Representation of n- BiHom Γ-Lie algebra

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Abstract— *The purpose of this paper, is to define and discuss a representation on an n-Bi-Hom Γ-Lie algebra and give some results about them.*

Keywords— Algebra, Lie algebra, Representation.

I. INTRODUCTION

Kitoune – Makhlouf – Silvestrov in [1] define n-Bi-Hom Lie algebra. The $representation$ notion of algebra is very essential since it defect some of its difficult structure invisible below. In [2,5] they study the theory of a $representation$ of 3-Bi-Hom $Lie\ algebra$ inserted and construction of n-Lie algebras. Rezaei and Davvaz in [3] define Γ- algebra , A Γ-algebra is an algebraic structure composed of a vector-space V, a groupoid Γ jointly with function from V× Γ × V to V. Then, on every associative Γ-algebra V and for all $\alpha \in \Gamma$,they establish an α-Lie algebra. The purpose of this paper, is to introduced a new concept which is representation of an n-Bi-Hom Γ-Lie algebra and equivalent two representation of an n-Bi-Hom Γ-Lie algebra . It is mentioned the reader that, and field K of characteristic 0 and every the vector-spaces are on K. Every place here after, the notation $\hat{s}$ attened that s is except, for example, record $f(s_1, \dots, \hat{s}_i, \dots, s_n)$ for $f(s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$.

Now, we will recall the followings concepts which are necessary in this paper.

Definition 1.1 :- [1]

An n- Bi-Hom Lie-algebra be a vector-space g, supplied with n-linear process [. , … , .] with 2-linear function $\underset{\wedge}{D}$, ϖsatisfying the following conditions:

1-$\underset{\wedge}{D} \circ \varpi = \varpi \circ \underset{\wedge}{D}$

2- For all s_1 ,…, s_n $\in g$, $\underset{\wedge}{D}$ ($[s_1, \dots, s_n]_g$) = $[\underset{\wedge}{D}(s_1), \dots, \underset{\wedge}{D}(s_n)]_g$ and

$\varpi([s_1, \dots, s_n]_g) = [\varpi(s_1), \dots, \varpi(s_n)]_g$.

3- Bi-Hom- skew symmetry: for all s_1 ,…, $s_n \in g$,

$$[\varpi(s_1), \dots, \varpi(s_k), \varpi(s_{k+1}), \dots, \varpi(s_{n-1}), \underset{\wedge}{D}(s_n)]_g$$
$$= -[\varpi(s_1), \dots, \varpi(s_{k+1}), \varpi(s_k), \dots, \varpi(s_{n-1}), \underset{\wedge}{D}(s_n)]_g$$
$$= -[\varpi(s_1), \dots, \varpi(s_{n-2}), \varpi(s_n), \underset{\wedge}{D}(s_{n-1})]_g$$

when $k = 1, 2, \dots, n-2$.

4- n- Bi-Hom Jacobi-identity:for all $s_1, \dots, s_{n-1}, t_1, \dots, t_n \in g$,

$$\left[\varpi^2(s_1), \dots, \varpi^2(s_{n-1}), [\varpi(t_1), \dots, \varpi(t_{n-1}), \underset{\wedge}{D}(t_n)]_g\right]_g$$
$$= \sum_{k=1}^{n}(-1)^{n-k}\left[\varpi^2(t_1), \dots, \widehat{\varpi^2(t_k)}, \dots, \varpi^2(t_n), \left[\begin{matrix}\varpi(s_1), \dots \\ , \varpi(s_{n-1}), \underset{\wedge}{D}(t_k)\end{matrix}\right]_g\right]_g$$

Note 1.2 [8]-[9]

1- when $\underset{\wedge}{D} = \varpi = id$, we obtain n-Lie algebra .

2- when n = 2, the Bi-Hom Jacobi-identity be:

$$\left[\varpi^2(s), [\varpi(t_1), \underset{\wedge}{D}(t_2)]_g\right]_g = -\left[\varpi^2(t_2), [\varpi(s), \underset{\wedge}{D}(t_1)]_g\right]_g + \left[\varpi^2(t_1), [\varpi(s), \underset{\wedge}{D}(t_2)]_g\right]_g \quad (2.1)$$

and the identity (2.1) is equivalent to $\sum_{s,t_1,t_2}\left[\varpi^2(s), [\varpi(t_1), \underset{\wedge}{D}(t_2)]_g\right]_g = 0$

3- For n = 3, the 3 Bi-Hom Jacobi-identity be:

$$\left[\varpi^2(s_1), \varpi^2(s_2), [\varpi(t_1), \varpi(t_2), \underset{\wedge}{D}(t_3)]_g\right]_g =$$
$$\left[\varpi^2(t_2), \varpi^2(t_3), [\varpi(s_1), \varpi(s_2), \underset{\wedge}{D}(t_1)]_g\right]_g$$
$$-\left[\varpi^2(t_1), \varpi^2(t_3), [\varpi(s_1), \varpi(s_2), \underset{\wedge}{D}(t_2)]_g\right]_g$$
$$+\left[\varpi^2(t_1), \varpi^2(t_2), [\varpi(s_1), \varpi(s_2), \underset{\wedge}{D}(t_3)]_g\right]_g$$

Definition 1.3:-[4]

Suppose Γ be a groupoid and V is a vector-space on a field F. Then , V be invited a Γ- algebra on the field F if there exist function $V \times \Gamma \times V \to V$ (the image be $s\alpha t$ for $s, t \in V$ and $\alpha \in \Gamma$) such the following satisfy:

(1) $(s+t)\alpha z = s\alpha z + t\alpha z,\ s\alpha(t+z) = s\alpha t + s\alpha z,$

(2) $s(\alpha+\beta)t = s\alpha t + s\beta t,$

(3) $(cs)\alpha t = c(s\alpha t) = s\alpha(ct),$

(4) $0\alpha t = t\alpha 0 = 0$, for all s, y, z ∈ V, c∈ F and $\alpha \in \Gamma$.

Moreover, a Γ- algebra is called associative if

(5) $(s\alpha t)\beta z = s\alpha(t\beta z).$

Example 1.4 :- [4]

Suppose that V is a vector-space and Γ a groupoid. For all $s, t \in V$ and $\alpha \in \Gamma$ we define $s\alpha t = 0$. There after, V is a Γ-algebra.

Definition 1.5 :- [3]

Suppose V is an associative Γ-algebra on a field F.There after, for all $\lambda \in \Gamma$ one can construct an Γ-Lie-algebra $L_\lambda(V)$. As a vector-space, $L_\lambda(V)$ is the like with V. The Lie Γ- bracket of 2-elements of $L_\lambda(V)$ be define to be their commentator in V, $[r, v]_\lambda = r\cdot_\lambda v - v\cdot_\lambda r$. Note that $[r, v]_\lambda = -[v, r]_\lambda$.

Theorem 1.6 :- [9]-[7]

Let $(g, [., ..., .]_g)$ be an n-Lie-algebra, and assume D, ϖ : g→ g is algebra morphism such $D \circ \varpi = \varpi \circ D$. The algebra $(g, [., ..., .]_{D\varpi}, D, \varpi)$, when $[., ..., .]_{D\varpi}$ be define $[s_1, ..., s_n]_{D\varpi} = [D(s_1), ..., D(s_{n-1}), \varpi(s_n)]_g$, is an Bi-Hom-Lie-algebra.

Definition 1.7 :- [9]

A homomorphism $f: (g, [., ..., .]_g, D, \varpi) \rightarrow (\acute{g}, [., ..., .]_{\acute{g}}, \acute{D}, \acute{\varpi})$ form an n-Bi-Hom Lie-algebra is a linear-function $f: g \rightarrow \acute{g}$ such $f \circ D = \acute{D} \circ f$, $f \circ \varpi = \acute{\varpi} \circ f$ and for any $s_k \in g$. $f([s_1, ..., s_n]_g) = [f(s_1), ..., f(s_n)]_{\acute{g}}$

Definition 1.8 :- [9]-[6]

Let $(g, [., ..., .]_g, D, \varpi)$ be an n- Bi-Hom Lie-algebra. We define, for every $s = s_1 \wedge ... \wedge s_{n-1}$, $t = t_1 \wedge ... \wedge t_{n-1}$ $in\ \wedge^{n-1} g\ and\ z \in g$ the nest :

1- the action fundamental subject over g:
$s \cdot z = ad_s(z) = [s_1, ... s_n, z]_g$

2- The linear functions $\bar{D}, \bar{\varpi} : \wedge\,^\wedge(n-1)\ g \rightarrow \wedge\,^\wedge(n-1)\ g$:
$\tilde{D}(s) = D(s_1) \wedge ... \wedge D(s_{n-1})$ and $\tilde{\varpi}(s) = \varpi(s_1) \wedge ... \wedge \theta_2(s_{n-1})$

II. MAIN RESULTS

In this section, we give a definition of n- Bi-Hom Γ – Lie-algebra, a representation on n- Bi-Hom Γ-Lie algebra, two equivalent representation of an n-Bi-Hom Γ-Lie algebra.

Definition 2.1:-

A representation an n- Bi-Hom Γ-Lie algebra $(g, [., ..., .]_{\lambda g}, D, \varpi)$ over a $vector\ space$ V regarding endomorphisms $D_v, \varpi_v \in End\ (V)$is a linear-function ρ: $\wedge^{n-1}$ g→ $End(V)$ such for every
$s = s_1 \wedge ... \wedge s_{n-1}$, $t = t_1 \wedge ... \wedge t_{n-1} \in \wedge^{n-1} g$,
$\tilde{D}(s) = D(s_1) \wedge ... \wedge D(s_{n-1})$, $\tilde{\varpi}(s) = \varpi(s_1) \wedge ... \wedge \varpi(s_{n-1})$, we have

1- $\rho\left(\tilde{D}(s)\right) \circ D_v = D_v \circ \rho(s)$;

2- $\rho\left(\tilde{\varpi}(s)\right) \circ \varpi_v = \varpi_v \circ \rho(s)$;

3- $\rho\left(\widetilde{D\varpi}(s)\right) \circ \rho(t) - \rho\left(\tilde{\varpi}(t)\right) \circ \rho(\tilde{D}(s))$

$$= \sum_{i=1}^{n-1} \rho \begin{pmatrix} \varpi(t_1), ..., \varpi(t_{i-1}), [\varpi(s_1), ..., \varpi(s_{n-1}), t_i]_{\lambda g}, \\ \varpi(t_{i+1}), ..., \varpi(t_{n-1}) \end{pmatrix} \circ \varpi_v;$$

4- $\rho\left(\varpi(t_2), ..., \varpi(t_{n-1}), [\varpi(s_1), ..., \varpi(s_{n-1}), t_1]_{\lambda g}\right) \circ \varpi_v$

$$= \sum_{i=1}^{n-1} (-1)^{n-i} \rho\left(D\,\varpi(s_1), ..., \widehat{D\varpi}(s_i), ..., D\,\varpi(s_{n-1}), \varpi(t_1)\right) \circ \rho(t_2, ..., t_{n-1}, D(s_i)) + \rho\left(\widetilde{D\varpi}(s)\right) \circ \rho(t)$$

We symbolize a representation by (v, ρ, D_v, ϖ_v).

Proposition 2.2:-

Let $(g, [., ...,]_{\lambda g}, D, \varpi)$ is an n-Bi-Hom $\Gamma - Lie\ algebra$. Define ad : $\wedge^{n-1} g \rightarrow End(g)$ by ad_s(y) $[s_1, ..., s_{n-1}, t]_{\lambda g}$, for every $s = s_1 \wedge ... \wedge s_{n-1} \in \wedge^{n-1} g$, $t \in g$. Then (g, ad, D, ϖ) is a $representation$ of the n-Bi-Hom Γ $Lie\ algebra$ $(g, [., ...,]_{\lambda g}, D, \varpi)$ on g, called adjoin representation.

Proof:-

An n- Bi-Hom Γ-Lie algebra $(g, [., ..., .]_\lambda, D, \varpi)$ over a vector space V regarding endomorphisms $D\ \varpi \in End\ (g)$is a linear function ad: $\wedge^{n-1}$ g→ $End(V)$ such for all
$s = s_1 \wedge ... \wedge s_{n-1}$, $t = t_1 \wedge ... \wedge t_{n-1} \in \wedge^{n-1}$g ,
$\tilde{D}(s) = D(s_1) \wedge ... \wedge D(s_n)$, $\tilde{\varpi}(s) = \varpi(s_1) \wedge ... \wedge \varpi(s_n)$, we have

1- $ad\left(\tilde{D}(s)\right) \circ D = D \circ ad(s)$;

2- $ad\left(\tilde{\varpi}(s)\right) \circ \varpi = \varpi \circ ad(s)$;

3- $ad\left(\widetilde{D\varpi}(s)\right) \circ ad(t) - ad\left(\tilde{\varpi}(t)\right) \circ ad(\tilde{D}(s)) = \sum_{i=1}^{n-1} ad \begin{pmatrix} \varpi(t_1), ..., \varpi(t_{i-1}), [\varpi(s_1), ..., \varpi(s_{n-1}), t_i]_{\lambda g} \\ , \varpi(t_{i+1}), ..., \varpi(t_{n-1}) \end{pmatrix} \circ \varpi$;

4- $ad\left(\varpi(t_2), ..., \varpi(t_{n-1}), [\varpi(s_1), ..., \varpi(s_{n-1}), t_1]_{\lambda g}\right) \circ \varpi = \sum_{i=1}^{n-1} (-1)^{n-i}\ ad\left(D\varpi(s_1), ..., \widehat{D\varpi}(s_i), ..., D\varpi(s_{n-1}), \varpi(t_1)\right) \circ ad(t_2, ..., t_{n-1}, D(s_i)) + ad\left(\widetilde{D\varpi}(s)\right) \circ ad(t)$ .We denote a representation by (g, ad, D, ϖ)

Proposition 2.3:-

Let (v, ρ, D_v, ϖ_v) be a $representation$ of n- Bi-Hom Γ-Lie algebra $(g, [., ..., .]_{\lambda g}, D, \varpi)$.
suppose the $maps$ D and ϖ_v are bijective. There after $(g \oplus v, [., ..., .]_{\lambda\rho}, D + D_v, \varpi + \varpi_v)$ is an n-Bi-Hom Γ-Lie algebra,when $D + D_v$, $\varpi + \varpi_v$: $g \oplus v \rightarrow g \oplus v$ are defined by $(D + D_v)(s + r) = D(s) + D_v(r)$ and $(\varpi + \varpi_v)(s + r) = \varpi(s) + \varpi_v(r)$ and the bracket operation $[., ..., .]_{\lambda\rho}: \wedge^n(g \oplus v) \rightarrow g \oplus v$ be define by $[s_1 + r_1, ..., s_n + r_n]_{\lambda\rho} = [s_1, ..., s_n]_{\lambda\rho}$

$$+ \sum_{i=1}^{n-1} (-1)^{n-i} \rho(s_1, ..., \hat{s_i}, ..., s_{n-1}, D^{-1}\varpi(s_n))(D_v \varpi_v^{-1}(r_i)) + \rho(s_1, ..., s_{n-1})(r_n)$$

for all $s_i \in g, r_i \in V$. We symbolize this semi direct n- Bi-Hom Γ-$Lie\ algebra$ simply by $g \propto v$.

Proof :-

It is clear that

$(\underset{\frown}{D} + \underset{\frown}{D}_v) \circ (\varpi + \varpi_v) = (\varpi + \varpi_v) \circ (\underset{\frown}{D} + \underset{\frown}{D}_v)$

from the fact $\underset{\frown}{D} \circ \varpi = \varpi \circ \underset{\frown}{D}$, $\underset{\frown}{D}_v \circ \varpi_v = \varpi_v \circ \underset{\frown}{D}_v$.

Now, we clear that $\underset{\frown}{D} + \varpi$ be an algebra morphism. On the other hand, we have

$$(\underset{\frown}{D} + \underset{\frown}{D}_v)[s_1 + r_1, \dots, s_n + r_n]_{\lambda\rho} = (\underset{\frown}{D} + \underset{\frown}{D}_v)\left([s_1, \dots, s_n]_{\lambda g} + \sum_{i=1}^{n}(-1)^{n-i}\rho(s_1, \dots, \widehat{s_i}, \dots, s_{n-1}, \underset{\frown}{D}^{-1}\varpi(s_n))(\underset{\frown}{D}_v\, \varpi_v^{-1}(r_i))\right)$$

$$= \underset{\frown}{D}([s_1, \dots, s_n]_{\lambda g}) + \sum_{i=1}^{n}(-1)^{n-i}\underset{\frown}{D}_v \circ \rho(s_1, \dots, \widehat{s_i}, \dots, s_{n-1}, \underset{\frown}{D}^{-1}\varpi(s_n))\, \underset{\frown}{D}_v\, (\varpi_v^{-1}(r_i))$$

on the other hand, we get

$$[(\underset{\frown}{D} + \underset{\frown}{D}_v)(s_1 + r_1), \dots, (\underset{\frown}{D} + \underset{\frown}{D}_v)(s_n + r_n)]_{\lambda\rho}$$
$$= [\underset{\frown}{D}(s_1) + \underset{\frown}{D}_v(r_1), \dots, \underset{\frown}{D}(s_n) + \underset{\frown}{D}_v(r_n)]_{\lambda\rho}$$
$$= [\underset{\frown}{D}(s_1), \dots, \underset{\frown}{D}(s_n)]_{\lambda g} + \sum_{i=1}^{n}(-1)^{n-i}\rho\left(\underset{\frown}{D}(s_1), \dots, \widehat{\underset{\frown}{D}}(s_i), \dots, \underset{\frown}{D}^{-1}\varpi(\underset{\frown}{D}(s_n))\right)\underset{\frown}{D}_v\left(\varpi_v^{-1}(\underset{\frown}{D}_v(r_i))\right)$$

since $\underset{\frown}{D}$ an algebra-morphism ρ and $\underset{\frown}{D}_v$ satisfies the condition (i) in definition (2.1), so that $\underset{\frown}{D} + \underset{\frown}{D}_v$ is an algebra- morphism regarding the bracket $[.,\dots,]_{\lambda\rho}$. The same way, we get $\varpi + \varpi_v$ is an algebra-morphism regarding the bracket $[.,\dots,]_{\lambda\rho}$.

follow we clear that the bracket $[.,\dots,]_{\lambda\rho}$ satisfying Bi-Hom Γ -Lie algebra skew-symmetry,

for every $s_i \in g, r_i \in V$

$$[(\varpi + \varpi_v)(s_1 + r_1), \dots, (\varpi + \varpi_v)(s_i + r_i), (\varpi + \varpi_v)(s_{i+1} + r_{i+1}), \dots, \underset{\frown}{D} + \underset{\frown}{D}_v(s_n + r_n]_{\lambda\rho}$$
$$= [\varpi(s_1) + \varpi_v(r_1), \dots, \varpi(s_i) + \varpi_v(r_i), \varpi(s_{i+1}) + \varpi_v(r_{i+1}), \dots, \underset{\frown}{D}(s_n) + \underset{\frown}{D}_v(r_n)]_{\lambda\rho}$$
$$= [\varpi(s_1), \dots, \varpi(s_i), \varpi(s_{i+1}), \dots, \underset{\frown}{D}(s_n)]_{\lambda g} + \sum_{k=1}^{n-1}(-1)^{n-k}\rho\begin{pmatrix}\varpi(s_1), \dots, \varpi(\widehat{s_k}), \dots, \varpi(s_i), \varpi(s_{i+1}),\\ \dots, \varpi(s_{n-1}), \underset{\frown}{D}^{-1}\varpi(\alpha(s_n))\end{pmatrix}\left(\underset{\frown}{D}_v\varpi_v^{-1}(\varpi_v(r_k))\right)$$
$$+ \rho(\varpi(s_1), \dots, \varpi(s_i), \varpi(s_{i+1}), \dots, \varpi(s_{n-1}))(\underset{\frown}{D}_v(r_n))$$
$$= -[\varpi(s_1), \dots, \varpi(s_{i+1}), \varpi(s_i), \dots, \underset{\frown}{D}(s_n)]_{\lambda g} - \sum_{k=1}^{n-1}(-1)^{n-k}\rho\begin{pmatrix}\varpi(s_1), \dots, \widehat{\varpi(s_k)}, \dots, \varpi(s_{i+1}), \varpi(s_i), \dots, \varpi(s_{n-1}),\\ \underset{\frown}{D}^{-1}\varpi(\underset{\frown}{D}(s_n))\end{pmatrix}\left(\underset{\frown}{D}_v\varpi_v^{-1}(\varpi_v(r_k))\right)$$
$$-\rho(\varpi(s_1), \dots, \varpi(s_{i+1}), \varpi(s_i), \dots, \varpi(s_{n-1}))(\underset{\frown}{D}_v(r_n))$$
$$= -[\varpi(s_1) + \varpi_v(r_1), \dots, \varpi(s_{i+1}) + \varpi_v(r_{i+1}), \varpi(s_i) + \varpi_v(r_i), \dots, \underset{\frown}{D}(s_n) + \underset{\frown}{D}_v(r_n)]_{\lambda\rho}$$
$$= -[(\varpi + \varpi_v)(s_1 + r_1), \dots, (\varpi + \varpi_v)(s_{i+1} + r_{i+1}), (\varpi + \varpi_v)(s_i + r_i), \dots, (\underset{\frown}{D} + \underset{\frown}{D}_v)(s_n + r_n)]_{\lambda\rho}.$$

The end, for every $s_i, t_i \in g, r_i, v_i \in V$ we get

$$\left[(\varpi + \varpi_v)^2(s_1 + r_1), \dots, (\varpi + \varpi_v)^2(s_{n-1} + r_{n-1}), [(\varpi + \varpi_v)(t_1 + v_1), \dots, (\underset{\frown}{D} + \underset{\frown}{D}_v)(t_n + v_n)]_{\lambda\rho}\right]_{\lambda\rho}$$
$$= \left[\varpi^2(s_1) + \varpi_v^2(r_1), \dots, \varpi^2(s_{n-1}) + \varpi_v^2(r_{n-1}), [\varpi(t_1) + \varpi_v(v_1), \dots, \underset{\frown}{D}(t_n) + \underset{\frown}{D}_v(v_n)]_{\lambda\rho}\right]_{\lambda\rho}$$
$$= \begin{bmatrix}\varpi^2(s_1) + \varpi_v^2(r_1), \dots, \varpi^2(s_{n-1}) + \varpi_v^2(r_{n-1}), [\varpi(t_1), \dots, \underset{\frown}{D}(t_n)]_{\lambda g}\\ + \sum_{i=1}^{n-1}(-1)^{n-i}\rho\left(\varpi(t_1), \dots, \widehat{\varpi(t_i)}, \dots, \varpi(t_{n-1}), \underset{\frown}{D}^{-1}\varpi(\underset{\frown}{D}(t_n))\right)\\ \left(\underset{\frown}{D}_v\varpi_v^{-1}(\varpi_v(v_i))\right) + \rho(\varpi(t_1), \dots, \varpi(t_{n-1}))\,(\underset{\frown}{D}_v(v_n))\end{bmatrix}_{\lambda\rho}$$
$$= \left[\varpi^2(s_1), \dots, \varpi^2(s_{n-1}), [\varpi(t_1), \dots, \underset{\frown}{D}(t_n)]_{\lambda\rho}\right]_{\lambda\rho} + \sum_{i=1}^{n-1}(-1)^{n-i}\rho\begin{pmatrix}\varpi^2(s_1), \dots, \widehat{\varpi^2(s_i)}, \dots, \varpi^2(s_{n-1}), \underset{\frown}{D}^{-1}\\ \varpi([\varpi(t_1), \dots, \underset{\frown}{D}(t_n)]_{\lambda g})(\underset{\frown}{D}_v\varpi_v(r_i))\end{pmatrix}$$
$$+ \rho(\varpi^2(s_1), \dots, \varpi^2(s_{n-1}))\left(\sum_{i=1}^{n-1}(-1)^{n-i}\rho(\varpi(t_1), \dots, \widehat{\varpi(t_i)}, \dots, \varpi(t_{n-1}), \underset{\frown}{D}^{-1}\varpi(\underset{\frown}{D}(t_n))\right)(\underset{\frown}{D}_v(v_i)) + \rho(\varpi(t_1), \dots, \varpi(t_{n-1}))(\underset{\frown}{D}_v(v_n))$$
$$= \sum_{i=1}^{n}(-1)^{n-i}\left[\varpi^2(t_1), \dots, \varpi^2(t_i), \varpi^2(t_n), [\varpi(s_1), \dots, \varpi(s_{n-1}), \underset{\frown}{D}(t_i)]_{\lambda g}\right]_{\lambda g}$$
$$+ \rho(\varpi(t_1), \dots, \varpi(t_{n-1}))(\underset{\frown}{D}_v(v_n)) + \sum_{i=1}^{n}(-1)^{n-i}\rho\left(\varpi^2(t_1), \dots, \widehat{\varpi^2(t_i)}, \dots, \varpi^2(t_{n-1}), \varpi^2(t_n)\right)$$
$$\rho(\varpi(s_1), \dots, \varpi(s_{n-1}))(\underset{\frown}{D}_v(v_i))$$
$$+ \sum_{i=1}^{n-1}(-1)^{n-i}\rho\left(\varpi^2(s_1), \dots, \widehat{\varpi^2(s_i)}, \dots, \varpi^2(s_{n-1})\right)\underset{\frown}{D}^{-1}\varpi$$
$$([\varpi(t_1), \dots, \underset{\frown}{D}(t_n)]_{\lambda g})(\underset{\frown}{D}_v\varpi_v(r_i))$$
$$= \sum_{k=1}^{n}(-1)^{n-k}\begin{bmatrix}\varpi^2(t_1), \dots, \widehat{\varpi^2(t_k)}, \dots, \varpi^2(t_n)\\ , [\varpi(s_1), \dots, \varpi(s_{n-1}), \underset{\frown}{D}(t_k)]_{\lambda g}\end{bmatrix}_{\lambda g}$$
$$+ \sum_{k=1}^{n}(-1)^{n-k}\left(\sum_{i=1}^{n-1}(-1)^{n-i}\rho\begin{pmatrix}\varpi^2(t_1), \dots, \widehat{\varpi^2(t_k)}, \dots, \widehat{\varpi^2(t_i)}, \dots, \varpi^2(t_n),\\ \underset{\frown}{D}^{-1}\varpi([\varpi(s_1), \dots, \varpi(s_{n-1}), \underset{\frown}{D}(t_k)]_{\lambda g})\end{pmatrix}\underset{\frown}{D}_v\varpi_v(v_i)\right)$$
$$+ \sum_{k=1}^{n}(-1)^{n-k}\rho(\varpi^2(t_1), \dots, \varpi^2(t_k), \dots, \varpi^2(t_n))\left(\sum_{i=1}^{n-1}(-1)^{n-i}\rho\begin{pmatrix}\varpi(s_1), \dots, \widehat{\varpi(s_i)}, \dots,\\ \varpi(s_{n-1}), \underset{\frown}{D}^{-1}\varpi(\underset{\frown}{D}(t_k))\end{pmatrix}(\underset{\frown}{D}_v(r_i)) + \rho(\varpi(s_1), \dots, \varpi(s_{n-1}))(\underset{\frown}{D}_v(v_k))\right)$$

$$= \sum_{k=1}^{n} (-1)^{n-k} \big[(\varpi+\varpi_v)^2(\text{ŧ}_1+v_1), \dots, (\varpi+\varpi_v)^2(\text{ŧ}_{k-1} + v_{k-1}), (\varpi+\varpi_v)^2(\text{ŧ}_{k+1} + v_{k+1}), \dots, (\varpi+\varpi_v)^2(\text{ŧ}_n + v_n), [(\varpi+\varpi_v)(\mathrm{s}_1+\text{ɍ}_1), \dots, (\varpi+\varpi_v)(\mathrm{s}_{n-1}+\text{ɍ}_{n-1}), (\text{Ḓ}+\text{Ḓ}_v)(\text{ŧ}_1 + v_1)]_{\lambda\rho}\big]_{\lambda\rho}$$

Then $g \propto v \coloneqq \big(g \oplus v, [.,\dots,]_{\lambda\rho}, \text{Ḓ}+\text{Ḓ}_v, \varpi+\varpi_v\big)$ is an n-Bi-Hom Γ-*Lie algebra*.

Definition 2.4:-

Assume $(v_1,\rho_1,\text{Ḓ}_1,\varpi_1)$ and $(v_2,\rho_2,\text{Ḓ}_2,\varpi_2)$ is 2-representation from n-Bi-Hom Γ-Lie-algebra $\big(g,[.,\dots,]_{\lambda g},\text{Ḓ},\varpi\big)$. They are said to be equivalent if there exist an isomorphism of *vector spaces* $T: v_1 \to v_1$ such that $T\rho_1(s)(\text{ɍ}) = \rho_2(s)(T\text{ɍ})$, $T \circ \text{Ḓ}_1 = \text{Ḓ}_2 \circ T$ and $T \circ \varpi_1 = \varpi_2 \circ T$ for every $s = \mathrm{s}_1 \wedge \dots \wedge \mathrm{s}_{n-1}$, $\text{ɍ} \in v_1$.

In terms from diagrams, we get

$$\begin{tikzcd} \wedge^{n-1} g \times V_1 \arrow[r, "\rho_1"'] \arrow[d, "id \times T"'] & V_1 \arrow[d, "T"] \\ \wedge^{n-1} g \times V_1 \arrow[r] & V_2 \end{tikzcd} , \quad \begin{tikzcd} V_1 \arrow[r, "\text{Ḓ}_1"] \arrow[d, "T"'] & V_1 \arrow[d, "T"] \\ V_2 \arrow[r, "\text{Ḓ}_2"] & V_2 \end{tikzcd} \quad \begin{tikzcd} V_1 \arrow[r, "\varpi_1"] \arrow[d, "T"'] & V_1 \arrow[d, "T"] \\ V_2 \arrow[r, "\varpi_2"] & V_2 \end{tikzcd}$$

Theorem 2.5:-

Let $\big(g,[.,\dots,]_{\lambda g}\big)$ be a Γ-Lie algebra and (V,ρ) be a representation of g. Let $\text{Ḓ},\varpi: g \to g$ be two endo morphisms of g and any two of the functions $\text{Ḓ},\varpi$, replace, and assume Ḓ_v , $\varpi_v : v \to v$ is 2-linear-functions from v and any 2-functions Ḓ_v , ϖ_v commute such $\text{Ḓ}_v \circ \rho(\mathrm{s}) = \rho\big(\tilde{\text{Ḓ}}(\mathrm{s})\big) \circ \text{Ḓ}_v$ and $\varpi_v \circ \rho(\mathrm{s}) = \rho\big(\tilde{\varpi}(\mathrm{s})\big) \circ \varpi_v$. Then $(V, \tilde{\rho} \coloneqq \varpi_v \circ \rho\ , \text{Ḓ}_v, \varpi_v)$ be *a representation* from an n-Bi-Hom Γ-*Lie algebra* $\big(g,[.,\dots,.]_{\lambda\text{Ḓ}=,\lambda\varpi}[.,\dots,]_{\lambda g} \circ (\alpha \otimes \dots \otimes \text{Ḓ} \otimes \varpi), \text{Ḓ}, \varpi\big)$.

Proof:-

For every $s = \mathrm{s}_1 \wedge \dots \wedge \mathrm{s}_{n-1}$, $Y = \text{ŧ}_1 \wedge \dots \wedge \text{ŧ}_{n-1} \in \wedge^{n-1} g$, we get

1- $\tilde{\rho}\big(\tilde{\text{Ḓ}}(\mathrm{s})\big) \circ \text{Ḓ}_v = \varpi_v \circ \rho\big(\tilde{\text{Ḓ}}(\mathrm{s})\big) \circ \text{Ḓ}_v$
$= \varpi_v \circ \text{Ḓ}_v \circ \rho(\mathrm{s})$
$= \text{Ḓ}_v \circ \varpi_v \circ \rho(\mathrm{s})$
$= \text{Ḓ}_v \circ \tilde{\rho}(\mathrm{s})$

2- $\tilde{\rho}\big(\tilde{\varpi}(\mathrm{s})\big) \circ \varpi_v = \varpi_v \circ \rho\big(\tilde{\varpi}(\mathrm{s})\big) \circ \varpi_v$
$= {\varpi_v}^2 \circ \rho(\mathrm{s})$
$= \varpi_v \circ \tilde{\rho}(\mathrm{s})$

3- $\tilde{\rho}\big(\widetilde{\text{Ḓ}\varpi}(\mathrm{s})\big) \circ \tilde{\rho}(\text{ŧ}) - \tilde{\rho}\big(\tilde{\varpi}(\text{ŧ})\big) \circ \tilde{\rho}\big(\tilde{\text{Ḓ}}(\mathrm{s})\big)$

$$= \varpi_v\, \rho\big(\widetilde{\text{Ḓ}\varpi}(\mathrm{s})\big) \circ \varpi_v\, \rho(\text{ŧ}) - \varpi_v\, \rho\big(\tilde{\varpi}(\text{ŧ})\big) \circ \varpi_v\, \rho\big(\tilde{\text{Ḓ}}(\mathrm{s})\big) - \sum_{i=1}^{n-1} \varpi_v\, \rho\begin{pmatrix} \tilde{\varpi}(\text{ŧ}_1), \dots, \tilde{\varpi}(\text{ŧ}_{n-1}), [\text{Ḓ}\tilde{\varpi}(\mathrm{s}_1), \dots, \text{Ḓ}\,\tilde{\varpi}(\mathrm{s}_{n-1}), \varpi(\text{ŧ}_i)]_{\lambda g} \\ , \tilde{\varpi}(\text{ŧ}_{i+1}), \dots, \tilde{\varpi}(\text{ŧ}_{n-1}) \end{pmatrix} \circ \varpi_v$$

$$= {\varpi_v}^2 \rho\big(\tilde{\text{Ḓ}}(\mathrm{s})\big) \circ \rho(\text{ŧ}) - {\varpi_v}^2 \rho(\text{ŧ}) \circ \rho\big(\tilde{\text{Ḓ}}(\mathrm{s})\big) - \sum_{i=1}^{n-1} {\varpi_v}^2\, \rho\begin{pmatrix} (\text{ŧ}_1), \dots, (\text{ŧ}_{i-1}), [\text{Ḓ}(\mathrm{s}_1), \dots, \text{Ḓ}(\mathrm{s}_{n-1}), (\text{ŧ}_i)]_{\lambda g}, \\ (\text{ŧ}_{i+1}), \dots, (\text{ŧ}_{n-1}) \end{pmatrix}$$

$$= {\varpi_v}^2\big(\rho(\tilde{\text{Ḓ}}(\mathrm{s}) \circ \rho(\text{ŧ}) - \rho(\text{ŧ}) \circ \rho(\tilde{\text{Ḓ}}(\mathrm{s}) - \sum_{i=1}^{n-1} \rho\big((\text{ŧ}_1), \dots, (\text{ŧ}_{n-1})\big)[\text{Ḓ}(\mathrm{s}_1), \dots, \text{Ḓ}(\mathrm{s}_{n-1}), (\text{ŧ}_i)]_{\lambda g}, (\text{ŧ}_{i+1}), \dots, (\text{ŧ}_{n-1})\big)$$

$= 0$

4-

$$\tilde{\rho}\big(\tilde{\varpi}(\text{ŧ}_2), \dots, \tilde{\varpi}(\text{ŧ}_{n-1}), [\tilde{\varpi}(\mathrm{s}_1), \dots, \tilde{\varpi}(\mathrm{s}_{n-1}), \text{ŧ}_1]_{\lambda\text{Ḓ}\varpi}\big) \circ \varpi_v - \sum_{i=1}^{n-1} (-1)^{n-i}\, \tilde{\rho}\Big(\text{Ḓ}\,\tilde{\varpi}(\mathrm{s}_1), \dots, \widehat{\text{Ḓ}\tilde{\varpi}}(\mathrm{s}_i), \dots, \text{Ḓ}\,\tilde{\varpi}(\mathrm{s}_{n-1}), \tilde{\varpi}(\text{ŧ}_1)\Big) \circ \tilde{\rho}\big(\text{ŧ}_2, \dots, \text{ŧ}_{n-1}, \text{Ḓ}(\mathrm{s}_i)\big) - \tilde{\rho}\Big(\widetilde{\text{Ḓ}\varpi}(\mathrm{s})\Big) \circ \tilde{\rho}(\text{ŧ})$$

$$= - \sum_{i=1}^{n-1} (-1)^{n-i}\, \varpi_v\, \rho\Big(\text{Ḓ}\,\tilde{\varpi}(\mathrm{s}_1), \dots, \widehat{\text{Ḓ}\tilde{\varpi}}(\mathrm{s}_i), \dots, \text{Ḓ}\,\tilde{\varpi}(\mathrm{s}_{n-1}), \tilde{\varpi}(\text{ŧ}_1)\Big) \circ \varpi_v\, \rho\big(\text{ŧ}_2, \dots, \text{ŧ}_{n-1}, \text{Ḓ}(\mathrm{s}_i)\big) - \varpi_v\, \rho\Big(\widetilde{\text{Ḓ}\varpi}(\mathrm{s})\Big) \circ \varpi_v\, \rho(\text{ŧ})$$

$$= {\varpi_v}^2 \rho\Big((\text{ŧ}_2), \dots, (\text{ŧ}_{n-1}), [\text{Ḓ}(\mathrm{s}_1), \dots, \text{Ḓ}(\mathrm{s}_{n-1}), \text{ŧ}_1]_{\lambda g}\Big) - \sum_{i=1}^{n-1} (-1)^{n-i}\, {\varpi_v}^2\, \rho\big(\text{Ḓ}(\mathrm{s}_1), \dots, \widehat{\text{Ḓ}}(\mathrm{s}_i), \dots, \text{Ḓ}(\mathrm{s}_{n-1}), \text{ŧ}_1\big) \circ \rho\big(\text{ŧ}_2, \dots, \text{ŧ}_{n-1}, \text{Ḓ}(\mathrm{s}_i)\big) - {\varpi_v}^2\, \rho\Big(\tilde{\text{Ḓ}}(\mathrm{s})\Big) \circ \rho(\text{ŧ})$$

$$= {\varpi_v}^2 \left(\rho\Big((\text{ŧ}_2), \dots, (\text{ŧ}_{n-1}), [\text{Ḓ}(\mathrm{s}_1), \dots, \text{Ḓ}(\mathrm{s}_{n-1}), \text{ŧ}_1]_{\lambda g}\Big) - \sum_{i=1}^{n-1} (-1)^{n-i}\, \rho\big(\text{Ḓ}(\mathrm{s}_1), \dots, \widehat{\text{Ḓ}}(\mathrm{s}_i), \dots, \text{Ḓ}(\mathrm{s}_{n-1}), \text{ŧ}_1\big) \rho\big(\text{ŧ}_2, \dots, \text{ŧ}_{n-1}, \text{Ḓ}(\mathrm{s}_i)\big) - \rho\Big(\tilde{\text{Ḓ}}(\mathrm{s})\Big) \circ \rho(\text{ŧ}) \right) = 0$$

Definition 2.6:-

Let $\big(g, [.,\dots,]_{\lambda g}, \text{Ḓ}, \varpi\big)$ be an n-Bi-Hom Γ-Lie algebra. A bilinear form f over g is told to be nondegenerate if $g^{\perp} = \{\mathrm{s} \in g / f(\mathrm{s},\text{ŧ}) = 0, \forall\, \text{ŧ} \in g\} = 0$;

$\text{Ḓ}\varpi$-invariant if for every $\mathrm{s}_1, \dots, \mathrm{s}_{n+1} \in g$,

$$f\Big([\varpi(\mathrm{s}_1), \dots, \varpi(\mathrm{s}_{n-1}), \text{Ḓ}(\mathrm{s}_n)]_{\lambda g}, \text{Ḓ}(\mathrm{s}_{n+1})\Big) = -f\big(\text{Ḓ}(\mathrm{s}_n), [\varpi(\mathrm{s}_1), \dots, \varpi(\mathrm{s}_{n-1}), \text{Ḓ}(\mathrm{s}_{n+1})]_{\lambda g}\big);$$

symmetric if $f(\mathrm{s},\text{ŧ}) = f(\text{ŧ},\mathrm{s})$. Ḓ is invite f symmetric , if $f(\text{Ḓ}(\mathrm{s}),\text{ŧ}) = f\big(\mathrm{s},\text{Ḓ}(\text{ŧ})\big)$ .The sub space I from g be invited isotropic if $I \subseteq I^{\perp}$.

Definition 2.7:-

Assume that $\big(g, [.,\dots,]_{\lambda g}, \text{Ḓ}, \varpi\big)$ is an n-Bi-Hom Γ-Lie-algebra on a field K. if g is admits a nondegenerate, $\text{Ḓ}\,\varpi$-

invariant and symmetric-bilinear form f such $\underset{\frown}{D}, \varpi$ are f-symmetrics, then we name $(g, f, \underset{\frown}{D}, \varpi)$ a quadratic n-Bi-Hom Γ-Lie algebra. Assume $\left(\acute{g}, [.,\dots,]_{\lambda\acute{g}}, \acute{\underset{\frown}{D}}, \acute{\varpi}\right)$ be another n-Bi-Hom Γ-$Lie\ algebra$. Two square n-Bi-Hom Γ-Liealgebras $(g, f, \underset{\frown}{D}, \varpi)\ and\ \left(\acute{g}, \acute{f}, \acute{\underset{\frown}{D}}, \acute{\varpi}\right)$ are told to be isometric if there exist algebra isomorphism $\emptyset: g \to \acute{g}$ such that
$f(\mathrm{s}, ŧ) = \acute{f}\left(\emptyset(\mathrm{s}), \emptyset(ŧ)\right)$, for all $\mathrm{s}, ŧ \in g$.

Theorem 2.8:-

Let $\left(g, [.,\dots,]_{\lambda g}, \underset{\frown}{D}, \varpi\right)$ be an n-Bi-Hom Γ-Lie-algebra and $(V, \rho, \underset{\frown}{D}_v, \varpi_v)$ a representation from g. Assume that V^* is the dual space of V and $\widetilde{\underset{\frown}{D}}_v, \widetilde{\varpi}_v: V^* \to V^*$ 2-Homomorphisms defined by $\widetilde{\underset{\frown}{D}}_v(f) = f \circ \underset{\frown}{D}_v$, $\widetilde{\varpi}_v(f) = f \circ \varpi_v$, $\forall f \in V^*$. Then the skew-symmetry linear-function $\tilde{\rho}: \wedge^{n-1} \to \mathrm{End}(V^*)$, define by

$\tilde{\rho}(\mathrm{s}_1, \dots, \mathrm{s}_{n-1})(f) = -f \circ \rho(\mathrm{s}_1, \dots, \mathrm{s}_{n-1})$, $\forall f \in V^*$, $(\mathrm{s}_1, \dots, \mathrm{s}_{n-1}) \in g$ be a representation from g over $\left(V^*, \tilde{\rho}, \widetilde{\underset{\frown}{D}}_v, \widetilde{\varpi}_v\right)$ if and only if for all $\mathrm{s} = \mathrm{s}_1 \wedge \dots \wedge \mathrm{s}_{n-1}$, $ŧ = ŧ_1 \wedge \dots \wedge ŧ_{n-1} \in \wedge^{n-1} g$,

1- $\underset{\frown}{D}_v \circ \rho\left(\widetilde{\underset{\frown}{D}}(\mathrm{s})\right) = \rho(\mathrm{s}) \circ \underset{\frown}{D}_v$,

2- $\varpi_v \circ \rho\left(\widetilde{\varpi}(\mathrm{s})\right) = \rho(\mathrm{s}) \circ \varpi_v$,

3- $\rho(ŧ) \circ \rho\left(\widetilde{\underset{\frown}{D}\varpi}(\mathrm{s})\right) - \rho\left(\widetilde{\underset{\frown}{D}}(\mathrm{s})\right) \circ \rho\left(\widetilde{\varpi}(ŧ)\right)$

$$= -\sum_{i=1}^{n-1} \varpi_v\, \rho\begin{pmatrix} \varpi(ŧ_1), \dots, \varpi(ŧ_{i-1}), [\varpi(\mathrm{s}_1), \dots, \varpi(\mathrm{s}_{n-1}), ŧ_i]_{\lambda g} \\ , \varpi(ŧ_{i+1}), \dots \varpi(ŧ_{n-1}) \end{pmatrix}$$

4- $\varpi_v\, \rho\left([\varpi(\mathrm{s}_1), \dots, \varpi(\mathrm{s}_{n-1}), ŧ_i]_{\lambda g}, \varpi(ŧ_2), \dots \varpi(ŧ_{n-1})\right) =$

$$-\sum_{i=1}^{n-1} (-1)^{n-i} \rho\left(ŧ_2, \dots, ŧ_{n-1}, \underset{\frown}{D}(\mathrm{s}_i)\right)$$
$$\circ\, \rho\left(\underset{\frown}{D}\,\varpi(\mathrm{s}_1), \dots, \widehat{\underset{\frown}{D}\,\varpi}(\mathrm{s}_i), \dots, \underset{\frown}{D}\,\varpi(\mathrm{s}_{n-1}), \varpi(ŧ_1)\right) - \rho(ŧ)$$
$$\circ\, \rho\left(\widetilde{\underset{\frown}{D}\varpi}(\mathrm{s})\right).$$

Proof :-

Let $f \in V^*, \mathrm{s} = \mathrm{s}_1 \wedge \dots \wedge \mathrm{s}_{n-1}$, $ŧ = ŧ_1 \wedge \dots \wedge ŧ_{n-1} \in \wedge^{n-1} g$.

First, we have

$\left(\tilde{\rho}\left(\underset{\frown}{D}(\mathrm{s}_1), \dots, \underset{\frown}{D}(\mathrm{s}_{n-1})\right) \circ \widetilde{\underset{\frown}{D}}_v\right)(f) = -\widetilde{\underset{\frown}{D}}_v(f) \circ \rho\left(\underset{\frown}{D}(\mathrm{s}_1), \dots, \underset{\frown}{D}(\mathrm{s}_{n-1})\right) = -f \circ \underset{\frown}{D}_v \circ \rho\left(\underset{\frown}{D}(\mathrm{s}_1), \dots, \underset{\frown}{D}(\mathrm{s}_{n-1})\right)$

and

$\widetilde{\underset{\frown}{D}}_v \circ \tilde{\rho}(\mathrm{s}_1, \dots, \mathrm{s}_{n-1})(f) = -\widetilde{\underset{\frown}{D}}_v\left(f \circ \rho(\mathrm{s}_1, \dots, \mathrm{s}_{n-1})\right) = -f \circ \rho(\mathrm{s}_1, \dots, \mathrm{s}_{n-1}) \circ \underset{\frown}{D}_v$, which implies

$\tilde{\rho}\left(\underset{\frown}{D}(\mathrm{s}_1), \dots, \underset{\frown}{D}(\mathrm{s}_{n-1})\right) \circ \widetilde{\underset{\frown}{D}}_v = \widetilde{\underset{\frown}{D}}_v \circ \tilde{\rho}(\mathrm{s}_1, \dots, \mathrm{s}_{n-1}) \Leftrightarrow \underset{\frown}{D}_v \circ \rho\left(\underset{\frown}{D}(\mathrm{s}_1), \dots, \underset{\frown}{D}(\mathrm{s}_{n-1})\right) = \rho(\mathrm{s}_1, \dots, \mathrm{s}_{n-1}) \circ \underset{\frown}{D}_v$.

Similarly ,

$\tilde{\rho}(\varpi(\mathrm{s}_1), \dots, \varpi(\mathrm{s}_{n-1})) \circ \widetilde{\varpi}_v = \widetilde{\varpi}_v \circ \tilde{\rho}(\mathrm{s}_1, \dots, \mathrm{s}_{n-1}) \Leftrightarrow \varpi_v \circ \rho\left(\underset{\frown}{D}(\mathrm{s}_1), \dots, \underset{\frown}{D}(\mathrm{s}_{n-1})\right) = \rho(\mathrm{s}_1, \dots, \mathrm{s}_{n-1}) \circ \varpi_v$.

Then we can get

$\tilde{\rho}\left(\underset{\frown}{D}\,\varpi(\mathrm{s}_1), \dots, \underset{\frown}{D}\,\varpi(\mathrm{s}_{n-1})\right) \circ \tilde{\rho}(ŧ_1, \dots, ŧ_{n-1})(f)$
$= -\tilde{\rho}\left(\underset{\frown}{D}\varpi(\mathrm{s}_1), \dots, \underset{\frown}{D}\varpi(\mathrm{s}_{n-1})\right)\left(f\rho(ŧ_1, \dots, ŧ_{n-1})\right)$
$= f\rho(ŧ_1, \dots, ŧ_{n-1})\, \rho\left(\underset{\frown}{D}\,\varpi(\mathrm{s}_1), \dots, \underset{\frown}{D}\,\varpi(\mathrm{s}_{n-1})\right)$ and
$\tilde{\rho}\left(\varpi(ŧ_1), \dots, \varpi(ŧ_{n-1})\right) \circ \tilde{\rho}\left(\underset{\frown}{D}(\mathrm{s}_1), \dots, \underset{\frown}{D}(\mathrm{s}_{n-1})\right)$

$$+\sum_{i=1}^{n-1} \tilde{\rho}\begin{pmatrix} \varpi(ŧ_1), \dots, \varpi(ŧ_{i-1}), [\varpi(\mathrm{s}_1), \dots, \varpi(\mathrm{s}_{n-1}), ŧ_i]_{\lambda g}, \varpi(ŧ_{i+1}), \dots, \\ \varpi(ŧ_{n-1}) \circ \widetilde{\varpi}_v \end{pmatrix}(f)$$

$= f\rho\left(\underset{\frown}{D}(\mathrm{s}_1), \dots, \underset{\frown}{D}(\mathrm{s}_{n-1})\right)\rho\left(\varpi(ŧ_1), \dots, \varpi(ŧ_{n-1})\right)$

$$-\sum_{i=1}^{n-1} f\varpi_v\, \rho\begin{pmatrix} \varpi(ŧ_1), \dots, \varpi(ŧ_{i-1}), [\varpi(\mathrm{s}_1), \dots, \varpi(\mathrm{s}_{n-1}), ŧ_i]_{\lambda g} \\ , \varpi(ŧ_{i+1}), \dots, \varpi(ŧ_{n-1}) \end{pmatrix}$$

which implies that

$\tilde{\rho}\left(\underset{\frown}{D}\varpi(\mathrm{s}_1), \dots, \underset{\frown}{D}\varpi(\mathrm{s}_{n-1})\right) \circ \tilde{\rho}(ŧ_1, \dots, ŧ_{n-1})$
$= \tilde{\rho}\left(\varpi(ŧ_1), \dots, \varpi(ŧ_{n-1})\right)$
$\circ\, \tilde{\rho}\left(\underset{\frown}{D}(\mathrm{s}_1), \dots, \underset{\frown}{D}(\mathrm{s}_{n-1})\right)$

$$+\sum_{i=1}^{n-1} \tilde{\rho}\begin{pmatrix} \varpi(ŧ_1), \dots, \varpi(ŧ_{i-1}), [\varpi(\mathrm{s}_1), \dots, \varpi(\mathrm{s}_{n-1}), ŧ_i]_{\lambda g}, \\ \varpi(ŧ_{i+1}), \dots, \varpi(ŧ_{n-1}) \circ \widetilde{\varpi}_v \end{pmatrix}$$

If and only if

$\rho(ŧ_1, \dots, ŧ_{n-1})\rho\left(\underset{\frown}{D}\varpi(\mathrm{s}_1), \dots, \underset{\frown}{D}\varpi(\mathrm{s}_{n-1})\right)$
$= \rho\left(\underset{\frown}{D}(\mathrm{s}_1), \dots, \underset{\frown}{D}(\mathrm{s}_{n-1})\right)\rho\left(\varpi(ŧ_1), \dots, \varpi(ŧ_{n-1})\right)$

$$-\sum_{i=1}^{n-1} \varpi_v\, \rho\begin{pmatrix} \varpi(ŧ_1), \dots, \varpi(ŧ_{i-1}), [\varpi(\mathrm{s}_1), \dots, \varpi(\mathrm{s}_{n-1}), ŧ_i]_{\lambda g} \\ , \varpi(ŧ_{i+1}), \dots, \varpi(ŧ_{n-1}) \end{pmatrix}$$

In the same way

$\tilde{\rho}\left([\varpi(\mathrm{s}_1), \dots, \varpi(\mathrm{s}_{n-1}), ŧ_1]_{\lambda g}, \varpi(ŧ_2), \dots, \varpi(ŧ_{n-1})\right)\widetilde{\varpi}_v$
$=$
$\sum_{i=1}^{n-1}(-1)^{n-i}\, \tilde{\rho}\left(\underset{\frown}{D}\varpi(\mathrm{s}_1), \dots, \widehat{\underset{\frown}{D}\varpi}(\mathrm{s}_i), \dots, \underset{\frown}{D}\varpi(\mathrm{s}_{n-1}), \varpi(ŧ_1)\right) \circ \tilde{\rho}\left(ŧ_2, \dots, ŧ_{n-1}, \underset{\frown}{D}(\mathrm{s}_i)\right) + \tilde{\rho}\left(\underset{\frown}{D}\varpi(\mathrm{s}_1), \dots, \underset{\frown}{D}\varpi(\mathrm{s}_{n-1})\right) \circ \tilde{\rho}(ŧ_1, \dots, ŧ_{n-1})$.

If and only if

$\varpi_v\, \rho\left([\varpi(\mathrm{s}_1), \dots, \varpi(\mathrm{s}_{n-1}), ŧ_1]_{\lambda g}, \varpi(ŧ_2), \dots, \varpi(ŧ_{n-1})\right)$

$$= -\sum_{i=1}^{n-1}(-1)^{n-i} \rho\left(ŧ_2, \dots, ŧ_{n-1}, \underset{\frown}{D}(\mathrm{s}_i)\right)$$
$$\circ\, \rho\left(\underset{\frown}{D}\varpi(\mathrm{s}_1), \dots, \widehat{\underset{\frown}{D}\varpi}(\mathrm{s}_i), \dots, \underset{\frown}{D}\varpi(\mathrm{s}_{n-1}), \varpi(ŧ_1)\right)$$
$$-\rho(ŧ_1, \dots, ŧ_{n-1}) \circ \rho\left(\underset{\frown}{D}\varpi(\mathrm{s}_1), \dots, \underset{\frown}{D}\varpi(\mathrm{s}_{n-1})\right).$$

It is clear that the theorem has been verified

Corollary 2.9:-

Assume ad is the adjoint $representation$ from an n-Bi-Hom Γ- Liealgebra $\left(g, [.,\dots,]_{\lambda g}, \underset{\frown}{D}, \varpi\right)$ .Assume that the bilinear function $ad^*: \wedge^{n-1} g \to \mathrm{End}(g^*)$ defined by $ad^*(\mathrm{s}_1, \dots, \mathrm{s}_{n-1})(f) = -f \circ ad(\mathrm{s}_1, \dots, \mathrm{s}_{n-1})$, for all $\mathrm{s}_1, \dots, \mathrm{s}_{n-1} \in g$ .Then $ad^*\ is\ a$ representation of g on $(g^*, ad^*, \overline{\underset{\frown}{D}}, \overline{\varpi})$ if and only if

1- $\underset{\frown}{D} \circ ad\left(\widetilde{\underset{\frown}{D}}(\mathrm{s})\right) = ad(\mathrm{s}) \circ \underset{\frown}{D}$,

2- $\varpi \circ ad\left(\widetilde{\varpi}(\mathrm{s})\right) = ad(\mathrm{s}) \circ \varpi$,

3- $ad(ŧ) \circ ad\left(\widetilde{\underset{\frown}{D}\varpi}(\mathrm{s})\right) - ad\left(\widetilde{\underset{\frown}{D}}(\mathrm{s})\right) \circ ad\left(\widetilde{\varpi}(ŧ)\right)$

4- $\varpi\ ad\left([\varpi(\mathrm{s}_1), \dots, \varpi(\mathrm{s}_{n-1}), ŧ_i]_{\lambda g}, \varpi(ŧ_2), \dots \varpi(ŧ_{n-1})\right) =$

$$-\sum_{i=1}^{n-1}(-1)^{n-i} ad\left(ŧ_2, \dots, ŧ_{n-1}, \underset{\frown}{D}(\mathrm{s}_i)\right)$$
$$\circ\, ad\left(\underset{\frown}{D}\varpi(\mathrm{s}_1), \dots, \widehat{\underset{\frown}{D}\varpi}(\mathrm{s}_i), \dots, \underset{\frown}{D}\varpi(\mathrm{s}_{n-1}), \varpi(ŧ_1)\right) - ad(ŧ)$$
$$\circ\, ad\left(\widetilde{\underset{\frown}{D}\varpi}(\mathrm{s})\right).$$

Proof:-

Let $f \in g^*, \mathrm{s} = \mathrm{s}_1 \wedge \dots \wedge \mathrm{s}_{n-1}$, $ŧ = ŧ_1 \wedge \dots \wedge ŧ_{n-1} \in \wedge^{n-1} g$. First, we have

$\left(ad^*\left(\underset{\wedge}{D}(s_1),\dots,\underset{\wedge}{D}(s_{n-1})\right)\circ\tilde{\underset{\wedge}{D}}\right)(f) = -\tilde{\underset{\wedge}{D}}(f)\circ ad\left(\underset{\wedge}{D}(s_1),\dots,\underset{\wedge}{D}(s_{n-1})\right) = -f\circ\underset{\wedge}{D}\circ ad\left(\underset{\wedge}{D}(s_1),\dots,\underset{\wedge}{D}(s_{n-1})\right)$
and
$\tilde{\underset{\wedge}{D}}\circ ad^*(s_1,\dots,s_{n-1})(f) = -\tilde{\underset{\wedge}{D}}\left(f\circ ad(s_1,\dots,s_{n-1})\right) = -f\circ ad(s_1,\dots,s_{n-1})\circ\underset{\wedge}{D}$. Then
$ad^*\left(\underset{\wedge}{D}(s_1),\dots,\underset{\wedge}{D}(s_{n-1})\right)\circ\tilde{\underset{\wedge}{D}} = \tilde{\underset{\wedge}{D}}\circ ad^*(s_1,\dots,s_{n-1}) \Leftrightarrow \underset{\wedge}{D}\circ ad\left(\underset{\wedge}{D}(s_1),\dots,\underset{\wedge}{D}(s_{n-1})\right) = ad(s_1,\dots,s_{n-1})\circ\underset{\wedge}{D}$.
The same way ,
$ad^*\left(\varpi(s_1),\dots,\varpi(s_{n-1})\right)\varpi(s_1)\circ\tilde{\varpi} = \tilde{\varpi}\circ ad^*(s_1,\dots,s_{n-1}) \Leftrightarrow \varpi\circ ad\left(\underset{\wedge}{D}(s_1),\dots,\underset{\wedge}{D}(s_{n-1})\right) = ad(s_1,\dots,s_{n-1})\circ\varpi$. There after we have

$$ad^*\left(\underset{\wedge}{D}\varpi(s_1),\dots,\underset{\wedge}{D}\varpi(s_{n-1})\right)\circ ad^*(t_1,\dots,t_{n-1})(f)$$
$$= -ad^*\left(\underset{\wedge}{D}\varpi(s_1),\dots,\underset{\wedge}{D}\varpi(s_{n-1})\right)\left(f\ ad(t_1,\dots,t_{n-1})\right)$$
$$= f\ ad(t_1,\dots,t_{n-1})\ ad\left(\underset{\wedge}{D}\varpi(s_1),\dots,\underset{\wedge}{D}\varpi(s_{n-1})\right)$$
and
$$ad^*\left(\varpi(t_1),\dots,\varpi(t_{n-1})\right)\circ ad^*\left(\underset{\wedge}{D}(s_1),\dots,\underset{\wedge}{D}(s_{n-1})\right)$$
$$+\sum_{i=1}^{n-1} ad^*\begin{pmatrix}\varpi(t_1),\dots,\varpi(t_{i-1}),[\varpi(s_1),\dots,\varpi(s_{n-1}),t_i]_{\lambda g},\\ \varpi(t_{i+1}),\dots,\varpi(t_{n-1})\circ\tilde{\varpi}\end{pmatrix}(f)$$
$$= f\ ad\left(\underset{\wedge}{D}(s_1),\dots,\underset{\wedge}{D}(s_{n-1})\right)ad\left(\varpi(t_1),\dots,\varpi(t_{n-1})\right)$$
$$-\sum_{i=1}^{n-1} f\ \varpi\ ad\begin{pmatrix}\varpi(t_1),\dots,\varpi(t_{i-1}),[\varpi(s_1),\dots,\varpi(s_{n-1}),t_i]_{\lambda g}\\ ,\varpi(t_{i+1}),\dots,\varpi(t_{n-1})\end{pmatrix}$$
which implies that
$$ad^*\left(\underset{\wedge}{D}\varpi(s_1),\dots,\underset{\wedge}{D}\varpi(s_{n-1})\right)\circ ad^*(t_1,\dots,t_{n-1}) = ad^*\left(\varpi(t_1),\dots,\varpi(t_{n-1})\right)\circ ad^*\left(\underset{\wedge}{D}(s_1),\dots,\underset{\wedge}{D}(s_{n-1})\right)$$
$$+\sum_{i=1}^{n-1} ad^*\begin{pmatrix}\varpi(t_1),\dots,\varpi(t_{i-1}),[\varpi(s_1),\dots,\varpi(s_{n-1}),t_i]_{\lambda g},\\ \varpi(t_{i+1}),\dots,\varpi(t_{n-1})\circ\tilde{\varpi}\end{pmatrix}$$
If and only if
$$ad(t_1,\dots,t_{n-1})\ ad\left(\underset{\wedge}{D}\varpi(s_1),\dots,\underset{\wedge}{D}\varpi(s_{n-1})\right)$$
$$= ad\left(v(s_1),\dots,\underset{\wedge}{D}(s_{n-1})\right)ad\left(\varpi(t_1),\dots,\varpi(t_{n-1})\right)$$
$$-\sum_{i=1}^{n-1}\varpi\ ad\begin{pmatrix}\varpi(t_1),\dots,\varpi(t_{i-1}),[\varpi(s_1),\dots,\varpi(s_{n-1}),t_i]_{\lambda g}\\ ,\varpi(t_{i+1}),\dots,\varpi(t_{n-1})\end{pmatrix}$$
In the same way
$$ad^*\left([\varpi(s_1),\dots,\varpi(s_{n-1}),t_1]_{\lambda g},\varpi(t_2),\dots,\varpi(t_{n-1})\right)\tilde{\varpi}$$
$$= \sum_{i=1}^{n-1}(-1)^{n-i}\ ad^*\left(\underset{\wedge}{D}\varpi(s_1),\dots,\widehat{\underset{\wedge}{D}\varpi}(s_i),\dots,\underset{\wedge}{D}\varpi(s_{n-1}),\varpi(t_1)\right)$$
$$\circ\ ad^*\left(t_2,\dots,t_{n-1},\underset{\wedge}{D}(s_i)\right) + ad^*\left(\underset{\wedge}{D}\varpi(s_1),\dots,\underset{\wedge}{D}\varpi(s_{n-1})\right)\circ ad^*(t_1,\dots,t_{n-1}).$$
If and only if
$$\varpi\ ad\left([\varpi(s_1),\dots,\varpi(s_{n-1}),t_1]_{\lambda g},\varpi(t_2),\dots,\varpi(t_{n-1})\right)$$
$$= -\sum_{i=1}^{n-1}(-1)^{n-i}\ ad\left(t_2,\dots,t_{n-1},\underset{\wedge}{D}(s_i)\right)$$
$$\circ\ ad\left(\underset{\wedge}{D}\varpi(s_1),\dots,\widehat{\underset{\wedge}{D}\varpi}(s_i),\dots,\underset{\wedge}{D}\varpi(s_{n-1}),\varpi(t_1)\right)$$
$$-ad(t_1,\dots,t_{n-1})\circ ad\left(\underset{\wedge}{D}\varpi(s_1),\dots,\underset{\wedge}{D}\varpi(s_{n-1})\right).$$

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