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ON K – MONOTONE APPROXIMATION IN L_P

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ABSTRACT

In 1995 Kopotun [4], introduced a paper on k-monotone polynomial and spline approximation in L_P, $0 < p < \infty$ quasi norm . In this paper, we discuss the errors of approximation of k-monotone function by k-monotone interpolation . It turns out that any two k-monotone functions f and g, whose graphs intersect each other at certain (sufficiently many) points in $[a,b]$, have to be "close" to each other in the sense that $\|f-g\|_P$, has to be small .

1. Notations and Definitions.

Let $L_p[a,b], 0 < p < \infty$, be the set of all measurable functions on $[a,b]$ such that $\|f\|_{L_p[a,b]} < \infty$, where

$$\|f\|_{L_P[a,b]} := \left(\int_a^b |f(x)|^p \, dx \right)^{\frac{1}{P}} .$$

Let us recall some definitions of moduli of smoothness used throughout this paper . The *k th symmetric difference* of f is given by :

$$\Delta_h^k(f,x,[a,b]) := \begin{cases} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} f\left(x - \frac{kh}{2} + ih\right) & x \pm \frac{kh}{2} \in [a,b] \\ 0 & o.w. \end{cases} , o < h < 1 .$$

The *k th usual modulus of smoothness* of $f \in L_p[a,b]$ is defined by :

$$\omega_k(f,\delta,[a,b])_p := \sup_{0 \le h \le \delta} \left\| \Delta_h^k(f,\cdot,[a,b]) \right\|_{L_p[a,b]} .$$

It will be omitted for the sake of simplicity ,

$$\omega_k(f,[a,b])_P = \omega_k(f,\delta,[a,b])_P$$

For $f \in L_p[a,b]$, let

$$E_n(f)_P = \inf_{p_n \in \Pi_n} \|f - p_n\|_P ,$$

denote the *degree of unconstrained approximation*, where Π_n, the set of all polynomials of degree $\le n$, and n is natural , i.e., $n \in \mathrm{N}$.

2. Local Estimates.

The main aim of this section is to introduce a direct theorem for k-monotone function by k-monotone interpolating function .

Firstly , let us introduce the Lagrange polynomial $L_{k-1}(f,\cdot \mid t_1,\ldots,t_k)$ of degree $\le k-1$, interpolating f, at the points t_j, $1 \le j \le k$, as :

$$L_{k-1}(f,x\,|\,t_1,...,t_k)= f(x)\prod_{\substack{i=1\\ i\neq j}}^{k-1}\frac{x-t_i}{t_j-t_i}\ .$$

Recall that , if q is a polynomial of degree $\leq m$, then $L_{k-1}(q)=q$ [6] .

If the interval $I=[\alpha,\beta]$ is denote $\Psi_\mu(I)=[\alpha-\mu(\beta-\alpha),\beta+\mu(\beta-\alpha)]$, i.e., $\Psi_\mu(I)$ is the interval of length $(1+2\mu)|I|$, such that I is in its center .

To prove theorem 2.2, we need this Lemma from [5] .

Lemma 2.1

Let f in $\Delta^k[a,b]$ f $L_P[a,b]$, $k\geq 1$. If an interval $I\subset[a,b]$ is such that $dist(I,\{a,b\})>0$, then , for any set $\{t_1,...,t_k\}$, of k points in I, and any $\mu>0$, such that $\Psi_\mu(I)\subset[a,b]$, we have

$$\|f-p_{k-1}(f,\cdot\,|\,t_1,...,t_k)\|_{L_P(\Psi_\mu(I))}\leq C(k,\mu)\omega_k(f,\Psi_\mu(I))_P\,,$$

where the constant C, which depends only on k and μ can be chosen to be a nonincreasing function of μ, $C(k,\mu_1)\leq C(k,\mu_2)$, for $\mu_1\geq\mu_2$.

Now , let us introduce our first theorem in this paper .

Theorem 2.2

Let f in $\Delta^k[a,b]$ f $L_P[a,b]$. Suppose that an interval $I\subset[a,b]$ is such that $b-a\leq A.dist(I,\{a,b\})$, for some $A\in R$. Also , let $\{t_1,...,t_{k-1}\}$ be a set of any (not necessarily distinct), $k-1$ points in I, and let q_{k-1} be a polynomial of degree $\leq k-1$, which interpolates f at $\{t_1,...,t_{k-1}\}$. Then the following inequality is valid ,

$$\|f-l_{k-1}(f)\|_P\leq C\|f-q_{k-1}\|_P\,,$$

where $l_{k-1}(f)$ is $L_{k-1}(f,\cdot\,|\,a,t_1,...,t_{k-1})$ or $L_{k-1}(f,\cdot\,|\,t_1,...,t_{k-1},b)$, and the constant C, depends only on k and A .

Proof. Since

$$\begin{aligned}\|f-l_{k-1}(f)\|_P&=\|f-q_{k-1}+q_{k-1}-l_{k-1}(f)\|_P\\&=\|f-q_{k-1}+l_{k-1}(q_{k-1})-l_{k-1}(f)\|_P\\&=\|f-q_{k-1}-l_{k-1}(f-q_{k-1})\|_P\\&\leq\|f-q_{k-1}\|_P+\|l_{k-1}(f-q_{k-1})\|_P\,,\end{aligned}$$

we recall that

$$l_{k-1}(f-q_{k-1},x)=(f(x)-q_{k-1}(x))\prod_{\substack{i=1\\ i\neq j}}^{k-1}\frac{x-t_i}{t_j-t_i}\,,$$

and hence

$$\|l_{k-1}(f-q_{k-1})\|_P=\left(\int_I\left|(f(x)-q_{k-1}(x))\prod_{\substack{i=1\\ i\neq j}}^{k-1}\frac{x-t_i}{t_j-t_i}\right|^P dx\right)^{1/P}$$

$$\le c(A)\frac{\left(b-a\big/2k\right)^{k-1}}{(b-a)^{k-2}(t_j-t_i)}\left(\int_I \left|(f(x)-q_{k-1}(x))\right|^P dx\right)^{1/P}$$

$$= c(A)\frac{b-a}{(t_j-t_i)(2k)^{k-1}}\left(\int_I \left|(f(x)-q_{k-1}(x))\right|^P dx\right)^{1/P}$$

$$\le c(A,k)\|f-q_{k-1}\|_P,$$

thus

$$\|f-l_{k-1}(f)\|_P \le \|f-q_{k-1}\|_P+\|l_{k-1}(f-q_{k-1})\|_P$$

$$\le C(A,k)\|f-q_{k-1}\|_P .$$

Corollary 2.3

If we assume q_{k-1} is a best approximation of f . *Then* $\|f-l_{k-1}(f)\|_P \le CE_{k-1}(f)_P$.

In this result we obtain a relationship between Lagrange polynomial and any interpolating polynomial .

Theorem 2.4

Let $k \ge 2$. Let p_{k-1} be a polynomial of degree $\le k-1$, interpolating f at the points $a \le t_1 \le \ldots \le t_{k-1} < b$. Then there is a function f in $\Delta^k[a,b]$ f $L_P[a,b]$, satisfying

$$\frac{\|p_{k-1}\|_P}{\|f\|_P} \ge c\left(1-c(p,k)\frac{1}{b-a}\right)$$

where c is a constant of $c(p,k)$ is a constant depends only on p and k .

Proof. Let $t_- = a$ and $t_k = b$, and define $I_i = [t_i, t_{i+1}]$, for $0 \le i \le k-1$. Now , let

$$f(x)=\begin{cases}(b-t_{k-1})^{1-k}(x-t_{k-1})_+^{k-1}; x\in[t_i,t_{i+1}],\\ 0 \quad ; \qquad\qquad\qquad\qquad x\notin[t_i,t_{i+1}],\end{cases}$$

where $(x-t_{k-1})_+^{k-1}=\max\{(x-t_{k-1})^{k-1},0\}$. And note that f in $\Delta^k[a,b]$ f $L_P[a,b]$, and

$$\|f\|_P^P=(b-t_{k-1})^{(1-k)P}\left.\frac{(x-t_{k-1})^{(k-1)P+1}}{(k-1)p+1}\right|_{t_i}^{t_i+1}=C$$

$$\|f\|_P=\left(\int_{t_i}^{t_{i+1}}\left|(b-t_{k-1})^{1-k}(x-t_{k-1})_+^{k-1}\right|^P dx\right)^{1/P}$$, so

Then since $\|f-L_{k-1}(f)\|_\infty \le c(k)\omega_k(f,I)_\infty$ [2], we have

$$\|f\|_\infty - \|L_{k-1}(f)\|_\infty \le \|f - L_{k-1}(f)\|_\infty$$
$$\le c(k)\omega_k(f,I)_\infty \le c(k)|I|^k \|f^{(k)}\|_\infty \ .$$

Then using the inequality

$$\|q^{(k)}\|_\infty \le c(p,r,k)|I|^{-k-1}\|q\|_P \ [3],$$

where q_r is a polynomial of degree $= r$. We have

$$\|f\|_\infty - \|L_{k-1}(f)\|_\infty \le c(p,k)|I|^k |I|^{-k-1}\|f\|_P$$
$$\le c(p,k)|I|^{-1}\|f\|_P \ .$$

Thus

$$\|L_{k-1}(f)\|_\infty \ge \|f\|_\infty - c(p,k)|I|^{-1}\|f\|_P$$
$$= \left(1 - c(p,k)|I|^{-1}\right)\|f\|_P . \qquad (2.1)$$

Since $L_{k-1}(f)$ is a Lagrange polynomial of degree $\le k-1$, and p_{k-1} is also a polynomial of degree $\le k-1$, defined on the interval $[a,b]$. So that there exists c_1 and c_2, such that

$$c_2\|L_{k-1}(f)\|_P \le \|p_{k-1}\|_P \le c_1\|L_{k-1}(f)\|_P \ .$$

This , implies

$$\frac{\|p_{k-1}\|_P}{\|f\|_P} \ge c_2 \frac{\|L_{k-1}(f)\|_P}{\|f\|_P} \ge c_2\left(1 - c(p,k)|I|^{-1}\right)$$
$$\ge c_2\left(1 - c(p,k)\frac{1}{b-a}\right),$$

where c is a positive constant and $c(p,k)$ is a constant depending on p and k .

Example 2.5

Let $f(x) = (1-\xi)^{1-k}(x-\xi)_+^{k-1}$ in $\Delta^k[0,1]$ f $L_P[0,1]$, such that ξ is the midpoint of $[0,1]$, and T_n is a Chebyshev polynomials of the first kind ,
we have

$$\|f(x)\|_P < \|f(x)\|_\infty = \sup_{x\in[0,1]}\left|\left(2\left(x-\frac{1}{2}\right)\right)^{k-1}\right| = 1. \qquad (2.2)$$

Since $k|\tilde{I}| \ge \sum_{i=0}^{k-1}|I_i| = 1$, where $\tilde{I}$ is the largest interval of the subintervals I_i, $0 \le i \le k-1$, satisfying $\sum_{i=0}^{k-1}|I_i| = 1$,

$$|\xi - t_i| \ge \frac{|\tilde{I}|}{2} \ge \frac{1-0}{2k}.$$

And using theorem 2.4

$$\|T_n(x)\|_P \ge \|L_{k-1}(f,x\,|\,t_1,\ldots,t_{k-1},1)\|_\infty$$
$$\ge |L_{k-1}(f,x\,|\,t_1,\ldots,t_{k-1},1)|$$

$$=\left|\left(2\left(x-\frac{1}{2}\right)\right)^{k-1}\prod_{i=1}^{k-1}\frac{x-t_i}{1-t_i}\right|$$

$$\geq\frac{\left(2\left(x-\frac{1}{2}\right)\right)^{k-1}\left(\frac{1}{2k}\right)^{k-1}}{(1-0)^{k-2}(1-t_{k-1})}\ .$$

Then

$$\frac{\|T_n\|_P}{\|f\|_P}\geq\frac{\|T_n\|_P}{\|f\|_\infty}\geq c(p,k)\frac{\left(2\left(x-\frac{1}{2}\right)\right)^{k-1}\left(\frac{1}{2k}\right)^{k-1}}{(1-t_{k-1})}$$

$$=c(p,k)\frac{1}{1-t_{k-1}}$$

Then , let us introduce the following auxiliary Lemma .

Lemma 2.6 [3]

Let $k\in\mathrm{N}$ and $f\in\Delta^k(a,b)$, and let $l_k(f,x\,|\,x_1,\ldots,x_k)$ be Lagrange polynomial of degree $\leq k-1$, interpolating f at the points x_i, $1\leq i$
$\leq k$, where $a=x_{-}<x_1\leq\ldots\leq x_k<x_{k+1}=b$. Then ,

$$(-1)^{k-i}\left(f(x)-l_k(f,x\,|\,x_1,\ldots,x_k)\right)\geq 0\,,\ x\in(x_i,x_{i+1}),\ 0\leq i\leq k\ .$$

Let a_i, denote the number of points t_i, such that $t_i=t_j$ with $i\leq j$.
Now , let us introduce our second theorem .

Theorem 2.7

Let $k\geq 2$, and an interval $I\subset[a,b]$ be such that $b-a\leq \mathrm{A}.dist(I,\{a,b\})$, for some $A\in R$, and $\{t_1,\ldots,t_{k-1}\}$ be a set of any $k-1$, points in I . If f,g in $\Delta^k[a,b]$ f $L_P[a,b]$ are such that $f^{(a_j-1)}(t_j)=g^{(a_j-1)}(t_j)$, for all $0\leq j\leq k$, (where $t_{-}=a$, $t_k=b$, and $a_j=a_j\left(\{t_i\}_{i=0}^k\right)$), then

$$\|f-g\|_P\leq C\min\{\omega_k(f,[a,b])_P,\omega_k(g,[a,b])_P\},$$

where the constant C, depends only on k and A .

Proof. Without loss of generality , assume that

$$\omega_k(f,[a,b])_P\leq\omega_k(g,[a,b])_P\ .$$

It was shown (see also Lemma 2.6) that , if f is k-monotone , then $f-L_{k-1}(f,\cdot\,|\,x_1,\ldots,x_k)$, changes sign at $x_1,\ldots,x_k$.

More precisely , let $k\in\mathrm{N}$, f *in* $\Delta^k[a,b]$ f $L_P[a,b]$, and recall that $L_{k-1}(f,\cdot\,|\,x_1,\ldots,x_k)$ is the Lagrange polynomial of degree $\leq k-1$, interpolating f (or f, together with its derivatives) at the points x_i, $1\leq i\leq k$, where $a=x_{-}\leq x_1\leq\ldots\leq x_k\leq x_{k+1}=b$. Then , for $0\leq i\leq k$,

$$(-1)^{k-i}\left(f(x)-L_{k-1}(f,x\,|\,x_1,\ldots,x_k)\right)\geq 0\,,\ x\in(x_i,x_{i+1})\qquad(2.3)$$

Now , let $q_{k-1}(x)=L_{k-1}(f,x\,|\,t_{\text{-}},\dots,t_{k-1})=L_{k-1}(g,x\,|\,t_{\text{-}},\dots,t_{k-1})$, and $\tilde{q}_{k-1}(x)=L_{k-1}(f,x\,|\,t_1,\dots,t_k)=L_{k-1}(g,x\,|\,t_1,\dots,t_k)$.

Inequalities (2.3), imply that for every $0\le i\le k-1$, and $x\in[t_i,t_{i+1}]$, the following inequalities are valid

$$(-1)^{k-1-i}\left(f(x)-q_{k-1}(x)\right)\ge 0,\ (-1)^{k-1-i}\left(g(x)-q_{k-1}(x)\right)\ge 0,$$

$$(-1)^{k-i}\left(f(x)-\tilde{q}_{k-1}(x)\right)\ge 0,\ (-1)^{k-i}\left(g(x)-\tilde{q}_{k-1}(x)\right)\ge 0,$$

and therefore every $x\in[a,b]$, the values $f(x)$ and $g(x)$, lie between $q_{k-1}(x)$ and $\tilde{q}_{k-1}(x)$. This implies that

$$\|f-g\|_P\le\|q_{k-1}-\tilde{q}_{k-1}\|_P\le\|f-q_{k-1}\|_P+\|f-\tilde{q}_{k-1}\|_P,$$

by theorem 2.2, we have

$$\begin{aligned}\|f-g\|_P &\le C\|f-\overset{\ulcorner}{q}_{k-1}\|_P+C\|f-\overset{\ulcorner}{\tilde{q}}_{k-1}\|_P\\ &\le C\omega_k(f,[a,b])_P,\end{aligned}$$

where $\overset{\ulcorner}{q}_{k-1}$, $\overset{\ulcorner}{\tilde{q}}_{k-1}$ are a polynomial of degree $\le k-1$, which interpolates f at $\{t_1,\dots,t_{k-1}\}$, and C, depends only on k and A .

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حول التقريب الرتيب –K في Lp

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الخلاصة

في هذا البحث ، نبحث اخطاء التعريب لدوال رتبة –K باندراج رتب - K وظهر لنا ان أي دالتين رتيبتين –K F و g يتقاطع بينهما في عدد كافي من النقاط في [a,b] تكونان متقاربين غير ان $\|f-g\|_P$ يكون صغير جداً.