COMPACT-F AND LINDELÔ F- C SPACES

Hadi J. Mustafa Inaam Razzaq AL-Siaq Hussein K. Asker
Dept. of Math\ College of Mathematics and Computer Science Kufa University

ABSTRACT:
In this paper, we introduce and study compact-F and Lindelô f-C Spaces. Compact-F space is a topological space in which every compact subset is finite, Lindelô f-C Space is a topological space in which every Lindelô f subsets is countable several properties of these spaces are proved.

1. Introduction:
It is known that every finite subset of a topological space \((X, \tau)\) is compact, and every countable subset of \((X, \tau)\) is Lindelô f.
In this paper, we study the convers proplems, that is study spaces in which every compact subset is finite which we call compact-F spaces also we study spaces in which every Lindelô f subset is countable which we call Lindelô f –C spaces.

1.1 Examples;
Consider \((\mathbb{R}, \tau_u)\) where \(\tau_u\) is the usual topology on the set of real numbers \(\mathbb{R}\). notice that \(W=[0, 1]\) is compact but \(W\) is not finite, so \((\mathbb{R}, \tau_u)\) is not compact-F space also \(W=[0, 1]\) is Lindelô f which is not countable, so \((\mathbb{R}, \tau_u)\) is not Lindelô f-C space.

2. Basic definitions and results:
In this section, we introduce and recall the basic definitions needed in this work.

Definition:
Let \((X, \tau)\) be a topological space, we say that

(i) \(X\) is compact-F if every compact subset of \(X\) is finite.
(ii) \(X\) is Lindelô f-C if every Lindelô f subset of \(X\) is countable

2.2 Examples:
(i) \((\mathbb{R}, \tau_u)\) is not compact-F also \((\mathbb{R}, \tau_u)\) is not Lindelô f-C
(ii) \((\mathbb{R}, \tau_d)\) (where \(\tau_d\) is the discrete topology on \(\mathbb{R}\)) is compact-F, also \((\mathbb{R}, \tau_d)\) is Lindelô f –C.

2.3 Remarks:
(i) if \(X\) is compact, then \(X\) is not necessarily compact-F consider \(X=[-3, 3]\) as a subspace of \((\mathbb{R}, \tau_d)\) \(X\) is compact but \(X\) is not compact-F for example \(W=[0, 1]\) a compact subset of \(X\) which is not finite.
Also if X is compact-F then X is not compact in general consider (R, τ_0) which is an example of a compact-F space which is not compact.

(ii) if X is Lindelöf, then X is not necessarily Lindelöf-C

consider (R, τ_0) which is Lindelöf but not Lindelöf-C, also if X is Lindelöf-C, then X is not necessarily Lindelöf consider (R, τ_0) which is Lindelöf-C but not Lindelöf.

3. Main Results:

In this section, we prove several theorems about compact-F and Lindelöf-C spaces.

3.1 Theorem:
(i) let X be a compact-F space and let Y be a subspace of X then Y is also a compact-F space.

(ii) let X be a Lindelöf-C space and let Y be a subspace of X, then Y is also Lindelöf-C space.

Proof
(i) let W ⊆ Y be compact in Y, then W is compact in X.

so W is finite, which means that Y is compact-F space.

(ii) let W ⊆ Y be Lindelöf in Y, then W is Lindelöf in X so W is countable, which means that Y is Lindelöf-C space.

3.2 Remark: Theorem (3.1):

Shows that the property of being compact-F is a hereditary property similarly the property of being Lindelöf-C is a hereditary property.

3.3 Theorem:
i) the property of being compact-F is a topological property.

ii) the property of being Lindelöf-C is a topological property.

Proof:
(i) let f: X → Y be a homeomorphism from a compact-F space X onto a topological space Y.

we will show that Y is a compact-F space

let W be a compact subset of Y

now f⁻¹(W) is a compact subset of X so f⁻¹(W) is finite

so W=f(f⁻¹(W)) is also finite which means that Y is a compact-F space

(ii) let f: X → Y be a homeomorphism from a Lindelöf-C space X onto a topological space Y.

we will show that Y is a Lindelöf-C space

let W be a Lindelöf subset of Y

now f⁻¹(W) is a Lindelöf subset of X so f⁻¹(W) is countable
now \( W = f( f^{-1}(w) ) \) is also countable which means that \( Y \) is Lindelöf –C space.

3.4 Remark:
(i) The continuous image of compact-F space is not necessarily compact-F
consider \( f: (\mathbb{R}, \tau_0) \to (\mathbb{R}, \tau_0) \) where \( f(x) = x, x \in \mathbb{R} \) now \( f \) is continuous
\((\mathbb{R}, \tau_0)\) is compact –F but \((\mathbb{R}, \tau_0)\) is not a compact- F space

ii) The continuous image of a Lindelöf –C space is not necessarily Lindelöf –C
the same example in (i) shows this fact.

3.5 Theorem:
(i) If \( X \times Y \) is compact –F, then both of \( X, Y \) are compact –F spaces

(ii) If \( X \times Y \) is Lindelöf –C, then both of \( X, Y \) are Lindelöf –C spaces

Proof:
(i) Suppose that \( X \times Y \) is compact-F
We will show that \( X \) is compact-F
Let \( W \) be a compact subset of \( X \)
Let \( y_0 \in Y \)
Now \( W \times \{ y_0 \} \) is compact in \( X \times Y \) so \( W \times \{ y_0 \} \) is finite
But \( W \) is homeomorphic to \( W \times \{ y_0 \} \)
Which means that \( W \) is also finite
Hence \( X \) is a compact-F space
Similarly we can prove that \( Y \) is a compact-F space

(ii) Suppose that \( X \times Y \) is Lindelöf –C we will show that \( X \) is a Lindelöf –C space

Let \( W \) be a Lindelöf subset of \( X \) and let \( y_0 \in Y \)
Now \( W \times \{ y_0 \} \) is Lindelöf in \( X \times Y \)
So \( W \times \{ y_0 \} \) is countable
Which means that \( W \) is also countable
So \( X \) is a Lindelöf –C space
Similarly we can prove that \( Y \) is a Lindelöf –C space.

3.6 Theorem
(i) If either of \( X \) or \( Y \) is compact-F, then \( X \times Y \) is compact-F

(ii) If either \( X \) or \( Y \) is Lindelöf –C, then \( X \times Y \) is Lindelöf –C

Proof:
(i) Suppose \( X \) is compact-F
Let \( G \) be a compact subset of \( X \times Y \)
Consider \( P_1: X \times Y \to X \) where \( P_1(\alpha, y) = x \)
Now \( P_1(G) \) is a compact subset of \( X \)
So $P_1(G)$ is finite
Consider $i: X \rightarrow X \times Y$
Where $i(x) = (x, y_0)$, $y_0 \in Y$
Now $i(P_1(G))$ is finite and homeomorphic to $G$ so $G$ is finite
So $X \times Y$ is compact-F

(ii) suppose $X$ is Lindel"of-C

let $G$ be a Lindel"of subset of $X \times Y$
consider $P_1: X \times Y \rightarrow X$ where $P_1(\alpha, y) = x$

now $P_1(G)$ is a Lindel"of subset of $X$, so $P_1(G)$ is countable
consider $i: X \rightarrow X \times Y$ where $i(x) = (x, y_0)$
now $i(P_1(G))$ is countable and homeomorphic to $G$
so $G$ is countable

hence $X \times Y$ is Lindel"of-C

References