

COMPACT-F AND LINDELÖF-C SPACES

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ABSTRACT:

In this paper, we introduce and study compact-F and Lindelöf-C Spaces. Compact-F space is a topological space in which every compact subset is finite, Lindelöf-C Space is a topological space in which every Lindelöf subsets is countable several properties of these spaces are proved.

1. Introduction:

It is known that every finite subset of a topological space (X, τ) is compact, and every countable subset of (X, τ) is Lindelöf. In this paper, we study the convers problems, that is study spaces in which every compact subset is finite which we call compact-F spaces also we study spaces in which every Lindelöf subset is countable which we call Lindelöf-C spaces.

1.1 Examples;

Consider (\mathbb{R}, τ_u) where τ_u is the usual topology on the set of real numbers \mathbb{R} . notice that $W=[0, 1]$ is compact but W is not finite, so (\mathbb{R}, τ_u) is not compact-F space also $W=[0, 1]$ is Lindelöf which is not countable, so (\mathbb{R}, τ_u) is not Lindelöf-C space.

2. Basic definitions and results:

In this section, we introduce and recall the basic definitions needed in this work.

Definition:

Let (X, τ) be a topological space, we say that

- (i) X is compact-F if every compact subset of X is finite.
- (ii) X is Lindelöf-C if every Lindelöf subset of X is countable

2.2 Examples:

- (i) (\mathbb{R}, τ_u) is not compact-F also (\mathbb{R}, τ_u) is not Lindelöf-C
- (ii) (\mathbb{R}, τ_d) (where τ_d is the discrete topology on \mathbb{R}) is compact-F, also (\mathbb{R}, τ_d) is Lindelöf-C.

2.3 Remarks:

- (i) if X is compact, then X is not necessarily compact-F consider $X=[-3, 3]$ as a subspace of (\mathbb{R}, τ_u) X is compact but X is not compact-F for example $W=[0, 1]$ a compact subset of X which is not finite.

Also if X is compact-F then X is not compact in general consider (\mathbb{R}, τ_d) which is an example of a compact-F space which is not compact.

(ii) if X is Lindelöf, then X is not necessarily Lindelöf-C

consider (\mathbb{R}, τ_u) which is Lindelöf but not Lindelöf-C, also if X is Lindelöf-C, then X is not necessarily Lindelöf consider (\mathbb{R}, τ_d) which is Lindelöf-C but not Lindelöf.

3. Main Results:

In this section, we prove several theorems about compact-F and Lindelöf-C spaces.

3.1 Theorem:

(i) let X be a compact-F space and let Y be a subspace of X then Y is also a compact-F space.

(ii) let X be a Lindelöf-C space and let Y be a subspace of X , then Y is also Lindelöf-C space.

Proof

(i) let $W \subseteq Y$ be compact in Y , then W is compact in X
so W is finite, which means that Y is compact-F space.

(ii) let $W \subseteq Y$ be Lindelöf in Y , then W is Lindelöf in X so W is countable, which means that Y is Lindelöf-C space.

3.2 Remark : Theorem(3.1):

Shows that the property of being compact-F is a hereditary property similarly the property of being Lindelöf-C is a hereditary property.

3.3 Theorem:

i) the property of being compact-F is a topological property.

ii) the property of being Lindelöf-C is a topological property.

Proof:

(i) let $f: X \rightarrow Y$ be a homeomorphism from a compact-F space X onto a topological space Y .

we will show that Y is a compact-F space

let W be a compact subset of Y

now $f^{-1}(w)$ is a compact subset of X so $f^{-1}(w)$ is finite

so $W = f(f^{-1}(w))$ is also finite which means that Y is a compact-F space

(ii) let $f: X \rightarrow Y$ be a homeomorphism from a Lindelöf-C space X onto a topological space Y

we will show that Y is a Lindelöf-C space

let W be a Lindelöf subset of Y

now $f^{-1}(w)$ is a Lindelöf subset of X so $f^{-1}(w)$ is countable

now $W=f(f^{-1}(w))$ is also countable which means that Y is Lindelöf-C space.

3.4 Remark:

- (i) The continuous image of compact-F space is not necessarily compact-F
consider $f: (\mathbb{R}, \tau_d) \rightarrow (\mathbb{R}, \tau_u)$ where $f(x)=x$, $x \in \mathbb{R}$ now f is continuous
 (\mathbb{R}, τ_d) is compact-F but (\mathbb{R}, τ_u) is not a compact-F space
- (ii) The continuous image of a Lindelöf-C space is not necessarily Lindelöf-C
the same example in (i) shows this fact.

3.5 Theorem:

- (i) If $X \times Y$ is compact-F, then both of X, Y are compact-F spaces
- (ii) If $X \times Y$ is Lindelöf-C, then both of X, Y are Lindelöf-C spaces

Proof:

- (i) Suppose that $X \times Y$ is compact-F
We will show that X is compact-F
Let W be a compact subset of X
Let $y_0 \in Y$
Now $W \times \{y_0\}$ is compact in $X \times Y$ so $W \times \{y_0\}$ is finite
But W is homeomorphic to $W \times \{y_0\}$
Which means that W is also finite
Hence X is a compact-F space
Similarly we can prove that Y is a compact-F space
- (ii) Suppose that $X \times Y$ is Lindelöf-C we will show that X is a Lindelöf-C space
Let W be a Lindelöf subset of X and let $y_0 \in Y$
Now $W \times \{y_0\}$ is Lindelöf in $X \times Y$
So $W \times \{y_0\}$ is countable
Which means that W is also countable
So X is a Lindelöf-C space
Similarly we can prove that Y is a Lindelöf-C space.

3.6 Theorem

- (i) If either of X or Y is compact-F, then $X \times Y$ is compact-F
- (ii) If either X or Y is Lindelöf-C, then $X \times Y$ is Lindelöf-C

Proof:

- (i) Suppose X is compact-F
Let G be a compact subset of $X \times Y$
Consider $P_1: X \times Y \rightarrow X$ where $P_1(\alpha, y) = x$
Now $P_1(G)$ is a compact subset of X

So $P_1(G)$ is finite

Consider $i: X \rightarrow X \times Y$

Where $i(x) = (x, y_0)$, $y_0 \in Y$

Now $i(P_1(G))$ is finite and homeomorphic to G so G is finite

So $X \times Y$ is compact-F

(ii) suppose X is Lindelöf-C

let G be a Lindelöf subset of $X \times Y$

consider $P_1: X \times Y \rightarrow X$ where $P_1(x, y) = x$

now $P_1(G)$ is a Lindelöf subset of X , so $P_1(G)$ is countable

consider $i: X \rightarrow X \times Y$ where $i(x) = (x, y_0)$

now $i(P_1(G))$ is countable and homeomorphic to G

so G is countable

hence $X \times Y$ is Lindelöf-C

References

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الفضاءات المتراسة -F والفضاءات اللندلوفية -C

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الخلاصة:

في هذا البحث قدمنا ودرسنا الفضاءات المتراسة -F والفضاءات اللندلوفية -C. الفضاء المتراس -F هو الفضاء الذي يكون فيه كل مجموعة متراسة منتهية ، الفضاء اللندلوف -C هو الفضاء الذي يكون فيه كل مجموعة لندلوفية معدودة . برهنا عدة خصائص لهذه الفضاءات.