

## COMPARISON OF MINQUE AND SIMPLE ESTIMATOR OF THE ERROR VARIANCE IN THE GAUSS MARKOFF MODEL

Abdul-Hussein Saber AL-MOUEL

Waala Khazal Salim

Dept. of Math.\ College of Education\ University of Basrah

**ABSTRACT**

The problem of estimation of variance components occurs in many areas of research. This paper is devoted to study the comparison between Minimum Norm Quadratic Unbiased Estimator (MINQUE) and Ordinary Least Square Estimator (OLSE) of  $\sigma^2$  in the Gauss Markoff Model  $\{Y, X\beta, \sigma^2 V\}$ , under mean square errors criterion, where the model matrix  $X$  need not have full rank and the dispersion matrix  $V$  can be singular.

A necessary and sufficient condition is obtained for that MINQUE is superior to simple estimator, in particular, a simple sufficient condition is that the degree of freedom of errors is equal to or greater than 4.

**Keywords:** Gauss Markoff Model; MINQUE; Mean Square error; Simple Estimator.

**1.Introduction**

We consider the Gauss Markoff Model

$$y = X\beta + e, \quad E(e) = 0, \quad Cov(e) = \sigma^2 V, \quad (1)$$

Where  $y$  is an  $n \times 1$  observable random vector, an  $n \times p$  matrix  $X$  and  $n \times n$  nonnegative definite matrix  $V$  is known, while  $\beta$  is a  $p \times 1$  vector of unknown parameter, the positive scalar  $\sigma^2$  is also unknown. The error vector  $e$  has the normal distribution  $N(0, \sigma^2 V)$ . The matrices  $X$  and  $V$  are both allowed to be of arbitrary rank. But it is assumed, throughout this chapter, that the model is consistent, [27]

i.e.  $y \in M(X : V)$ , where  $M(A)$  stands for the range of a matrix  $A$  and  $(A^B)$  denotes the partitioned matrix with  $A$  and  $B$  placed next to each other.

In the statistical literature, there are two important estimators of  $\sigma^2$ . One of which is the MINQUE (Minimum Norm Quadratic Unbiased Estimator)

$$\hat{\sigma}_m^2 = y' M (MVM)^+ My / k, \quad (2)$$

suggested by Rao(1974), where  $M = I - X(X'X)^+ X'$ ,  $A^+$  stands for the Moore-Penrose inverse of a matrix  $A$ ,

$$k = rank(X^V) - rank(X).$$

According to Theorem 3.4 in Rao (1974) [28], the MINQUE can be represented in several different forms.

Another estimator of  $\sigma^2$  is given by

$$\hat{\sigma}_s^2 = y' My / k, \quad (3)$$

which is obtained simply by replacing  $V$  by  $I$  in (2), and is called simple estimator or the ordinary least squares estimator. Some authors studied statistical properties of  $\hat{\sigma}_s^2$  when  $V$  has some special structures, see, for example, Neudecker (1977) and Dufour (1986)[7,21]. Groß(1997)[11] established some necessary and sufficient conditions for the equality  $\hat{\sigma}_m^2 = \hat{\sigma}_s^2$  when  $X$  and  $V$  can be deficient in rank, without the normality assumption of error distribution.

The object of the present work is to make further comparison of these two estimators. Obviously in the general case  $\hat{\sigma}_s^2$  need not even be unbiased.

Thus the mean square error (MSE) criterion is adopted, where the mean square error of an estimator  $\hat{\theta}$  of a scalar parameter  $\theta$  is defined by

$$MSE(\hat{\theta}) = E(\hat{\theta} - \theta)^2.$$

A necessary and sufficient condition for the following inequality

$$MSE(\hat{\sigma}_m^2) \leq MSE(\hat{\sigma}_s^2), \quad (4)$$

is obtained. Our results show that although when  $V$  is nonsingular and  $k \geq 4$  (4) always holds, there are some cases, such that the reversed inequality of (4) holds.

## 2 Comparison of the Estimators

The following lemmas are necessary for the proof of our main theorem.

**Lemma 1.** Let  $\Sigma$  be  $n \times n$  nonnegative definite matrix with rank  $r$ . Random vector  $X \sim N_p(\mu, \Sigma)$  if and only if  $X = \mu + AU$ , where  $A$  is  $p \times r$  matrix with rank  $r$  and  $AA' = \Sigma$ ,  $U \sim N_r(0, I_r)$ .

A proof can be found in [27].

**Lemma 2.** Let  $X$  be an  $n \times p$  matrix and  $V$   $n \times n$  nonnegative definite matrix. Then

$$rank(VM) = rank(V \wedge X) - rank(X),$$

where  $M = I - X(X'X)^+ X'$ .

**Proof .** Denote by  $\dim(S)$  the dimension of a linear space  $S$ .

We have

$$\begin{aligned} rank(VM) &= \dim(VM) = \dim\{VMt, \text{ for any } t_{n \times 1}\} \\ &= \dim\{Vu, X'u = 0\} \\ &= rank(V \wedge X) - rank(X). \end{aligned}$$

The last equality follows from Theorem 2.1.4 in [32].  $\square$

**Lemma 3.**

$$\hat{\sigma}_m^2 = \sum_{i=1}^k u_i^2 / k, \quad (5)$$

$$\hat{\sigma}_s^2 = \sum_{i=1}^k \lambda_i u_i^2 / k, \quad (6)$$

where  $u_i \sim N(0, \sigma^2)$ ,  $i=1, \dots, k$  are independent and  $\lambda_1 \geq \dots \geq \lambda_k > 0$  are the positive eigenvalues of  $MV$ .

**Proof.** Since  $MX = 0$  thus  $\hat{\sigma}_m^2$  and  $\hat{\sigma}_s^2$  can be rewritten as

$$\begin{aligned} \hat{\sigma}_m^2 &= e'M(MVM)^+ Me/k, \\ \hat{\sigma}_s^2 &= e'Me/k, \end{aligned}$$

In view of Lemma 1 and  $e \sim N(0, \sigma^2 V)$ ,  $r = \text{rank}(V)$ , we note that there is an  $n \times r$  matrix  $A$  such that

$$e = A\varepsilon, \quad \varepsilon \sim N(0, \sigma^2 I), \quad V = AA',$$

thus

$$\hat{\sigma}_m^2 = \varepsilon' Q_1 \varepsilon / k, \quad (7)$$

$$\hat{\sigma}_s^2 = \varepsilon' Q_2 \varepsilon / k, \quad (8)$$

where

$$Q_1 = A'M(MAA'M)^+ MA,$$

$$Q_2 = A'MA,$$

It is easy to verify that  $Q_1 Q_2 = Q_2 Q_1$ , which implies [see for example 27] that there is an  $r \times r$  orthogonal matrix  $T$  such that both  $T'Q_1 T$  and  $T'Q_2 T$  are diagonal. By using lemma 2, it can be shown that

$$\begin{aligned} \text{rank}(Q_1) &= \text{rank}(A'M) = \text{rank}(A'MA) = \text{rank}(VM) \\ &= \text{rank}(V^{\wedge} X) - \text{rank}(X) = k. \end{aligned} \quad (9)$$

We note that  $Q_1$  is a projection matrix, thus

$$T'Q_1 T = \text{diag}(I_k, 0), \quad (10)$$

$$T'Q_2 T = \text{diag}(\Lambda_k, 0), \quad (11)$$

where  $\Lambda_k = (\lambda_1, \dots, \lambda_k)$ . Denote  $u = T'\varepsilon$ , then

$$u \sim N_r(0, \sigma^2 I_r). \quad (12)$$

Substituting (10), (11) and (12) in (7) and (8) yields (5) and (6). The proof of Lemma 3 is completed.

Denote

$$f(MV, k) = \frac{\text{tr}(MV)^2}{k} + \frac{(\text{tr}(MV))^2}{2k} - \text{tr}(MV) + \frac{k}{2}$$

where  $k$  is defines in (9), that is, the number of the nonzero eigenvalues of  $MV$ .

### Theorem 1.

(i) If  $f(MV, k) < 1$ , then  $MSE(\hat{\sigma}_m^2) > MSE(\hat{\sigma}_s^2)$

(ii) If  $f(MV, k) = 1$ , then  $MSE(\hat{\sigma}_m^2) = MSE(\hat{\sigma}_s^2)$

(iii) If  $f(MV, k) > 1$ , then  $MSE(\hat{\sigma}_m^2) < MSE(\hat{\sigma}_s^2)$

### Proof.

Since  $\hat{\sigma}_m^2$  is unbiased and from (5) we have

$$\begin{aligned} MSE(\hat{\sigma}_m^2) &= \text{var}(\hat{\sigma}_m^2) = \text{var}\left(\sum_{i=1}^k u_i^2 / k\right) = \frac{1}{k^2} \text{var}\left(\sum_{i=1}^k u_i^2\right) \\ &= \frac{\sigma^4}{k^2} \text{var}\left(\sum_{i=1}^k \frac{u_i^2}{\sigma^2}\right) = \frac{2k\sigma^4}{k^2} = \frac{2\sigma^4}{k} \end{aligned}$$

Now, from (6) we have

$$\begin{aligned} \text{var}(\hat{\sigma}_s^2) &= \text{var}\left(\sum_{i=1}^k \lambda_i u_i^2 / k\right) = \frac{1}{k^2} \left(\sum_{i=1}^k \lambda_i^2 \text{var}(u_i^2)\right) \\ &= \frac{\sigma^4}{k^2} \left(\sum_{i=1}^k \lambda_i^2 \text{var}\left(\frac{u_i^2}{\sigma^2}\right)\right) = \frac{\sigma^4}{k^2} \left[2\lambda_1^2 + \dots + 2\lambda_k^2\right] \\ &= \frac{2\sigma^4}{k^2} \sum_{i=1}^k \lambda_i^2 \end{aligned}$$

thus

$$\begin{aligned} \text{MSE}(\hat{\sigma}_s^2) &= E(\hat{\sigma}_s^2 - \sigma^2)^2 = E(\hat{\sigma}_s^4 - 2\sigma^2 \hat{\sigma}_s^2 + \sigma^4) \\ &= E(\hat{\sigma}_s^4) - 2\sigma^2 E(\hat{\sigma}_s^2) + \sigma^4 \\ &= E(\hat{\sigma}_s^4) - (E(\hat{\sigma}_s^2))^2 + (E(\hat{\sigma}_s^2))^2 - 2\sigma^2 E(\hat{\sigma}_s^2) + \sigma^4 \\ &= \text{var}(\hat{\sigma}_s^2) + (E(\hat{\sigma}_s^2))^2 - 2\sigma^2 E(\hat{\sigma}_s^2) + \sigma^4 \\ &= \frac{2\sigma^4}{k^2} \sum_{i=1}^k \lambda_i^2 + \frac{\sigma^4}{k^2} \left(\sum_{i=1}^k \lambda_i\right)^2 - \frac{2\sigma^4}{k} \sum_{i=1}^k \lambda_i + \sigma^4 \\ &= \frac{2\sigma^4}{k} \left[ \frac{\sum_{i=1}^k \lambda_i^2}{k} + \frac{\left(\sum_{i=1}^k \lambda_i\right)^2}{2k} - \sum_{i=1}^k \lambda_i + \frac{k}{2} \right] \end{aligned}$$

since  $\text{tr}(MV) = \sum_{i=1}^k \lambda_i$  and  $\text{tr}(MV)^2 = \sum_{i=1}^k \lambda_i^2$  we have

$$\begin{aligned} &= \frac{2\sigma^4}{k} \left[ \frac{\text{tr}(MV)^2}{k} + \frac{(\text{tr}(MV))^2}{2k} - \text{tr}(MV) + \frac{k}{2} \right] \\ &= \frac{2\sigma^4}{k} f(MV, k). \end{aligned}$$

Therefore the proof is completed.

**Lemma 4.** For any nonnegative definite matrix  $A$ ,

$$(\text{tr}A)^2 \leq r \cdot \text{tr}(A)^2,$$

where  $r$  is the number of the positive eigenvalues of  $A$ , and the equality holds when all positive eigenvalues are equal.

**Proof .** Let  $\lambda_1, \dots, \lambda_r$  be the positive eigenvalues of  $A$ .

Consider

$$S = \sum_{i=1}^r (\lambda_i - \bar{\lambda})^2$$

where

$$\bar{\lambda} = \frac{1}{r} \sum_{i=1}^r \lambda_i, \text{ but } \sum_{i=1}^r \lambda_i = \sum_{i=1}^n \lambda_i = \text{tr}(A), \quad \bar{\lambda} = \frac{\text{tr}(A)}{r}.$$

It is clear that  $S \geq 0$ ; also  $S = 0$  if and only if  $\lambda_1 = \lambda_2 = \dots = \lambda_r = \bar{\lambda}$ , that is, if and only if each and every  $\lambda_i$  is equal to  $\bar{\lambda}$  for  $i=1, 2, \dots, r$

$$S = \sum_{i=1}^r (\lambda_i^2 - 2\lambda_i\bar{\lambda} + \bar{\lambda}^2) \Rightarrow S = \sum_{i=1}^r \lambda_i^2 - r\bar{\lambda}^2 = \text{tr}(A^2) - r\left(\frac{\text{tr}(A)}{r}\right)^2 \geq 0,$$

and hence

$$(\text{tr}A)^2 \leq r \cdot \text{tr}(A)^2$$

**Corollary 1.** If  $k \geq 4$ , then

$$MSE(\hat{\sigma}_m^2) \leq MSE(\hat{\sigma}_s^2)$$

**Proof.** According to the proof of Theorem 1, it is sufficient to show that  $k \geq 4$  implies that

$$g(x) = \sum_{i=1}^k x_i^2 - k \sum_{i=1}^k x_i + \frac{1}{2} \left( \sum_{i=1}^k x_i \right)^2 + \frac{k^2}{2} - k \geq 0$$

In fact, it is easy to show that at  $x_o = (k/(k+2))1_k$

where  $1_k$  is a  $k \times 1$  vector of ones,  $g(x)$  has its minimum

$$g(x_o) = \frac{-2k}{k+2} = \frac{(k-4)(k+1)}{(k+2)}$$

which completes the proof.

**Corollary 2.** Suppose that matrix  $V$  is nonsingular and  $\text{rank}(X) \leq n-4$ .

Then

$$MSE(\hat{\sigma}_m^2) \leq MSE(\hat{\sigma}_s^2).$$

**Proof.** since  $V$  is nonsingular, thus

$$\begin{aligned} k &= \text{rank}(MV) = \text{rank}(M) = \text{rank}(I - X(X'X)^+ X') \\ &= n - \text{rank}(X), \end{aligned}$$

from which and corollary 1, the proof is completed.

There two corollaries show that in most of cases the performance of  $\hat{\sigma}_m^2$  is superior to  $\hat{\sigma}_s^2$ .

Denote

$$c = \frac{1 + \frac{1}{2} \sqrt{\frac{15}{4} + \frac{1}{k^2}}}{2 \left( \frac{1}{k^2} + \frac{1}{2k} \right)}.$$

**Corollary 3.** If  $\text{tr}(MV) \geq c$ , then

$$MSE(\hat{\sigma}_m^2) \leq MSE(\hat{\sigma}_s^2)$$

**Proof.** since

$$\text{tr}(MV) = \text{tr}(MVM),$$

$$tr(MV)^2 = tr(MVM)^2,$$

applying lemma 4 yields

$$f(MV, k) \geq \left( \frac{1}{k^2} + \frac{1}{2k} \right) (tr(MV))^2 - tr(MV) + \frac{k}{2}.$$

Thus a sufficient condition for  $f(MV, k) \geq 1$  is

$$\left( \frac{1}{k^2} + \frac{1}{2k} \right) (tr(MV))^2 - tr(MV) + \frac{k}{2} - 1 \geq 0,$$

which holds when  $tr(MV) \geq c$ .

The proof is completed.

### Example

The estimate of  $\sigma^2$  are often need in the estimation of variance of estimable functions. In what follows we will give simple example to illustrate applications of the result, obtained in this paper.

Consider the following linear model

$$Y = \mu 1_n + e, \quad E(e) = 0, \quad Cov(e) = \sigma^2 V \quad (11)$$

This model has been found useful in certain statistical inference problems on the mean  $\mu$  of a population when the observations  $y_1, y_2, \dots, y_n$  are not independent. For some examples of applications in medical data and animal genetic selection.

For the model (11), if the matrix  $V$  is nonsingular and  $n \geq 5$ , we know from Corollary I that  $\hat{\sigma}_m^2$  is better than  $\hat{\sigma}_s^2$ .

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## المقارنة بين مينك والتقدير البسط لتباين الخطأ في نموذج كاوس ماركوف

عبد الحسين صبر المولى  
ولاء خزعل سالم  
قسم الرياضيات/ كلية التربية/ جامعة البصرة

### الخلاصة

بان مسألة التقدير لمرتبات التباين تحصل في مجالات لبحوث كثيرة . في هذا البحث ركزنا على دراسة المقارنة بين مينك وبين تقدير المربعات الصغرى الاعتيادية الى  $\alpha 2$  في نموذج كاوس ماركوف في ظل اخطاء المربع الوسط.