Studying of Energy-Like Statistic Under Sine-Model Samples by Chord Circular Distance

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Abstract— *Statistical properties of the energy-like statistic under bivariate Sine-Model circular distribution and chord circular distance is presented in this paper. Simulation study is consider as well to examine the properties that we computed by comparing the true expected value and variance of the Sine-Model with the estimated expected value and variance under different sample sizes and different values of the parameters using R-Studio software.*

Keywords— ***Circular data, Sine-model circular distribution, Energy statistic, chord distance***

I. Introduction (Heading 1) (Times New Roman 11 Bold)

In various scientific fields including medicine, biology etc. there are many circular variables where a cyclic scale is used to register. This data, for example wind direction, circular data, such as time of day or compass direction, cannot be evaluated using conventional statistical techniques. As a result, to handle this type of data, we are looking for circular statistical approaches. [1, 2].

Watson in 1964 [3] presented the statistical test to test the difference amidst two circular distributions, which is the most common test. It's a non-rank-based test. This test is performed in most software implementations such as MATLAB.

Distinguishing between two multivariate samples is important however this can be hard. In 1984-1985, Gabor and Székely introduced a new statistic based on distance which is called energy statistic (ε-statistic), where this statistic was presented in a chain of lectures [4, 13].

Energy statistic is a function of the metric space separations between observations. In recent years, applications for -statistic have expanded to include [5, 6]. Let $X = \{x_1, \dots, x_n\}$ & $Y = \{y_1, \dots, y_m\}$ be independent samples, then the ε-statistic is defined by [16]

$$\varepsilon(X,Y) = \frac{nm}{n+m}\left(\frac{2}{nm}A - \frac{1}{n^2}B - \frac{1}{m^2}C\right)$$

where

$$A = \sum_{i=1}^{n}\sum_{j=1}^{m} D(x_i, y_j) \quad , \quad B = \sum_{i=1}^{n}\sum_{j=1}^{n} D(x_i, x_j) \quad ,$$

$$C = \sum_{i=1}^{m}\sum_{j=1}^{m} D(y_i, y_j) \quad .$$

Because the distribution of the ε-statistic is unknown and difficult to compute, it could be used in conjunction with a randomization test (which takes a very long time) to identify homogeneity between the distributions of two groups of data. Ali and Abushilah therefore suggested that in 2021 [7], a new statistical test to detect homogeneity amidst two groups of circular data. Let $G_1 = \{(\vartheta_{11}, \varphi_{11}), \dots, (\vartheta_{1n}, \varphi_{1n})\}$ and $G_2 = \{(\vartheta_{21}, \varphi_{21}), \dots, (\vartheta_{2m}, \varphi_{2m})\}$ be indpendent non-empty sets of bivariate angles. We seek to test the null hypothesis

$$H_0: F_1 = F_2$$

against

$$H_1: F_1 \neq F_2$$

where the circular distribution of the it is population is F_i of which G_i is a sample and energy-like statistic is define by the following form:

$$\mathcal{T}_d(G_1, G_2) = \left(\frac{nm}{n+m}\right)^{0.5} \sum_{i=1}^{n}\sum_{j=1}^{m} D(\vartheta_i, \varphi_j) \qquad (1)$$

where $D(\vartheta_i, \varphi_j)$ measure of distance between angles ϑ_i and φ_j.

In this paper the properties of the energy like statistic is calculated theoretically under Sine-Model and chord circular distance.

II. BASIC CONCEPTS

Basic ideas are introduced in this section.

A. Definition (2.1) [8] [12]

Circular statistics which are also called angular statistics or directional statistics is a branch of statistics specifically designed to deal with observations that are circular data. This data can be appeared as points on the unit circular, $\vartheta \epsilon [0, 2\pi)$, or $[-\pi, \pi$), or by the unit vector $(\cos(\theta), \sin(\theta))$.

B. Definition (2.2)

The Von -Mises distribution is one of the most important circular distributions. The Von -Mises distribution provided by Von-Mises (1918) [9]. The probability density function for this distribution, which is presented in Figure 1 with different concentration parameter, is given by

$$f(\vartheta; \mu, k) = \frac{1}{2\pi I_0(k)} e^{k\cos(\vartheta-\mu)}, \quad 0 \leq \vartheta < 2\pi,$$

Where $\kappa \geq 0$ is the concentration parameter, μ is the circular mean, and $I_0(k)$ is the modified Bessel function of the first kind and order zero [1-14] which is defined by

$$I_0(k) = \frac{1}{2\pi}\int_0^{2\pi} e^{k\cos\vartheta} d\vartheta,$$

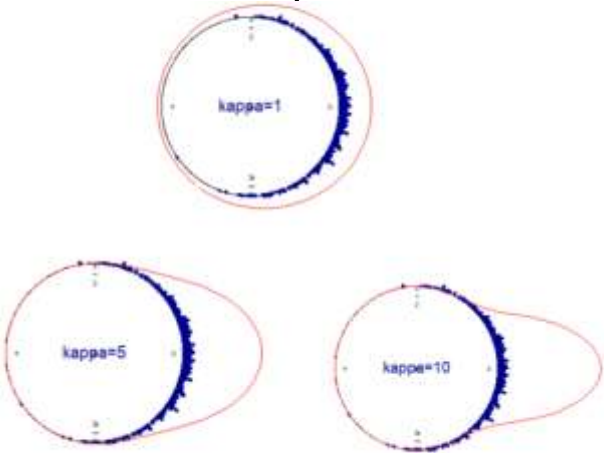

Fig 1: Von-Mises distribution which has been used in this analysis under different values of the concentration parameters and with sample size 1000 for each.

The bivariate Von Mises distribution with eight parameters was presented by Mardia [10], before Rivest a six parameters version in 1988. sine-model with five parameters of Rivest's distribution was introduced by Singh et al. in 2002 [11]. For two random circular variables, ϑ and φ. The probability density function of the bivariate von -Mises distribution of sine-model is given by

$$f(\vartheta, \varphi) = C_{Sine}\, exp\{\kappa_1 \cos(\vartheta - \mu_1) + \kappa_2 \cos(\varphi - \mu_2) + \delta \sin(\vartheta - \mu_1) \sin(\varphi - \mu_2)\},$$

for $0 \leq \vartheta, \varphi \leq 2\pi, k_1, k_2 \geq 0, 0 \leq \mu_1, \mu_2 \leq 2\pi$, C_{Sine} is a normalization constant which is defined by

$$C_{Sine} = \left[4\pi^2 \sum_{r=0}^{\infty} \binom{2r}{r} \left(\frac{\delta^2}{4\kappa_1\kappa_2}\right)^r I_r(\kappa_1) I_r(\kappa_2)\right]^{-1}.$$

III. PROPERTIES OF THE STATISTIC $\mathcal{T}_d$ UNDER SINE-MODEL SAMPLES AND CHORD CIRCULAR DISTANCE.

Let Φ and Ψ be two independent random variables such that $\Phi \sim BvM_{sine}(\mu, \Lambda_1)$ and $\Psi \sim BvM_{sine}(\mu, \Lambda_2)$ where $\mu = \binom{0}{0}$, $\Lambda_1 = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$ & $\Lambda_2 = \begin{pmatrix} \kappa_3 & 0 \\ 0 & \kappa_4 \end{pmatrix}$.
Let$G_1 = \{(\vartheta_{11}, \varphi_{11}), \dots, (\vartheta_{1n}, \varphi_{1n})\}andG_2 = \{(\vartheta_{21}, \varphi_{21}), \dots, (\vartheta_{2m}, \varphi_{2m})\}$ be random samples from these random variables, the circular distance amidst each pair of angle by using distance at that which is defined by

$$D(\vartheta_i, \varphi_j) = D(\vartheta_{1i}, \vartheta_{2j})^2 + D(\varphi_{1i}, \varphi_{2j})^2, \quad (2)$$
$$i = 1,2, \dots, n; j = 1,2, \dots, m$$

We calculate the expected value for the statistic $\mathcal{T}_d$ as follows:

$$E(\mathcal{T}_d(G_1, G_2)) = E[\left(\frac{nm}{n+m}\right)^{0.5} \sum_{i=1}^{n} \sum_{j=1}^{m} D(\theta_i, \theta_j)] \quad (3)$$

$$= \left(\frac{nm}{n+m}\right)^{0.5} \sum_{i=1}^{n} \sum_{j=1}^{m} E[D(\vartheta_{1i}, \vartheta_{2j})] + E[D(\varphi_{1i}, \varphi_{2j})].$$

Where

$$D(\theta_i, \theta_j) = \sqrt{2(1 - \cos(\theta_i - \theta_j))}, \theta = \vartheta, \varphi.$$

We comput the part $E[D(\vartheta_{1i}, \vartheta_{2j})]$ in eqution (3) as follows:

$$\sum_{i=1}^{n} \sum_{j=1}^{m} E[D(\vartheta_{1i}, \vartheta_{2j})] = \sum_{i=1}^{n} \sum_{j=1}^{m} E[2(1 - \cos(\vartheta_{1i} - \vartheta_{2j}))]$$

$$= 2nm - 2\sum_{i=1}^{n} \sum_{j=1}^{m} E[\cos(\vartheta_{1i} - \vartheta_{2j})]. (4)$$

The quantity $E[\cos(\vartheta_{1i} - \vartheta_{2j})]$ in Equation (4) is computed as follows:

$$E[\cos(\vartheta_1 - \vartheta_2)] = \int_0^{2\pi}\int_0^{2\pi} \cos(\vartheta_1 - \vartheta_2) f(\vartheta_1, \vartheta_2) d\vartheta_1 d\vartheta_2$$

$$= \int_0^{2\pi}\int_0^{2\pi} [\cos\vartheta_1 \cos\vartheta_2 + \sin\vartheta_1 \sin\vartheta_2] f(\vartheta_1) f(\vartheta_2) d\vartheta_1 d\vartheta_2$$

$$= \frac{V_{f1}}{V_{f0}}, \quad (5)$$

where

$$V_{f0} = I_0(\kappa_1) I_0(\kappa_3) \quad \& \quad V_{f1} = I_1(\kappa_1) I_1(\kappa_3).$$

As a result, we get the following

$$\sum_{i=1}^{n} \sum_{j=1}^{m} E[D(\vartheta_{1i}, \vartheta_{2j})] = 2nm - 2nm\frac{V_{f1}}{V_{f0}}.$$

In the same way extract

$$\sum_{i=1}^{n}\sum_{j=1}^{m} E[D(\varphi_{1i},\varphi_{2j})] = 2nm - 2nm\frac{V_{S1}}{V_{S0}},$$

where

$$V_{S0} = I_0(\kappa_2)I_0(\kappa_4) \ \& \ V_{S1} = I_1(\kappa_2)I_1(\kappa_4).$$

As a result, we get the following

$$E(\tau_d(G_1,G_2)) = (\frac{nm}{n+m})^{0.5}[2nm - 2nm\frac{V_{f1}}{V_{f0}} + 2nm - 2nm\frac{V_{S1}}{V_{S0}}].$$

So, the expected value of $\mathcal{T}_d$ is equal to this amount

$$E(\mathcal{T}_d(G_1,G_2)) = 2(\frac{(nm)^3}{n+m})^{0.5}[2 - \frac{V_{f1}}{V_{f0}} - \frac{V_{S1}}{V_{S0}}]. \quad (6)$$

The variance of $\mathcal{T}_d$ is calculated as follows:

$$var(\mathcal{T}_d(G_1,G_2)) = var[(\frac{nm}{n+m})^{0.5}\sum_{i=1}^{n}\sum_{j=1}^{m} D(\theta_i,\theta_j)]$$

$$= \frac{nm}{n+m} var[\sum_{i=1}^{n}\sum_{j=1}^{m} D(\theta_i,\theta_j)]$$

$$= \frac{nm}{n+m} var[\sum_{i=1}^{n}\sum_{j=1}^{m} D(\vartheta_{1i},\vartheta_{2j}) + D(\varphi_{1i},\varphi_{2j})]. (7)$$

To calculate the right-hand side of the equation (7) we must calculate the following quantities

- $var[D(\vartheta_1,\vartheta_2)]$.
- $cov[D(\vartheta_{11},\vartheta_{21}), D(\vartheta_{11},\vartheta_{22})]$.
- $cov[D(\vartheta_{11},\vartheta_{21}), D(\vartheta_{12},\vartheta_{21})]$.

The quantity $var[D(\vartheta_1,\vartheta_2)]$ is caculated as follows:

$$var[D(\vartheta_1,\vartheta_2)] = var[2(1 - cos(\vartheta_1 - \vartheta_2))]$$

$$= 4var[cos(\vartheta_1 - \vartheta_2)]$$

$$= 4[E[cos^2(\vartheta_1 - \vartheta_2)] - (E[cos(\vartheta_1 - \vartheta_2)])^2]. \quad (8)$$

The first part in the right hand side of Equation (8) equal to

$$E[cos^2(\vartheta_1 - \vartheta_2)] = \frac{1}{2}[1 + E[cos(2\vartheta_1 - 2\vartheta_2)]]$$

$$= \frac{1}{2}[1 + \frac{I_2(\kappa_1)I_2(\kappa_3)}{V_{f0}}]. \quad (9)$$

Substituting the Equation (9) and (5) into the Equation (8) we get the following:

$$var[D(\vartheta_1,\vartheta_2)] = 4\left[\frac{V_{f0} + I_2(\kappa_1)I_2(\kappa_3)}{2V_{f0}} + \frac{V_{f1}^2}{V_{f0}^2}\right]$$

$$= 4\left[\frac{V_{f0}^2 + V_{f0}I_2(\kappa_1)I_2(\kappa_3) + 2V_{f1}^2}{2V_{f0}^2}\right]. (10)$$

The quantity $cov[D(\vartheta_{11},\vartheta_{21}), D(\vartheta_{11},\vartheta_{22})]$ is computed as following:

$$cov[D(\vartheta_{11},\vartheta_{21}), D(\vartheta_{11},\vartheta_{22})]$$

$$= E[D(\vartheta_{11},\vartheta_{21})D(\vartheta_{11},\vartheta_{22})] - E[D(\vartheta_{11},\vartheta_{21})]E[D(\vartheta_{11},\vartheta_{22})]$$

$$= E[D(\vartheta_{11},\vartheta_{21})D(\vartheta_{11},\vartheta_{22})] - [1 - \frac{V_{f1}}{V_{f0}}]^2. \quad (11)$$

The first part of the equation (11) is calculated as follows:

$$E[D(\vartheta_{11},\vartheta_{21})D(\vartheta_{11},\vartheta_{22})]$$

$$= E[2(1 - cos(\vartheta_{11} - \vartheta_{21}))2(1 - cos(\vartheta_{11} - \vartheta_{22}))]$$

$$= 4[1 - E[\cos(\vartheta_{11} - \vartheta_{21})] - E[\cos(\vartheta_{11} - \vartheta_{22})]$$

$$+ E[cos(\vartheta_{11} - \vartheta_{21})cos(\vartheta_{11} - \vartheta_{22})]]$$

$$= 4[1 - 2\frac{V_{f1}}{V_{f0}} + E[cos(\vartheta_{11} - \vartheta_{21})cos(\vartheta_{11} - \vartheta_{22})]]. (12)$$

In Equation (12) the quantity $E[cos(\vartheta_{11} - \vartheta_{21})cos(\vartheta_{11} - \vartheta_{22})]$ is equal to

$$= E[cos^2\vartheta_{11}cos\vartheta_{21}cos\vartheta_{22}$$

$$+ E[cos\vartheta_{11}cos\vartheta_{21}sin\vartheta_{11}sin\vartheta_{22}]$$

$$+ E[sin\vartheta_{11}sin\vartheta_{21}cos\vartheta_{11}sin\vartheta_{22}]$$

$$+ E[sin^2\vartheta_{11}sin\vartheta_{21}sin\vartheta_{22}], \quad (13)$$

where

$$E[cos^2\vartheta_{11}cos\vartheta_{21}cos\vartheta_{22} = \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} cos^2\vartheta_{11}cos\vartheta_{21}cos\vartheta_{22}f(\vartheta_{11})f(\vartheta_{21})f(\vartheta_{22})d\vartheta_{11}d\vartheta_{21}d\vartheta_{22}$$

$$= \frac{I_1^2(\kappa_2)}{2I_0^2(\kappa_2)}\int_0^{2\pi} 1 + cos2\vartheta_{11}f(\vartheta_{11})d\vartheta_{11}$$

$$= \frac{I_1^2(\kappa_3)I_0(\kappa_1) + I_1^2(\kappa_3)I_2(\kappa_1)}{2I_0(\kappa_3)V_{f0}}, \quad (14)$$

and $E[cos\vartheta_{11}cos\vartheta_{21}sin\vartheta_{11}sin\vartheta_{22}]E[sin\vartheta_{11}sin\vartheta_{21}cos\vartheta_{11}sin\vartheta_{22}]$,$E[sin^2\vartheta_{11}sin\vartheta_{21}sin\vartheta_{22}]$ are equal to zero.

Substituting Equation (14) in

$$E[cos(\vartheta_{11} - \vartheta_{21})cos(\vartheta_{11} - \vartheta_{22})] = \frac{I_1^2(\kappa_3)I_0(\kappa_1) + I_1^2(\kappa_3)I_2(\kappa_1)}{2I_0(\kappa_3)V_{f0}}, \quad (15)$$

substituting Equation (15) in (12) we ge

$$E[D(\vartheta_{11},\vartheta_{21})D(\vartheta_{11},\vartheta_{22})] = 4[1 - 2\frac{V_{f1}}{V_{f0}} + \frac{I_1^2(\kappa_3)I_0(\kappa_1) + I_1^2(\kappa_3)I_2(\kappa_1)}{I_0(\kappa_3)V_{f0}}], (16)$$

substituting Equation (16) in (11) we get

$$cov[D(\vartheta_{11},\vartheta_{21})D(\vartheta_{11},\vartheta_{22})] = 4[\frac{I_1^2(\kappa_3)I_0^2(\kappa_1) + I_1^2(\kappa_3)I_1(\kappa_1)I_0(\kappa_1) - 2V_{f1}^2}{2V_{f0}^2}]. \quad (17)$$

In the same way extract

$$cov[D(\vartheta_{11},\vartheta_{21})D(\vartheta_{12},\vartheta_{21})] = 4[\frac{I_1^2(\kappa_1)I_0^2(\kappa_3) + I_1^2(\kappa_1)I_1(\kappa_3)I_0(\kappa_3) - 2V_{f1}^2}{2V_{f0}^2}]. (18)$$

In the same way extract the left hand side of Equation (7)

- $var[D(\varphi_1,\varphi_2)]$.
- $cov[D(\varphi_{11},\varphi_{21}), D(\varphi_{11},\varphi_{22})]$.
- $cov[D(\varphi_{11},\varphi_{21}), D(\varphi_{12},\varphi_{21})]$.

Such that

$$var[D(\varphi_1,\varphi_2)] = 4\left[\frac{V_{S0}^2 + I_0(\kappa_2)I_0(\kappa_4)I_2(\kappa_2)I_2(\kappa_4) + 2V_{S1}^2}{2V_{S0}^2}\right] \quad (19)$$

$$cov[D(\varphi_{11},\varphi_{21})D(\varphi_{11},\varphi_{22})] = 4\left[\frac{I_1^2(\kappa_4)I_0^2(\kappa_2) + I_1^2(\kappa_4)I_1(\kappa_2)I_0(\kappa_2) - 2V_{S1}^2}{2V_{S0}^2}\right], \quad (20)$$

$$cov[D(\varphi_{11},\varphi_{21})D(\varphi_{12},\varphi_{21})] = 4\left[\frac{I_1^2(\kappa_2)I_0^2(\kappa_4) + I_1^2(\kappa_2)I_1(\kappa_4)I_0(\kappa_4) - 2V_{S1}^2}{2V_{S0}^2}\right]. \quad (21)$$

As a result, we get the following

$$var(\mathcal{T}_d(G_1, G_2)) = 4\frac{(nm)^2}{n+m}[[VA_1 + VA_2] + \frac{n+m-2}{2}[CR_1 + CR_2 + CR_3 + CR_4]], \quad (22)$$

where

$VA_1 = var[D(\phi_1, \phi_2)]$ & $VA_2 = var[D(\psi_1, \psi_2)]$,

$CR_1 = cov[D(\phi_{11}, \phi_{21}), D(\phi_{11}, \phi_{22})]$,

$CR_2 = cov[D(\phi_{11}, \phi_{21}), D(\phi_{12}, \phi_{21})]$,

$CR_3 = cov[D(\psi_{11}, \psi_{21}), D(\psi_{11}, \psi_{22})]$,

$CR_4 = cov[D(\psi_{11}, \psi_{21}), D(\psi_{12}, \psi_{21})]$.

After the expected value and the varance of the statistic $\mathcal{T}_d$ have been computed, we seek to examine the properties that we have computed using simulated data. In the next section we will compare between the empirical and the theoretical properties that we have calculated.

PERFORMANCE EVALUATION

In this section, we conducted a simulation study to examine the characteristics of the "expectation and standard deviation" of the energy-like statistics. We seek to compare the theoretical value of the standard deviation and expectation with the empirical value of the standard deviation and expectation of the statistic $\mathcal{T}_d$, under square chord circular distance and Sine-Model samples.

To do a comparison we generate groups from circular distribution of sine-model and the empirical standard deviation and expected value for each pair of groups is calculated and compare with theoretical properties.

The simulation summary is shown in the Algorithm 1.

Results are presented in Table 1 and Table 2 in this tables we can see compare the theoretical value of the standard deviation and expectation with the empirical value of the standard deviation and expectation of the statistic $\mathcal{T}_d$, under different parameters and sample sizes of Two-sample Sine-Model.

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Algorithm 1: Simulation study.

Data: Circular data $\{(\vartheta_{11}, \varphi_{11}), \ldots, (\vartheta_{1n}, \varphi_{1n})\}, \{(\vartheta_{21}, \varphi_{21}), \ldots, (\vartheta_{2m}, \varphi_{2m})\}$.

Result: Theoretical and Empirical value of $E(\mathcal{T}_d)$ and $Sd(\mathcal{T}_d)$

begin

input $\{\mu_1, \mu_2, \mu_3, \mu_4, \kappa_1, \kappa_2, \kappa_3, \kappa_4, n, m\}$

for $(i \text{ in } 1:1000)$ **do**

Generate G_1 from $BeM_{sine}(\binom{\mu_1}{\mu_2}, \begin{pmatrix}\kappa_1 & 0\\ 0 & \kappa_2\end{pmatrix})$

Generate G_2 from $BeM_{sine}(\binom{\mu_3}{\mu_4}, \begin{pmatrix}\kappa_3 & 0\\ 0 & \kappa_4\end{pmatrix})$

with pdf

$f_1(\vartheta_1, \varphi_1) = C_s exp\{\kappa_1 cos(\vartheta_1 - \mu_1) + \kappa_2 cos(\varphi_1 - \mu_2) + \lambda sin(\vartheta_1 - \mu_1) sin(\varphi_1 - \mu_2)\}$

$f_2(\vartheta_2, \varphi_2) = C_s exp\{\kappa_3 cos(\vartheta_2 - \mu_3) + \kappa_4 cos(\varphi_2 - \mu_4) + \lambda sin(\vartheta_2 - \mu_3) sin(\varphi_2 - \mu_4)\}$

calculate Theoretical value of $E(\mathcal{T}_d)$ ***and*** $Sd(\mathcal{T}_d)$

- square chord circular distance $[D(\theta_i, \theta_j)]$ usin Equation $D(\theta_i, \theta_j) = 2(1 - cos(\vartheta_{1i} - \vartheta_{2j})) + 2(1 - cos(\varphi_{1i} - \varphi_{2j}))$
- $\mathcal{T}_d[i] \leftarrow (\frac{nm}{n+m})^{0.5} \sum_{i=1}^{n} \sum_{j=1}^{m} D(\theta_i, \theta_j)$
- $V_{f0} \longleftarrow I_0(\kappa_1) I_0(\kappa_3)$ & $V_{f1} \longleftarrow I_1(\kappa_1) I_1(\kappa_3)$
  $V_{s0} \longleftarrow I_0(\kappa_2) I_0(\kappa_4)$ & $V_{s1} \longleftarrow I_1(\kappa_2) I_1(\kappa_4)$
- $\hat{E}(\mathcal{T}_d) \leftarrow 2(\frac{(nm)^2}{n+m})^{0.5}[2 - \frac{V_{f1}}{V_{f0}} - \frac{V_{s1}}{V_{s0}}]$
- $VA_1 \longleftarrow var[D(\vartheta_1, \vartheta_2)]$ & $VA_2 \longleftarrow var[D(\varphi_1, \varphi_2)]$
  $CR_1 \longleftarrow cov[D(\vartheta_{11}, \vartheta_{21}), D(\vartheta_{11}, \vartheta_{22})]$ & $CR_2 \longleftarrow cov[D(\vartheta_{11}, \vartheta_{21}), D(\vartheta_{12}, \vartheta_{21})]$
  $CR_3 \longleftarrow cov[D(\varphi_{11}, \varphi_{21}), D(\varphi_{11}, \varphi_{22})]$ & $CR_4 \longleftarrow cov[D(\varphi_{11}, \varphi_{21}), D(\varphi_{12}, \varphi_{21})]$
- $\widehat{var}(\mathcal{T}_d(G_1, G_2)) = 4\frac{(nm)^2}{n+m}\left[[VA_1 + VA_2] + \frac{n+m-2}{2}[CR_1 + CR_2 + CR_3 + CR_4]\right]$
- $\widehat{Sd}(\mathcal{T}_d) \leftarrow [\widehat{var}(\mathcal{T}_d)]^{0.5}$

calculate Empircal value of $E(\mathcal{T}_d)$ ***and*** $Sd(\mathcal{T}_d)$

- $E(\mathcal{T}_d) = mean(\mathcal{T}_d[i])$
- $Sd(\mathcal{T}_d) = [var(\mathcal{T}_d[i])]^{0.5}$

end

Results $\{\hat{E}(\mathcal{T}_d)$ & $\widehat{Sd}(\mathcal{T}_d)$ & $E(\mathcal{T}_d)$ & $Sd(\mathcal{T}_d)\}$

end

TABLE 1. Comparison between the empirical expected value of the statistic $\mathcal{T}_d$ and the theoretical values under different parameters and sample sizes of two-Sample bivariate Sine-Model the simulation is repeated 1000 time.

$\boldsymbol{\kappa}$				*Expect value*	
κ_1	κ_2	κ_3	κ_4	*Theoretical*	*Empirical*
n=50 , m=50					
0.5	0.5	0.5	0.5	47059.7	47069.95
1.0	1.5	0.7	2.0	35916.14	35932.06
3.0	2.5	3.0	1.0	25060.92	25090.38
2.0	3.5	4.0	5.0	16150.7	16040.9
6.0	6.0	6.0	6.0	8380.02	8393.25
n=100 , m=200					
0.5	0.5	0.5	0.5	614785.3	614809.3
1.0	1.5	0.7	2.0	469206.5	469967.1
3.0	2.5	3.0	1.0	327394.5	327095.4
2.0	3.5	4.0	5.0	210991.8	210402
6.0	6.0	6.0	6.0	109476.2	109271.7
n=500 , m=400					
0.5	0.5	0.5	0.5	11224393	11226985
1.0	1.5	0.7	2.0	8566499	8566108
3.0	2.5	3.0	1.0	5977379	5975396
2.0	3.5	4.0	5.0	3842166	3849953
6.0	6.0	6.0	6.0	1998753	1997733

TABLE 2. Comparison between the empirical Standard deviation of the statistic $\mathcal{T}_d$ and the theoretical values under different parameters and sample sizes of two-Sample bivariate Sine-Model the simulation is repeated 1000 time.

κ				*Standard deviation*	
κ_1	κ_2	κ_3	κ_4	*Theoretical*	*Empirical*
n=50 , m=50					
0.5	0.5	0.5	0.5	1250.21	1249.414
1.0	1.5	0.7	2.0	1973.79	1971.22
3.0	2.5	3.0	1.0	2034.46	2007.75
2.0	3.5	4.0	5.0	1588.17	1606.71
6.0	6.0	6.0	6.0	811.469	817.86
n=100 , m=200					
0.5	0.5	0.5	0.5	9517.11	9516.71
1.0	1.5	0.7	2.0	15607.48	15527.71
3.0	2.5	3.0	1.0	14192.4	14729.25
2.0	3.5	4.0	5.0	12652.48	12068.68
6.0	6.0	6.0	6.0	6453.248	6470.993
n=500 , m=400					
0.5	0.5	0.5	0.5	93500.14	91273.48
1.0	1.5	0.7	2.0	155460	151753
3.0	2.5	3.0	1.0	161645.5	161518.7
2.0	3.5	4.0	5.0	129348	129555.3
6.0	6.0	6.0	6.0	64403.63	64848.07

V. CONCLUSION

Theoretical properties "expectancy and standard deviation" of the energy-like statistic based on the chord circular distance are presented under samples taken from the bivariate sine-model. The results of the simulation study show that the empirical expected values and standard deviation of the statistic $\mathcal{T}_d$ are close to the theoretical values under different sample size, different parameters of the Sine-Model circular distribution. In addition, a simulation study is conducted to verify the validity of our properties, using the R's circular package.

REFERENCES

[1] Jammalamadaka, S. Rao, and Ashis SenGupta. *Topics in circular statistics*. Vol. 5. world scientific, 2001.

[2] Pewsey, Arthur, Markus Neuhäuser, and Graeme D. Ruxton. *Circular statistics in R*. OUP Oxford, 2013.

[3] Wheeler, Stanley, and Geoffrey S. Watson. "A distribution-free two-sample test on a circle." *Biometrika* 51.1/2 (1964): 256-257.

[4] Rizzo, Maria L., and Gábor J. Székely. "Energy distance." wiley interdisciplinary reviews: Computational statistics 8.1 (2016): 27-38.

[5] Benoist, Tristan, Vojkan Jakšić, and Claude-Alain Pillet. "Energy statistics in open harmonic networks." Journal of Statistical Physics 168 (2017): 1016-1030.

[6] Green, Earl, and Charmaine D. DeLisser. "Advancing a culture of effective energy statistics and information management: A study of the CARICOM economic space." *World Journal of Science, Technology and Sustainable Development* 15.1 (2018): 26-34

[7] Ali, Ahmed Jebur, and Samira Faisal Abushilah. "Distribution-free two-sample homogeneity test for circular data based on geodesic distance." *International Journal of Nonlinear Analysis and Applications* 13.1 (2022): 2703-2711.

[8] Glimm, Ekkehard. "Fisher, NI: Statistical Analysis Of Circular Data. Cambridge University Press, Cambridge, UK 1995. 277 pp.,£ 16.95." Biometrical Journal 38.3 (1996): 314-314.

[9] Mardia, K. V., and P. E. Jupp. "Directional statistics. john willey and sons." *Inc., Chichester* (2000).

[10] Mardia, K. V., and T. W. Sutton. "On the modes of a mixture of two von Mises distributions." Biometrika (1975): 699-701.

[11] Singh, Harshinder, Vladimir Hnizdo, and Eugene Demchuk. "Probabilistic model for two dependent circular variables." Biometrika 89.3 (2002): 719-723.

[12] Cremers, Jolien, and Irene Klugkist. "One direction? A tutorial for circular data analysis using R with examples in cognitive psychology." *Frontiers in psychology* 9 (2018): 2040.

[13] Nguyen, Hien Duy, et al. "Approximate Bayesian computation via the energy statistic." *IEEE Access* 8 (2020): 131683-131698.

[14] Zhao, Tie-Hong, Lei Shi, and Yu-Ming Chu. "Convexity and concavity of the modified Bessel functions of the first kind with respect to Hölder means." *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas* 114 (2020): 1-14.

[15] Nguyen, Hien Duy, et al. "Approximate Bayesian computation via the energy statistic." *IEEE Access* 8 (2020): 131683-131698.

[16] Gritsenko, Alexey, et al. "A spectral energy distance for parallel speech synthesis." *Advances in Neural Information Processing Systems* 33 (2020): 13062-13072.