there exist an isomorphism of vector spaces \(T: v_i \rightarrow v_i\) such that:

\[T \rho_1(t_i)(\nu) = \rho_2(T(t_i))(T \sigma_1 = \sigma_2 \circ T) \]

for every \( s \in \Lambda \) and \( s_{n-1}, t \in v_i\).

In terms of diagrams, we get:

\[\Lambda^{n-1} g \times v_i \rightarrow \Lambda^{n-1} g \times v_i \]

\[\nu_1 \circ \rho_1 \circ T \times t \rightarrow \nu_2 \circ \rho_2 \circ T \times t \]

\[D_1 \circ \sigma_1 \circ T \times t \rightarrow D_2 \circ \sigma_2 \circ T \times t \]

\[\nu_1 \circ \sigma_1 \circ T \times t \rightarrow \nu_2 \circ \sigma_2 \circ T \times t \]

\[\lambda^{n-1} g \times v_i \rightarrow \Lambda^{n-1} g \times v_i \]

\[\nu_1 \circ \rho_1 \circ T \times t \rightarrow \nu_2 \circ \rho_2 \circ T \times t \]

\[D_1 \circ \sigma_1 \circ T \times t \rightarrow D_2 \circ \sigma_2 \circ T \times t \]

\[\nu_1 \circ \sigma_1 \circ T \times t \rightarrow \nu_2 \circ \sigma_2 \circ T \times t \]

\[\lambda \rightarrow \sigma_2 \circ \rho_2 \circ T \times t \]

\[\sigma_2 \circ \rho_2 \circ T \times t \rightarrow \sigma_2 \circ \rho_2 \circ T \times t \]

\[\lambda \rightarrow \sigma_2 \circ \rho_2 \circ T \times t \]

\[\lambda \rightarrow \sigma_2 \circ \rho_2 \circ T \times t \]

\[\lambda \rightarrow \sigma_2 \circ \rho_2 \circ T \times t \]

**Theorem 2.5:-**

Let \((g, \ldots, \nu, \rho)\) be a \( \Gamma \)-Lie algebra and \((\nu, \rho)\) be a representation of \( g \). Let \( D, \sigma: g \rightarrow g \) be two endo morphisms of \( g \) and any two of the functions \( D, \sigma \), replace, and assume \( D_v, \sigma_v : v \rightarrow v \) is 2-linear-functions from \( v \) and any 2-functions \( D_v, \sigma_v \). Then:

\[\rho(s) \circ \rho(s) = \rho(D(s)) \circ D_v \] and \( \sigma_v \circ \rho(s) = \rho(\sigma(s)) \circ \sigma_v \).

Then \((V, \rho \circ \sigma_v, D_v, \sigma_v)\) is a representation from an n-Bi-Hom \( \Gamma \)-Lie algebra

\[\Lambda^{n-1} g \times v_i \rightarrow \Lambda^{n-1} g \times v_i \]

\[\nu_1 \circ \rho_1 \circ T \times t \rightarrow \nu_2 \circ \rho_2 \circ T \times t \]

\[D_1 \circ \sigma_1 \circ T \times t \rightarrow D_2 \circ \sigma_2 \circ T \times t \]

\[\nu_1 \circ \sigma_1 \circ T \times t \rightarrow \nu_2 \circ \sigma_2 \circ T \times t \]

\[\lambda \rightarrow \sigma_2 \circ \rho_2 \circ T \times t \]

\[\lambda \rightarrow \sigma_2 \circ \rho_2 \circ T \times t \]

\[\lambda \rightarrow \sigma_2 \circ \rho_2 \circ T \times t \]

\[\lambda \rightarrow \sigma_2 \circ \rho_2 \circ T \times t \]

**Proof:-**

For every \( s = s_1 \wedge ... \wedge s_{n-1}, t = t_1 \wedge ... \wedge t_{n-1} \in \Lambda^{n-1} g \), we get:

1. \[\rho(\sigma(s)) \circ D_v = \sigma_v \circ \rho(D_v) \circ \sigma_v \]

2. \[\sigma_v \circ \rho(s) = \rho(\sigma_v(s)) \circ \sigma_v \]

3. \[\rho(\sigma(s)) \circ \rho(t) = \rho(\sigma(s \wedge t)) \circ \rho(\sigma(t)) \]

\[= \sigma_v \circ \rho(D_v(s \wedge t)) \circ \sigma_v \rho(D_v(s \wedge t)) \]

\[- \sum_{i=1}^{n-1} \sigma_v \circ \rho(\sigma_v(s_i \wedge t_i), D_v(s_i \wedge t_i)) \]

\[= \sigma_v \circ \rho(\sigma_v(s_i \wedge t_i), D_v(s_i \wedge t_i)) \]

\[= \sigma_v \circ \rho(D_v(s_i \wedge t_i)) \]

\[= \sigma_v \circ \rho(D_v(s_i \wedge t_i)) \]

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\[= \sigma_v \circ \rho(D_v(s_i \wedge t_i)) \]

**Definition 2.6:-**

Let \((g, \ldots, \nu, \rho)\) be an n-Bi-Hom \( \Gamma \)-Lie algebra. A bilinear form \( f \) over \( g \) is told to be nondegenerate if \( g^\perp = \{ s \in g : f(s, t) = 0, \forall t \in g \} = 0 \).

\[\sigma_v \text{-invariant if for every } s_1, ... s_{n+1} \in g, \]

\[f(\sigma(s_1), ..., \sigma(s_{n+1}), D(s_{n+1})) = -f(D(s_n), \sigma(s_1), ..., \sigma(s_{n+1}), D(s_{n+1})) \]

\[= \sigma_v(f(s_1), ..., \sigma(s_{n+1}), D(s_{n+1})) \]

\[\text{symmetric if } f(s, t) = f(t, s) \text{ is symmetric, i.e. } f(D(s), s) = f(s, D(t)) \text{. The sub space } I \text{ from } g \text{ be invariant isotropic if } I \leq I^2 \].

**Definition 2.7:-**

Assume that \((g, \ldots, \nu, \rho)\) is an n-Bi-Hom \( \Gamma \)-Lie-algebra on a field \( K \). If \( g \) admits a nondegenerate, \( \sigma_v \)-
invariant and symmetric-bilinear form \( f \) such \( \mathfrak{D}, \sigma \) are \( f \)-symmetrics, then we name \((g, f, \mathfrak{D}, \sigma)\) a quadratic n-Bi-Hom \( \Gamma \)-Lie algebra. Assume \((\tilde{g}, \tilde{f}, \tilde{\mathfrak{D}}, \tilde{\sigma})\) be another n-Bi-Hom \( \Gamma \)-Lie algebra. Two square n-Bi-Hom \( \Gamma \)-Lie algebras \((g, f, \mathfrak{D}, \sigma)\) and \((\tilde{g}, \tilde{f}, \tilde{\mathfrak{D}}, \tilde{\sigma})\) are told to be isometric if there exist algebra isomorphism \( \tilde{g} \rightarrow g \) such that 

\[
 f(s, t) = \tilde{f}(\tilde{g}(s), \tilde{g}(t)) \quad \text{for all } s, t \in g.
\]

**Theorem 2.8:**

Let \((\tilde{g}, \tilde{f}, \tilde{\mathfrak{D}}, \tilde{\sigma})\) be an n-Bi-Hom \( \Gamma \)-Lie algebra and 
\( (V, \rho, \mathfrak{D}_v, \sigma_v) \) a representation from \( g \). Assume that \( V^* \) is the dual space of \( V \) and \( \mathfrak{D}_v, \sigma_v \) \( 2 \)-Homomorphisms defined by 
\[
 \tilde{g}(f) = f \circ \mathfrak{D}_v, \quad \tilde{\sigma}_v(f) = f \circ \sigma_v, \quad \forall f \in V^*.
\]

Then the skew-symmetric linear-function \( \hat{\rho} : \Lambda^{n-1} \rightarrow \text{End}(V^*) \), define by

\[
 \hat{\rho}(s_1, \ldots, s_{n-1}) = -f \circ \rho(s_1, \ldots, s_{n-1}), \quad \forall f \in V^*,
\]

\( (s_1, \ldots, s_{n-1}) \in g \) be a representation from \( g \) over 
\( (V^*, \hat{\rho}, \tilde{\mathfrak{D}}_v, \tilde{\sigma}_v) \) if and only if for all \( s = s_1 \ldots s_{n-1} \), 
\( t = t_1 \ldots t_{n-1} \in \Lambda^{n-1} \).

1. \( \sigma_v \circ \rho(\tilde{g}(s)) = \rho(\tilde{g}(s)) \circ \sigma_v \).
2. \( \sigma_v \circ \rho(\tilde{g}(s)) = \rho(\tilde{g}(s)) \circ \sigma_v \).
3. \( \rho(\tilde{g}(s)) \circ \rho(\tilde{g}(s)) = \rho(\tilde{g}(s)) \circ \rho(\tilde{g}(s)) \).

**Proof:**

Let \( f \in V^* \), \( s = s_1 \ldots \ldots s_{n-1} \), \( t = t_1 \ldots \ldots t_{n-1} \in \Lambda^{n-1} \).

First, we have 
\[
 \hat{\rho}(\tilde{g}(s_1), \ldots, \tilde{g}(s_{n-1})) = -f \circ \rho(\tilde{g}(s_1), \ldots, \tilde{g}(s_{n-1}))
\]
and 
\[
 \tilde{\mathfrak{D}}_v \circ \rho(\tilde{g}(s_1), \ldots, \tilde{g}(s_{n-1})) = -\tilde{\mathfrak{D}}_v \circ \rho(\tilde{g}(s_1), \ldots, \tilde{g}(s_{n-1})) = -f \circ (\sigma_v \circ \rho(\tilde{g}(s_1), \ldots, \tilde{g}(s_{n-1})) \circ \sigma_v).
\]

Similarly,

\[
 \hat{\rho}(\tilde{g}(s_1), \ldots, \tilde{g}(s_{n-1})) = -\hat{\rho}(\tilde{g}(s_1), \ldots, \tilde{g}(s_{n-1})) = -\hat{\rho}(\tilde{g}(s_1), \ldots, \tilde{g}(s_{n-1})) \circ \sigma_v.
\]

Then we can get 
\[
 \hat{\rho}(\tilde{g}(s_1), \ldots, \tilde{g}(s_{n-1})) \circ \hat{\rho}(t_1, \ldots, t_{n-1}) (f) = -\hat{\rho}(\tilde{g}(s_1), \ldots, \tilde{g}(s_{n-1})) \circ (f(t_1, \ldots, t_{n-1})).
\]

**Corollary 2.9:**

Assume ad is the adjoint representation from an n-Bi-Hom \( \Gamma \)-Lie algebra \((g, \mathfrak{D}, \sigma, \sigma_v)\). Assume that the bilinear function \( ad^* : \Lambda^{n-1} g \rightarrow \text{End}(g^*) \) defined by

\[
 ad^*(s_1, \ldots, s_{n-1})(f) = -f \circ ad(s_1, \ldots, s_{n-1}), \quad \forall s_1, \ldots, s_{n-1} \in g.
\]

Then \( ad^* \) is a representation of \( g \) on \( g^* \).

**Proof:**

Let \( f \in g^* \), \( s = s_1 \ldots \ldots s_{n-1} \), \( t = t_1 \ldots \ldots t_{n-1} \in \Lambda^{n-1} g \). First, we have
\[(ad^{*}(D(s_1), \ldots, D(s_{n-1})) \circ D)(f) = -D(f)\]
and
\[D \circ ad^{*}(s_2, \ldots, D(s_{n-1}))(f) = -D(f) \circ ad(s_1, \ldots, D(s_{n-1})) \circ D\].

Then
\[D \circ ad^{*}(s_2, \ldots, D(s_{n-1}))(f) \Rightarrow D \circ ad^{*}(s_1, \ldots, s_{n-1}) \Rightarrow D \circ ad(s_1, \ldots, D(s_{n-1}) \circ D.

The same way,
\[ad^{*}(\sigma(s_1), \ldots, \sigma(s_{n-1})) \circ \sigma(s_1) = \bar{\sigma} \circ \sigma \]
and
\[ad^{*}(s_1, \ldots, s_{n-1}) \Rightarrow \sigma \circ ad(s_1, \ldots, D(s_{n-1})) = ad(s_1, \ldots, D(s_{n-1}) \circ \sigma).\]

Then we have
\[\begin{align*}
D \circ ad^{*}(s_1, \ldots, D(s_{n-1})) \circ ad^{*}(t_1, \ldots, t_{n-1})(f) &= -ad^{*}(D(s_1), \ldots, D(s_{n-1}))(f \circ ad(t_1, \ldots, t_{n-1})) \\
&= f \circ ad(s_1, \ldots, s_{n-1}) \circ ad^{*}(s_1, \ldots, D(s_{n-1}))\]
and
\[\begin{align*}
&+ \sum_{i=1}^{n-1} ad^{*}(s_1, \ldots, s_{n-1}) \circ ad^{*}(t_1, \ldots, t_{n-1}) \\
&= f \circ ad(s_1, \ldots, D(s_{n-1})) \circ ad^{*}(t_1, \ldots, t_{n-1}) \circ \sigma \bar{\sigma} \\
&- \sum_{i=1}^{n-1} f \circ ad^{*}(s_1, \ldots, D(s_{n-1})) \circ ad^{*}(t_1, \ldots, D(s_{n-1})) \\
&+ \sum_{i=1}^{n-1} ad^{*}(t_1, \ldots, t_{n-1}) \circ \sigma \circ \sigma \bar{\sigma} \]
which implies that
\[ad^{*}(D(s_1), \ldots, D(s_{n-1}) \circ ad^{*}(t_1, \ldots, t_{n-1}) = ad^{*}(\sigma(s_1), \ldots, D(s_{n-1})).\]

If and only if
\[ad(t_1, \ldots, t_{n-1}) \circ D(\sigma(s_1), \ldots, D(s_{n-1}))(f) = -ad^{*}(D(s_1), \ldots, D(s_{n-1})) \circ D(f \circ ad(t_1, \ldots, t_{n-1})).\]

Then
\[\begin{align*}
&+ \sum_{i=1}^{n-1} ad^{*}(s_1, \ldots, s_{n-1}) \circ \sigma \circ \sigma \bar{\sigma} \\
&= \sum_{i=1}^{n-1} (-1)^{n-i} ad^{*}(D\sigma(s_1), \ldots, D\sigma(s_{n-1}) \circ \sigma(t_1, \ldots, t_{n-1}) \circ \sigma(t_1) \circ \sigma(s_1, \ldots, D(s_{n-1})) \circ \sigma(t_1).\]

\[\text{REFERENCES}\]


