
BANACH ALGEBRA VALUED – MEASURE

Noori F. Al- Mayahi

Dept. of Math.\ College of Computer Science and Mathematics\ University of Al-Qadisiya

ABSTRACT

In this paper, we introduce the concept of Banach Algebra valued measure, several properties of this measure are proved.

1. Preliminaries**Definition (1.1)**

Let X be a real vector space. A partial order relation \leq on X is called vector order, if the following axioms are satisfied

- (1) $x \leq y \Rightarrow x + z \leq y + z \quad \forall x, y, z \in X$
- (2) $x \leq y \Rightarrow \lambda x \leq \lambda y \quad \forall x, y \in X \text{ and } \lambda \geq 0$

A real vector space endowed with a vector order is called an **ordered vector space**. An element x of an ordered vector space X is said to be **positive** if $x \geq 0$, and negative if $x \leq 0$. The set of all positive elements of an ordered vector space X will be denoted by X_+ , i.e. $X_+ = \{x \in X : x \geq 0\}$, X_+ is called the **positive cone** of X . It is easy to show that

- (1) X_+ is a convex cone of X , i.e. $X_+ + X_+ \subseteq X_+$ and $\lambda X_+ \subseteq X_+$
- (2) $X_+ \cap (-X_+) = \{0\}$.

Definition (1.2)

Let X be a real vector space. A function $\|\cdot\|: X \rightarrow R$ is said to be norm on X if the following axioms are satisfied

- (1) $\|x\| \geq 0 \quad \forall x \in X$
- (2) $\|x\| = 0 \quad \text{iff } x = 0$
- (3) $\|\lambda x\| = |\lambda| \|x\| \quad \forall x \in X, \lambda \in R$
- (4) $\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in X$

A normed space is the pair $(X, \|\cdot\|)$ where X is a real vector space and $\|\cdot\|$.

A Banach space is a normed space which is complete in the metric defined by its norm.

Definition (1.3)

An algebra is a vector space in which a multiplication is defined that satisfies

- (1) $x(yz) = (xy)z$, $\forall x, y, z \in X$
- (2) $x(y+z) = xy + xz$, $(x+y)z = xz + yz$, $\forall x, y, z \in X$
- (3) $\lambda(xy) = (\lambda x)y = x(\lambda y)$, $\forall x, y, z \in X, \lambda \in R$

Definition (1.4)

A real vector space X is called Banach algebra if the following axioms are satisfied

- (1) X is a Banach space
 - (2) X is algebra
 - (3) $\exists e \in X$ s.t. $ex = xe = x \quad \forall x \in X$ and $\|e\| = 1$
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$$(4) \|xy\| \leq \|x\| \|y\| \quad \forall x, y \in X$$

A commutative algebra is an algebra where the multiplication satisfies the condition: $xy = yx, \quad \forall x, y \in X$

An algebra with identity is an algebra with the following property. There exists a non – zero element in the algebra, denoted by 1 and called the multiplication identity element , such that $|x = x| = x,$ for all x

A normed algebra X , is a normd space and also an algebra over F , and $\|xy\| \leq \|x\| \|y\| \quad \forall x, y \in X$.

Example (1.5)

Let X be a Banach space, and let $B(X)$ denote the set of all bounded (or continuous) linear function of X into itself, then $B(X)$ is a Banach algebra with the algebra operation

$$(1) (f + g)(x) = f(x) + g(x)$$

$$(2) (fg)(x) = f(g(x))$$

$$(3) (\lambda f)(x) = \lambda f(x)$$

And the operator norm $\|f\| = \sup\{\|f(x)\|: x \in X, \|x\| \leq 1\}$

Remark

If $X \neq \{0\}$, then the identity linear function I is the identity element of $B(X)$ such that $\|I\| = 1$.

2. The main results

Definition (2.1)

Let (Ω, F) be a measurable space , and let $X = (x, >, \cdot)$ be an ordered Banach algebra. A set function $\mu : F \longrightarrow X$ is said to be

$$(1) \text{ BA- finitely additive if } \mu(\bigcup_{k=1}^n A_k) = \sum_{k=1}^n \mu(A_k) , \text{ whenever } A_1, A_2, \dots, A_n \text{ disjoint sets in } F .$$

$$(2) \text{ BA-countably additive if } \mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n) , \text{ wherever } \{A_n\} \text{ is a sequence of disjoint sets in } F .$$

$$(3) \text{ BA- measure if } \mu \text{ is BA- countably additive and } \mu(A) \geq 0 \quad \forall A \in F , \text{ where } 0 \text{ is the identity element for the operation.}$$

Remark

Every BA- countably additive is BA – finitely additive but converse is not true.

Example (2.2)

Let $\Omega = (0,1]$, F the class of half – open intervals $(a,b]$ where $0 \leq a \leq b \leq 1$. Define $\mu : F \longrightarrow \mathfrak{R}$ by

$$\mu(a, b] = \begin{cases} b - a & , \quad a \neq 0 \\ 0 & \quad a = 0 \end{cases}$$

It is clear to show that μ is BA- finitely additive .

$$\text{Let } A_n = \left(\frac{1}{n+1}, \frac{1}{n}\right], n = 1, 2, 3, \dots$$

Then $\{A_n\}$ is disjoint sequence in F and $\bigcup_{n=1}^{\infty} A_n = (0, 1]$

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu(0, 1] = 1$$

$$\sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^{\infty} \mu\left(\frac{1}{n+1}, \frac{1}{n}\right] = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1$$

$$\Rightarrow \mu\left(\bigcup_{n=1}^{\infty} A_n\right) \neq \sum_{n=1}^{\infty} \mu(A_n)$$

$\Rightarrow \mu$ is not BA- countably additive

The following theorem is the main result

Theorem (2.3)

Let μ be an BA – finitely additive set function on measurable space (Ω, F) , and let $A, B \in F$

- (1) $\mu(A) = \mu(A \cap B) + \mu(A | B)$
- (2) $\mu(A \cup B) = \mu(A \cap B) + \mu(A | B) + \mu(B | A)$
- (3) $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$
- (4) $\mu(A \Delta B) = \mu(A | B) + \mu(B | A)$
- (5) If $A \subseteq B$, then $\mu(A) = \mu(A \cap B) + \mu(A | B)$.

Proof

- (1) Let $A, B \in F$ then $A \cap B \in F$ and $A | B \in F$

Since $A \cap B$ and $A | B$ are disjoint set , $(A \cap B) \cup (A | B) = A$

$$\mu(A) = \mu((A \cap B) \cup (A | B)) = \mu(A \cap B) + \mu(A | B)$$

- (2) $A, B \in F \Rightarrow A \cap B \in F$, $A | B \in F$ and $B | A \in F$

Since $A \cap B$, $A | B$ and $B | A$ are disjoint set , and

$$(A \cap B) \cup (A | B) \cup (B | A) = A \cup B$$

$$\mu(A \cup B) = \mu(A \cap B) \cup (A | B) \cup (B | A) = \mu(A \cap B) + \mu(A | B) + \mu(B | A)$$

- (3) $\mu(A) = \mu(A \cap B) + \mu(A | B)$, $\mu(B) = \mu(A \cap B) + \mu(B | A)$

$$\begin{aligned} \mu(A) + \mu(B) &= \mu(A \cap B) + \mu(A | B) + \mu(A \cap B) + \mu(B | A) \\ &= (\mu(A \cap B) + \mu(A | B) + \mu(B | A) + \mu(A \cap B)) \\ &= \mu(A \cup B) + \mu(A \cap B) \end{aligned}$$

- (4) $A, B \in F \Rightarrow A | B, B | A \in F$

$A | B$ and $B | A$ are disjoint sets , and $A \Delta B = (A | B) \cup (B | A)$

$$\mu(A \Delta B) = \mu((A | B) \cup (B | A)) = \mu(A | B) + \mu(B | A)$$

$$(5) B = (A \cap B) \cup (B | A)$$

$$A \subset B \Rightarrow A \cap B = A \Rightarrow B = A \cup (B | A) \Rightarrow \mu(B) = \mu(A \cup (B | A))$$

Since $A, B | A \in F$ and $A, B | A$ are disjoint set

$$\mu(A \cup (B | A)) = \mu(A) + \mu(B | A) \Rightarrow \mu(B) = \mu(A) + \mu(B | A)$$

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نوري فرحان المياحي

قسم الرياضيات/ كلية الرياضيات وعلوم الحاسبات/ جامعة القادسية

الخلاصة

في هذا البحث نقدم القياس ذات القيم في جبر بناخ ثم برهنة العديد من الخواص لهذا القياس.