FACTORIZATION OF SEPARABLE VARIABLE FUNCTION
FOR TWO AND THREE VARIABLES

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ABSTRACT
In this paper, we give an alternate proof to the main theorem of [1], then generalize the result to cover arbitrary functions, and we study the separation of any function with three variables. Also, we introduce a way to factorize such function.

Introduction 1
Scott [2] in 1985 gave a necessary and sufficient condition for a function $f(x,y)$ to be of separable variable. His result, of course, is very important; especially in the area of differential equations and integration theory. In this work we give a way to factorize such separable functions in the polynomial, as in the general case. Also we give a necessary and sufficient condition for a function $f(x,y,z)$ to be of separable variables and factorize such function. We start the theorem by Scott [2] and for the proof see [2] or ( [3] page 37 – 38 ).

Proposition 1.1
1) Suppose that $f$ is a separable variable function on a domain D; that is $f(x,y) = \varphi(x) \psi(y)$. If $\varphi$ and $\psi$ are differentiable, then
$$\frac{\partial^2 f(x,y)}{\partial x \partial y} = \left( \frac{\partial f(x,y)}{\partial x} \right) \left( \frac{\partial f(x,y)}{\partial y} \right)$$ (1)
2) Suppose that $f$, $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial^2 f}{\partial x \partial y}$ exist and are continuous in a domain D. If $f \neq 0$ and (1) above holds, then $f$ is a separable function.

Factorization of $f(x,y)$ 2:
We first begin with the theorem of [1], which gives a simple way how to factorize a polynomial function if it a separable variable and establish an alternative and easier proof.

Definition 2.1:
Let $m$ and $n$ be a nonnegative integers. The function
$$f(x) = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} x^i y^j$$
where $a_{ij}$ are constants, is called a polynomial function of degree $(m + n)$ in $x$ and $y$. 
Theorem 2.2:

Let \( f \) be a separable variable polynomial function such that \( a_{mn} = 1 \). Then 
\[
\varphi(x) = \frac{1}{n!} \left[ \partial^n f(x,y) / \partial y^n \right], \\
\psi(y) = \frac{1}{m!} \partial^m f(x,y) / \partial x^m.
\]

Proof: It is clear that 
\[
f(x,y) = \varphi(x). \psi(y) = \\
(x^m + b_{m-1} x^{m-1} + \ldots + b_1 x + b_0) (y^n + c_{n-1} + \ldots + c_1 y + c_0),
\]
since \( a_{mn} = 1 \). By taking the \( n \)th derivative of \( f \) with respect to \( y \), we get 
\[
\partial^n f(x,y) / \partial y^n = n! \varphi(x). 
\]
Similarly we obtain \( \partial^m f(x,y) / \partial x^m = m! \psi(y) \).

The proof is completed.

Next we generalize the above theorem to any separable variable function.

Theorem 2.3:

Let \( f \) be a separable variable function satisfying \( f(x_0,y_0) = 1 \), for some point \( (x_0,y_0) \) in a domain \( D \). Then 
\[
f(x,y) = \varphi(x). \psi(y), \\
\varphi(x) = f(x,y_0) \quad \text{and} \quad \psi(y) = f(x_0,y).
\]

Proof:

Since \( f(x_0,y_0) = 1 \), we can take \( \varphi(x_0) = 1 \) and \( \psi(y_0) = 1 \). Therefore 
\[
f(x,y_0) = \varphi(x) \psi(y_0) \quad \text{and} \quad f(x_0,y) = \varphi(x_0) \psi(y).
\]

Remark:

It is clear that the condition \( f(x_0,y_0) = 1 \) can be replaced by \( f(x_0,y_0) \neq 0 \).

Example:

Let \( f(x,y) = \frac{\cos(x - y) - \cos(x + y)}{2} \). It is easy to verify that \( f \) satisfies (1) and \( f(\pi/2, \pi/2) = 1 \).

Therefore \( \varphi(x) = f(x,\pi/2) = \frac{\cos(x - \pi/2) - \cos(x + \pi/2)}{2} = \frac{\sin(x) + \sin(x)}{2} = \sin(x) \) and 
\[
\psi(y) = f(\pi/2,y) = \frac{\cos(\pi/2 - y) - \cos(\pi/2 + y)}{2} = \frac{\sin(y) + \sin(y)}{2} = \sin(y).
\]

By using the trigonometric identities, we easily see that \( \varphi \) and \( \psi \) are the right factors of \( f \).

Separable of \( f(x,y,z) \): 

In this section, we introduce a theorem parallel to that in [2] but for three variables and factorize such function.
Theorem 3.1:

Let \( f \) be a three variable function defined and twice continuously differentiable on a region \( D \) of \( \mathbb{R}^3 \) and \( f(x,y,z) \neq 0 \) for each \((x,y,z) \in D\). Then \( f \) is of separable variable if and only if

\[
f_{xy} = f_x \cdot f_y, \quad f_{xz} = f_x \cdot f_z, \quad f_{yz} = f_y \cdot f_z. \tag{3.1}
\]

Proof:

If \( f \) is of separable, then \( f(x,y,z) = \mu(x) \cdot \nu(y) \cdot \omega(z) \) and it is easy to check that (3.1) hold. Conversely, suppose that (3.1) holds. From \( f_{xy} = f_x \cdot f_y \), we have \( f_{xy} / f_y = f_x / f \). By integrating both sides with respect to \( x \), we obtain

\[
\ln|f_y| = \ln|f| + C(y,z).
\]

This implies that

\[
f_y / f = \exp(C(y,z)).
\]

Again integrate with respect to \( y \) to get

\[
\ln|f| = C_2(y,z) + C_3(x,z),
\]

which yields \( f(x,y,z) = \varphi(y,z) \cdot \psi(x,z) \).

Next we use \( f_{xz} = f_x \cdot f_z \). Direct calculations show that

\[
\varphi \cdot \psi_x (\varphi_x \cdot \psi + \varphi \cdot \psi_x) = \varphi \cdot \psi (\varphi_x \cdot \psi_x + \varphi \cdot \psi_{xz})
\]

Hence we arrive, by simplification, at \( \psi_x = \psi \cdot \psi_{xz} \).

By applying Scott’s theorem (1) to \( \psi \), we get \( \psi(x,z) = \psi_1(x) \cdot \psi_2(z) \).

Similar calculation also lead to; \( \varphi(y,z) = \varphi_1(y) \cdot \varphi_2(z) \), by combing all, we conclude that \( f(x,y,z) = \mu(x) \cdot \nu(y) \cdot \omega(z) \).

Theorem 3.2:

Let \( f(x,y,z) \) be a variable separable function satisfying \( f(x_0,y_0,z_0) = 1 \) for some point \((x_0,y_0,z_0)\) in a domain \( D \). Then

\[
f(x,y,z) = \mu(x) \cdot \nu(y) \cdot \omega(z), \quad \text{where} \quad \mu(x) = f(x,y_0,z_0), \quad \nu(y) = f(x_0,y,z_0), \quad \omega(z) = f(x_0,y_0,z).
\]

Proof: The proof is similar to which given in (2.3).

Reference:


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في هذا البحث نتناول نموذج متعدد المتغيرات في البحث (1)، ثم نعمله لكل حالة واحدة فصل الدالة ثلاثة متغيرات والشروط التي يجب أن تكون في هذه الدالة، بالإضافة إلى إيجاد عواملها.