FACTORIZATION OF SEPARABLE VARIABLE FUNCTION FOR TWO AND THREE VARIABLES

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ABSTRACT

In this paper, we give an alternate proof to the main theorem of [1], then generalize the result to cover arbitrary functions, and we study the separation of any function with three variables. Also, we introduce a way to factorize such function.

Introduction 1

Scott [2] in 1985 gave a necessary and sufficient condition for a function f(x,y) to be of separable variable. His result, of course, is very important; especially in the area of differential equations and integration theory. In this work we give a way to factorize such separable functions in the polynomial, as in the general case. Also we give a necessary and sufficient condition for a function f(x,y,z) to be of separable variables and factorize such function. We start the theorem by Scott [2] and for the proof see [2] or ([3] page 37 – 38).

Proposition 1.1

1) Suppose that f is a separable variable function on a domain D; that is

$f(x,y) = \varphi(x)\,\psi(y)$. If φ and ψ are differentiable, then

$f(x,y).\ [\partial^2 f(x,y) / (\partial x\, \partial y)] = [(\partial f(x,y))/ (\partial x)].\ [(\partial f(x,y))/ (\partial y)]$ (1)

2) Suppose that f, $\partial f / \partial x$, $\partial f / \partial y$ and $\partial^2 f / \partial x \partial y$ exist and are continuous in a domain D. If $f \neq 0$ and (1) above holds, then f is a separable function.

Factorization of f(x,y) 2:

We first begin with the theorem of [1]. which gives a simple way how to factorize a polynomial function if it a separable variable and establish an alternative and easer proof.

Definition 2.1:

Let m and n be a nonnegative integers. The function

$$f(x) = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} x^i y^j$$

where a_{ij} are constants, is called a polynomial function of degree (m + n) in x and y.

Theorem 2.2:

Let f be a separable variable polynomial function such that $a_{mm} = 1$. Then $\varphi(x) = (1/n!), [\partial^n f(x,y) / \partial y^n]$, $\psi(y) = (1/m!) \partial^m f(x,y) / \partial x^m]$.

Proof: It is clear that

$f(x,y) = \varphi(x). \psi(y) =$

$(x^m + b_{m-1} x^{m-1} + \ldots + b_1 x + b_0)(y^n + c_{n-1} + \ldots + c_1 y + c_0)$,

since $a_{mn} = 1$. By taking the n^{th} derivative of f with respect to y, we get

$\partial^n f(x,y) / \partial y^n = n!\varphi(x)$. Similarly we obtain $\partial^m f(x,y) / \partial y^m = m!\psi(y)$.

The proof is completed.

Next we generalize the above theorem to any separable variable function.

Theorem 2.3:

Let f be a separable variable function satisfying $f(x_0,y_0) = 1$, for some point (x_0,y_0) in a domain D. Then $f(x,y) = \varphi(x). \psi(y)$, where $\varphi(x) = f(x,y_0)$ and $\psi(y) = f(x_0,y)$.

Proof:

Since $f(x_0,y_0) = 1$, we can take $\varphi(x_0) = 1$ and $\psi(y_0) = 1$. Therefore

$f(x,y_0) = \varphi(x) \psi(y_0)$ and $f(x_0,y) = \varphi(x_0) \psi(y)$.

Remark:

It is clear that the condition $f(x_0,y_0) = 1$ can be replaced by $f(x_0,y_0) \neq 0$.

Example:

Let $f(x,y) = [\cos(x - y) - \cos(x + y)] / 2$. It is easy to verify that f satisfies (1) and $f(\pi/2, \pi/2) = 1$.

Therefore $\varphi(x) = f(x,\pi/2) = [\cos(x - \pi/2) - \cos(x + \pi/2)] / 2 = [\sin(x) + \sin(x)]/2 = \sin(x)$ and $\psi(y) = f(\pi/2,y) = [\cos(\pi/2 - y) - \cos(\pi/2 + y)]/2 = [\sin(y) + \sin(y)]/2 = \sin(y)$.

By using the trigonometric identities, we easily see that φ and ψ are the right factors of f.

Separable of f(x,y,z) 3:

In this section, we introduce a theorem parallel to that in [2] but for three variables and factorize such function.

Theorem 3.1:

Let f be a three variable function defined and twice continuously differentiable on a region D of R^3 and $f(x,y,z) \neq 0$ for each $(x,y,z) \in D$. Then f is of separable variable if and only if

$$f.\ f_{xy} = f_x.\ f_y,\ f.\ f_{xz} = f_x.\ f_z\ ,\ f.\ f_{yz} = f_y.\ f_z\ . \qquad (3.1)$$

Proof:

If f is of separable, then $f(x,y,z) = \mu(x).\ \nu(y).\ \omega(z)$ and it is easy to check that (3.1) hold. Conversely, suppose that (3.1) holds. From $f.\ f_{xy} = f_x.\ f_y$, we have $f_{xy} / f_y = f_x / f$. By integrating both sides with respect to x, we obtain

$$\mathrm{Ln}\ |f_y| = \mathrm{Ln}\ |f| + C_1(y,z).$$

This implies that

$$f_y / f = \exp(\ C(y,z)).$$

Again integrate with respect to y to get

$\mathrm{Ln}\ |f| = C_2(y,z) + C_3(x,z)$, which yields $f(x,y,z) = \varphi(y,z).\ \psi(x,z)$.

Next we use f. fxz = fx. fz. Direct calculations show that

$\varphi.\ \psi_x\ (\varphi_z.\ \psi + \varphi.\ \psi_z\ ) = \varphi.\ \psi\ (\varphi_z.\ \psi_x + \varphi.\ \psi_{xz})$

Hence we arrive, by simplification, at $\psi_x.\ \psi_z = \psi.\ \psi_{xz}$.

By applying Scott's theorem (1) to ψ, we get $\psi(x,z) = \psi_1(x).\ \psi_2(z)$.

Similar calculation also lead to; $\varphi(y,z) = \varphi_1(y).\ \varphi_2(z)$, by combing all, we conclude that $f(x,y,z) = \mu(x).\ \nu(y).\ \omega(z)$.

Theorem 3.2:

Let f(x,y,z) be a variable separable function satisfying $f(x_0,y_0,z_0) = 1$ for some point (x_0,y_0,z_0) in a domain D. Then

$f(x,y,z) = \mu(x).\ \nu(y).\ \omega(z)$, where $\mu(x) = f(x,y_0,z_0)$, $\nu(y) = f(x_0,y,z_0)$.

$\omega(z) = f(x_0,y_0,z)$

Proof: The proof is similar to which given in (2.3).

Reference:

[1] Ali R. and G. Mosa; " Factorization of separable variable polynomials " Al Manarah J. 4 # 2 (1999), (43 – 49).

[2] Scott D.; " When is an ordinary differential equation separable " Ameri. Math. Monthly 92 (1985) (422 – 423).

[3] Derrick W. R. and S. I. Grossman; " A first course in differential equations " 3rd edition, West Publishing Company, U.S.A., 1987.

تحليل الدوال القابلة للفصل في متغيرين أو ثلاثة متغيرات

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الخلاصة

في هذا البحث سنتناول برهان موازي للمبرهنة الواردة في البحث (١)، ثم نعممه لكل دالة كذلك ندرس عملية فصل الدالة لثلاث متغيـــرات والشروط التي يجب أن تتوفر في هذه الدالة، بالإضافة إلى إيجاد عواملها.