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S^*-ORLICZ LATTICE

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ABSTRACT

In this paper, we review here some of the ideas we have encountered in Orlicz function and define S^*-Orlicz lattice. We have proved that S^*-Orlicz space $(X, \|.\|_F)$ is a normed S^*-Vector Lattice, complete and therefore, it's a Banach S^*-Vector Lattice.

1. **Introduction**

In this work a series of known notions, notations and facts of the theory of Boolean algebra, vector lattices [2,3,13], the integration theory for measures with values in semi-field [7,8,11,12] is cited.

Suppose that R is the set of real numbers and E is a partially ordered set ($E\subseteq R$). Our interest in this work is to define the S^*-Orlicz Lattice. The main result in this work is the following:

Theorem I: Suppose that X is an S^*-Orlicz modular on an S^*-vector lattice X, then $(X, \|.\|_F)$ is a normed S^*-vector lattice.

Theorem II: Suppose that X is an S^*-Orlicz modular on a S^*-vector lattice X, then $(X, \|.\|_F)$ is a Banach S^*-vector lattice.

2. The Basic Concepts

In this section, we shall review some of the ideas we have encountered in Orlicz function and define S^*- Orlicz lattice.

2.1. Definition [1]

A non-negative convex function M(u) defined on the semi-axis $[0,\infty)$ is called an Orlicz function, if $M(0) = 0$ and $M(u) > 0$ for $u \neq 0$.

2.2. Proposition [10]

Suppose $x_n, x\in C_\infty(Q(\nabla))$, $0 \leq x_n\uparrow x$, then $M(x_n) \uparrow M(x)$.

2.3. Proposition [4]

If M(u) is an Orlicz function, then M(u) is strictly increasing and $M(u)\to\infty$ as $u\to\infty$.

It is clear that any N-function on [0,1] is an Orlicz function. The converse is not generally true.

2.4. Definition[5]

We say that an Orlicz function M(u) satisfies the Δ_2-condition, if $M(2u)\leq cM(u)$ for all $u \geq 0$ and some constant $c>0$.

Suppose that ∇ is an arbitrary σ-complete Boolean algebra, m is a strictly positive measure on ∇ with values in S^*, (we assume that the Boolean algebra of idempotent of S^* is a regular Boolean subalgebra ∇_1 of ∇ and that m has the moduleness property, $m(e\, g) = e\, m(g)$ for all $e \in\nabla_1$, $g \in\nabla$).

Suppose that M(u) is an Orlicz function with the Δ_2-condition. For every $x\in C_\infty(Q(\nabla))$, we define M(x), and $L_M(\nabla, m) = L_M = \{x\in C_\infty(Q(\nabla)) : M(|x|)\in L_1(m)\}$.

2.5. Proposition [9]
Suppose that $\|.\|_{(M)}$ is an S^*-norm on L_M. Furthermore, it follows from $|x|\le|z|$, $x,z \in L_M$, that $\|x\|_{(M)} \le \|z\|_{(M)}$, thus $(L_M, \|.\|_{(M)})$ is a normed S^*-vector lattice.

2.6. Proposition [9]
Let $\|.\|_{(M)}$ be an S^*-norm on L_M, then $(L_M, \|.\|_{(M)})$ is a Banach S^*-vector lattice.

2.7. Definition [6]
Suppose that X is a S^* - vector lattice. A mapping $F: X \to S^*$ is called an S^*- Orlicz modular if:

1. $F(x) \ge 0$ for all $x \in X$ and $F(x) = 0$, if and only if, $x = 0$.
2. $F(x) \le F(y)$, if $|x| \le |y|$ for all $x,y\in X$.
3. $F((\propto x + (\hat{\mathbf{1}}-\propto) y) \le \propto F(x) + (\hat{\mathbf{1}}-\propto) F(y)$ for all $x,y\in X$, $\propto\in S^*$ and $0 \le \propto \le \hat{\mathbf{1}}$.
4. $F(2 x) \le c F(x)$ for all $x\in X$ and $c >0$.
5. $F(x + y) = F(x) + F(y)$, if $x,y\in X$ and $x \wedge y = 0$.
6. $F(e x) = e F(x)$ for all $x\in X$ and $e\in \nabla(S^*)$.

From the definition, we get that $F(x) = F(|x|)$ and $F(\propto x) \le \propto F(x)$ for all $x\in X$, $\propto\in S^*$ and $0 \le \propto \le \hat{\mathbf{1}}$.

2.8. Remark [6]
If $(L_M, \|.\|_{(M)})$ is an S^*-Orlicz space generated by the Orlicz function M(u) with the Δ_2-condition, then $F(x) = \mu(M(|x|))$ is an S^*-Orlicz modular on L_M .

Now, suppose that X is an S^*- Orlicz modular on a S^*-vector lattice X. For all $x\in X$, we set $B(x) = \{\lambda\in S^*_+ : F(\lambda^{-1}x) \le \hat{\mathbf{1}}$, λ is invertible$\}$. If $x\in X$, $\lambda = F (x) + \hat{\mathbf{1}}$, then $F(\lambda^{-1} x) \le \lambda^{-1} F (x) \le \hat{\mathbf{1}}$, it means, that $B(x) \ne \varnothing$. For each $x\in X$, we set $\|x\|_F=\inf \{\lambda: \lambda\in B(x)$.

3. The main result
In this section, we shall prove an important proposition related to the S^*-Orlicz lattice.

3.1. Proposition
Suppose that X is an S^*-Orlicz modular on an S^*-vector lattice X , then $(X, \|.\|_F)$ is a normed S^*-vector lattice.

Proof. It is clear that, $\|x\|_F \ge 0$ for all $x\in X$, so $F(x) = \mu(M(|x|))$ and therefore F is a S^*-Orlicz modular on the S^*-vector lattice X.

For all $x\in X$, we have $B(x)=\{\lambda\in S^*_+ : F (\lambda^{-1} x) \le \hat{\mathbf{1}}$, λ is invertible$\}$.

Choose an arbitrary element say $\lambda\in S^*_+$, such that $F(\lambda^{-1} x)=\mu(M(\lambda^{-1} |x|))\le\hat{\mathbf{1}}$ and $\|x\|_F=0$. From proposition (2.5), we have $F(x) = 0$, therefore $x = 0$. Suppose $B(x)=\{\lambda_1\in S^*_+ : F (\lambda_1^{-1} x)\le\hat{\mathbf{1}}$, λ_1 is invertible$\}$, and $B(y)=\{\lambda_2\in S^*_+ : F (\lambda_2^{-1} y)\le\hat{\mathbf{1}}$, λ_2 is invertible$\}$. Using the triangle inequality and the fact, that
$F(\propto x + (1-\propto) y) \le \propto F(x) + (1-\propto) F(y)$, we get

$$\begin{aligned} F((\lambda_1 + \lambda_2)^{-1} (x + y)) &= F(\lambda_1(\lambda_1 + \lambda_2)^{-1} (\lambda_1^{-1} x) + \lambda_2 (\lambda_1 + \lambda_2)^{-1} (\lambda_2^{-1} y)) \\ &\le \lambda_1 (\lambda_1 + \lambda_2)^{-1} F (\lambda_1^{-1} x) + \lambda_2 (\lambda_1 + \lambda_2)^{-1} F (\lambda_2^{-1} y) \\ &\le \lambda_1 (\lambda_1 + \lambda_2)^{-1} + \lambda_2 (\lambda_1 + \lambda_2)^{-1} \le \hat{\mathbf{1}}. \end{aligned}$$

Therefore, $B(x + y) = \{(\lambda_1 + \lambda_2)\in S_+^* : F((\lambda_1 + \lambda_2)^{-1}(x + y)) \leq \hat{\imath}$, $(\lambda_1 + \lambda_2)$ is invertible$\}$. Hence, $\|x + y\|_F = \inf\{(\lambda_1 + \lambda_2)\in S_+^* : (\lambda_1 + \lambda_2)\in B(x + y)\}$.

Now, since $F(x) = \mu(M(|x|))$, then by using the similar proof of the proposition (2.5), it can be shown that $\|x + y\|_F \leq \|x\|_F + \|y\|_F$, If $x,y\in X$, $\lambda\in B(x)$, $|x| \leq |y|$, then
$F(\lambda^{-1} x) \leq F(\lambda^{-1} y) \leq \hat{\imath}$.
It means, that $\lambda\in B(x)$. Since $\|x\|_F = \inf\{\lambda : \lambda\in B(x)\}$, $\|y\|_F = \inf\{\lambda : \lambda\in B(y)\}$. Then $\|x\|_F \leq \|y\|_F$. In the same way of proposition (2.5), we can prove the inequality $\|\propto x\|_F=|\propto|\ \|x\|_F$.
Thus, $(X, \|.\|_F)$ is a normed S^*-vector lattice.

3.2. Theorm

Suppose that X is an S^*-Orlicz modular on a S^*-vector lattice X , then $(X, \|.\|_F)$ is a Banach S^*-vector lattice.

Proof. Since $(L_M, \|.\|_{(M)})$ is an S^*- Orlicz space generated by the Orlicz function M(u) with the Δ_2-condition, then $F(x) = \mu(M(|x|))$ is an S^*- Orlicz modular on L_M, and
$B(x) = \{\lambda\in S_+^* : F(\lambda^{-1} x) \leq \hat{\imath}$, λ is invertible$\}$,
So, $\|x\|_F = \inf\{\lambda : \lambda\in B(x)\}$ for all $x\in X$. To show, that $(X, \|.\|_F)$ is complete, recall the same proof of proposition (2.6).
Thus, $(X, \|.\|_F)$ is a Banach S^*-vector lattice.

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الحزم البناخية S^*

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الخلاصة

في هذا البحث نستعرض بعض المفاهيم التي تواجهنا في دالة اورليسز إضافة إلى ذلك نعرف حزم اورليسز -S^* عـــلاوة على ذلك، فأننا سنبرهن بأن فضاء اورليسز -S^* المتمثل بـــ $(X, \|.\|_F)$ يكون حزمة متجه-S^* قياسية وكاملة وعليه يكون حزمة متجه - S^* بناخيه.