BANACH ALGEBRA VALUED – MEASURE

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ABSTRACT
In this paper, we introduce the concept of Banach Algebra valued measure, several properties of this measure are proved.

1. Preliminaries

Definition (1.1)
Let $X$ be a real vector space. A partial order relation $\leq$ on $X$ is called vector order, if the following axioms are satisfied

1. $x \leq y \Rightarrow x + z \leq y + z \quad \forall x, y, z \in X$
2. $x \leq y \Rightarrow \lambda x \leq \lambda y \quad \forall x, y \in X \quad \text{and} \quad \lambda \geq 0$

A real vector space endowed with a vector order is called an ordered vector space.

An element $x$ of an ordered vector space $X$ is said to be positive if $x \geq 0$, and negative if $x \leq 0$. The set of all positive elements of an ordered vector space $X$ with be denoted by $X_+$, i.e. $X_+ = \{x \in X : x \geq 0\}$, $X_+$ is called the positive cone of $X$. It is easy to show that

1. $X_+$ is a convex cone of $X$, i.e. $X_+ + X_+ \subseteq X_+$ and $\lambda X_+ \subseteq X_+$
2. $X_+ \cap (-X_+) = \{0\}$.

Definition (1.2)
Let $X$ be a real vector space. A function $\| \cdot \| : X \to R$ is said to be norm on $X$ if the following axioms are satisfied

1. $\|x\| \geq 0 \quad \forall x \in X$
2. $\|x\| = 0 \iff x = 0$
3. $\|\lambda x\| = |\lambda| \|x\| \quad \forall x \in X, \quad \lambda \in R$
4. $\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in X$

A normed space is the pair $(X, \| \cdot \|)$ where $X$ is a real vector space and $\| \cdot \|$. A Banach space is a normed space which is complete in the metric defined by its norm.

Definition (1.3)
An algebra is a vector space in which a multiplication is defined that satisfies

1. $x(yz) = (xy)z \quad \forall x, y, z \in X$
2. $x(y + z) = xy + xz \quad (x + y)z = xz + yz \quad \forall x, y, z \in X$
3. $\lambda(xy) = (\lambda x)y = x(\lambda y) \quad \forall x, y, z \in X, \quad \lambda \in R$

Definition (1.4)
A real vector space $X$ is called Banach algebra if the following axioms are satisfied

1. $X$ is a Banach space
2. $X$ is algebra
3. $\exists e \in X$ s.t $ex = xe = x \quad \forall x \in X$ and $\|e\| = 1$
A commutative algebra is an algebra where the multiplication satisfies the condition: \( xy = yx \), \( \forall x, y \in X \).

An algebra with identity is an algebra with the following property. There exists a non-zero element in the algebra, denoted by \( 1 \) and called the multiplication identity element, such that \( |x = 1| = x \), for all \( x \).

A normed algebra \( X \), is a normed space and also an algebra over \( F \), and \( \|xy\| \leq \|x\|\|y\| \quad \forall x, y \in X \).

**Example (1.5)**

Let \( X \) be a Banach space, and let \( B(X) \) denote the set of all bounded (or continuous) linear function of \( X \) into itself, then \( B(X) \) is a Banach algebra with the algebra operation:

1. \( (f + g)(x) = f(x) + g(x) \)
2. \( (fg)(x) = f(g(x)) \)
3. \( (\lambda f)(x) = \lambda f(x) \)

And the operator norm \( \|f\| = \sup\{\|f(x)\|: x \in X, \|x\| \leq 1\} \)

**Remark**

If \( X \neq \{0\} \), then the identity linear function \( I \) is the identity element of \( B(X) \) such that \( \|I\| = 1 \).

**2. The main results**

**Definition (2.1)**

Let \( (\Omega, F) \) be a measurable space, and let \( X = (x,\leq) \) be an ordered Banach algebra. A set function \( \mu : F \longrightarrow X \) is said to be

1. **BA-finitely additive** if \( \mu(\bigcup_{k=1}^{n} A_k) = \sum_{k=1}^{n} \mu(A_k) \), whenever \( A_1, A_2, \ldots, A_n \) disjoint sets in \( F \).
2. **BA-countably additive** if \( \mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n) \), whenever \( \{A_n\} \) is a sequence of disjoint sets in \( F \).
3. **BA-measure** if \( \mu \) is BA-countably additive and \( \mu(A) \geq 0 \quad \forall A \in F \), where \( 0 \) is the identity element for the operation.

**Remark**

Every BA-countably additive is BA-finitely additive but converse is not true.

**Example (2.2)**

Let \( \Omega = (0,1) \), \( F \) the class of half-open intervals \( (a,b] \) where \( 0 \leq a \leq b \leq 1 \). Define \( \mu : F \longrightarrow \mathbb{R} \) by
\[ \mu(a,b) = \begin{cases} b-a & , \ a \neq 0 \\ 0 & , \ a = 0 \end{cases} \]

It is clear to show that \( \mu \) is BA-finitely additive.

Let \( A_n = (\frac{1}{n+1}, \frac{1}{n}] \), \( n = 1,2,3, \ldots \)

Then \( \{A_n\} \) is disjoint sequence in \( F \) and \( \bigcup_{n=1}^{\infty} A_n = (0,1] \)

\[ \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu(0,1] = 0 \]

\[ \sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^{\infty} \mu\left(\frac{1}{n+1}, \frac{1}{n}\right] = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 \]

\[ \Rightarrow \mu\left(\bigcup_{n=1}^{\infty} A_n\right) \neq \sum_{n=1}^{\infty} \mu(A_n) \]

\[ \Rightarrow \mu \text{ is not BA-countably additive} \]

The following theorem is the main result

**Theorem (2.3)**

Let \( \mu \) be an BA-finitely additive set function on measurable space \( (\Omega, F) \), and let \( A,B \in F \)

(1) \( \mu(A) = \mu(A \cap B) + \mu(A \mid B) \)

(2) \( \mu(A \cup B) = \mu(A \cap B) + \mu(A \mid B) + \mu(B \mid A) \)

(3) \( \mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B) \)

(4) \( \mu(A \Delta B) = \mu(A \mid B) + \mu(B \mid A) \)

(5) If \( A \subseteq B \), then \( \mu(A) = \mu(A \cap B) + \mu(A \mid B) \).

**Proof**

(1) Let \( A,B \in F \) then \( A \cap B \in F \) and \( A \mid B \in F \)

Since \( A \cap B \) and \( A \mid B \) are disjoint set, \( (A \cap B) \cup (A \mid B) = A \)

\( \mu(A) = \mu((A \cap B) \cup (A \mid B)) = \mu(A \cap B) + \mu(A \mid B) \)

(2) \( A,B \in F \Rightarrow A \cap B \in F \) , \( A \mid B \in F \) and \( B \mid A \in F \)

Since \( A \cap B \), \( A \mid B \) and \( B \mid A \) are disjoint set, and \( (A \cap B) \cup (A \mid B) \cup (B \mid A) = A \cup B \)

\( \mu(A \cup B) = \mu((A \cap B) \cup (A \mid B) \cup (B \mid A)) = \mu(A \cap B) + \mu(A \mid B) + \mu(B \mid A) \)

(3) \( \mu(A) = \mu(A \cap B) + \mu(A \mid B) \), \( \mu(B) = \mu(A \cap B) + \mu(B \mid A) \)

\( \mu(A) + \mu(B) = \mu(A \cap B) + \mu(A \mid B) + \mu(A \cap B) + \mu(A \mid B) + \mu(B \mid A) \)

\( = (\mu(A \cap B) + \mu(A \mid B) + \mu(B \mid A)) + \mu(A \cap B) \)

\( = \mu(A \cup B) + \mu(A \cap B) \)

(4) \( A,B \in F \Rightarrow A \mid B \), \( B \mid A \in F \)

\( A \mid B \) and \( B \mid A \) are disjoint sets, and \( A \Delta B = (A \mid B) \cup (B \mid A) \)

\( \mu(A \Delta B) = \mu((A \mid B) \cup (B \mid A)) = \mu(A \mid B) + (B \mid A) \)
(5) \( B = (A \cap B) \cup (B \mid A) \)

\[ A \subset B \Rightarrow A \cap B = A \Rightarrow B = A \cup (B \mid A) \Rightarrow \mu(B) = \mu(A \cup (B \mid A)) \]

Since \( A, B \mid A \in F \) and \( A, B \mid A \) are disjoint set

\[ \mu(A \cup (B \mid A)) = \mu(A) + \mu(B \mid A) \Rightarrow \mu(B) = \mu(A) + \mu(B \mid A) \]

References

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الخلاصة

في هذا البحث نقدم القياس ذات القيم في جبر بناخ ثم برهنة العديد من الخواص لهذا القياس.