The w-Expansive With The Shadowing Property in g-Nonautonomous Discrete Dynamical Systems

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Abstract— *In this study, we define the w-expansive property of generic non-autonomous discrete dynamical systems. We also show how the w-expansive feature is preserved under conjugate, inverse limits, and iterate. In addition, we discuss the relationship between the w-expansive and shadowing properties in generic non-autonomous discrete dynamical systems.*

Keywords— general nonautonomous dynamic, expansive, w-expansive, shadowing properties, uniform homeomorphism, topologically conjugate .

I. INTRODUCTION

Expansion has been studied in mathematics since 1950, R.Bowen and P. Waiter wrote the basics for the theory of expansive flow [1]. course of studying expansion, many equivalent definitions and new types of expansion emerged, such as (n, m)-expansiveness [2]. been clarified n-expansiveness on a compact surface M in [3]. L. Snoha and S. Kolyada introduced nonautonomous discrete dynamic systems (NDS for short) in their work [4], In [5] Radhika and Ruchi are study some types of expansive for non-autonomous discrete dynamical systems, In [6] it was addressed the dynamics of n-expansive homeomorphisms with the shadowing property specified on compact metric spaces. The general-nonautonomous discrete dynamical systems is defined by Baraa A. and Iftichar M. and denoted by $(\boldsymbol{X}, \boldsymbol{\ell}_{n,\infty})$ when **X** is a compact topological space and $\boldsymbol{\ell}_n: \boldsymbol{X} \to \boldsymbol{X}$, $\forall \boldsymbol{n} \in \mathbb{N}$, such that $\boldsymbol{\ell}_{n,\infty}$ is the sequence of the continuous maps in [7]. in present study, we will definition the w-expansive of the g-non autonomous discrete dynamical systems in section two. In section three we will explain the relation between the w-expansive and the shadowing property in general non autonomous discrete dynamical systems.

II. DEFINITIONS

First, we define the general non-autonomous discrete dynamical systems

2.1. Assume that a metric space is $(\mathbf{X}, \boldsymbol{d})$, where (ℓ_n) be a sequence of a uniform homeomorphism maps,

such that $\ell_n: \mathbf{X} \longrightarrow \mathbf{X}$, for all $n \in \mathbb{Z}$, and the composition

$$\ell_n^m = \begin{cases} \ell_m \circ \ell_{m-1} \circ \dots \dots \circ \ell_n & for\ 0 \le n < m \\ \ell_{-m}^{-1} \circ \ell_{-(m-1)}^{-1} \circ \dots \dots \circ \ell_{-n}^{-1} & for\ m < n \le -1 \end{cases}$$

Is said to be The g-non autonomous discrete dynamical systems for short (g-NDS). its inverse map is given by $(\ell_n^m)^{-1} = \ell_{-m}^{-n}$, $where\ m < n \le 0$. And we define a (k th –iterate of ℓ_n^m)

$(\ell_n^m)^k = (g_m)_{mk}^{(m-1)k+1}$, where $0 \le \boldsymbol{n} < \boldsymbol{m} \in \mathbb{N}, k > 0$ on **X**, where

$$g_m = \ell_{mk} \circ \ell_{(m-1)k+k-1} \circ \dots \circ \ell_{(m-1)k+1.}$$

Thus $(\ell_n^m)^k = \ell_{(m-1)k+1,mk}$.

2.2. Assume that a metric space is $(\mathbf{X}, d)$ and $\ell_n: \mathbf{X} \to \mathbf{X}$ be a continuous maps $\forall n \in \mathbb{N}$, the sequence ℓ_n^m is said to be an expansive with constant expansive $e > 0$ in g-nonautonomous discrete dynamical systems when $\forall x, y \in X, x \neq y, \exists\, 0 \le n < m \in \mathbb{N}, d\big(\ell_n^m(x), \ell_n^m(y)\big) > e$. Equivalently,

if $x, y \in X, d\big(\ell_n^m(x), \ell_n^m(y)\big) \le e, \forall\, 0 \le n < m \in \mathbb{N}$,

then $x = y$.

2.3. A homeomorphism $\ell_n: \mathbf{X} \to \mathbf{X}$ is called a uniform homeomorphism If both in which ℓ_n^m, and $(\ell_n^m)^{-1}$is a uniform continuous on **X.**

2.4. Assume that two compact metric spaces are $(\mathbf{X}, d_1)$ and (Y, d_2) with non-autonomous mapping sequences $(\ell_n^m)_{0 \le n < m}$and$(g_n^m)_{0 \le n < m}$, respectively. If there is a uniform homeomorphism $h: \mathbf{X} \to Y$ such that $h \circ \ell_n^m = g_n^m \circ h$, for all$0 \le n < m \in \mathbb{Z}$, then ℓ_n^m and g_n^m are said to be topologically conjugate.

2.5. Assume that a metric space is (X, d), and $\ell_n: \mathbf{X} \longrightarrow \mathbf{X}$ be an uniform continuous map. The sequence ℓ_n^m is said to be w-expansive in g-nonautonomous discrete dynamical systems : if whether a constant exists $e > 0$ (is said an w-expansive constant) whereas for each $x \in \mathbf{X}$. $\Lambda(x, e)$ is the set that has w different points (where $w = m - n$), and let the set $\Lambda(x, e) = \{\, y \in \mathbf{X}: d(\ell_n^m(x), \ell_n^m(y) \le e, 0 \le n < m \in \mathbb{N}\,\}$, i.e. for every $x \in \mathbf{X}$, $\exists\; e > 0, \ni \Lambda(x, e) =$ $\mathbf{X} - \{y \in X: d(\ell_n^m(x), \ell_n^m(y) > e, 0 \le n < m \in \mathbb{N}\,\}$ which be a finite set of w different point.

2.6. Assume that a metric space is $(\mathbf{X}, d)$ and $(\ell_n^m)_{0 \le n < m}$is sequence on **X**. for $\boldsymbol{\delta} > \mathbf{0}$, the sequence $\{x_i\}_{i=0}^{\infty}$ in **X** is defined as an δ-pseudo orbit of ℓ_n^m in g-non autonomous discrete dynamical systems if

$$d\big(\ell_j(x_i), x_{i+1}\big) < \delta, \quad for\, n \le j < m \in \mathbb{N},$$

$$d(\ell^{-j}(x_{m+1}), x_m) < \delta \quad for\, m < j \le n.$$

And for $\varepsilon > 0$, a δ-pseudo orbit $\{x_i\}_{i=0}^{\infty}$ is deemed to be ε-traced in g-nonautonomous discrete dynamical systems

by $y \in X$ if

$$d((\ell_n^m)^i(y), x_i) < \varepsilon \text{ for all } 0 \le n < m \in \mathbb{N}.$$

then ℓ_n^m is considered to have pseudo orbit tracing property (P.O.T.P.) or shadowing property in g- non autonomous discrete dynamical systems if there exists $\delta > 0$ for each $\varepsilon > 0$ then every δ-pseudo orbit by some point of $\boldsymbol{X}$ is ε-traced.

III. THE RELATION BETWEEN THE W-EXPANSIVE AND THE SHADOWING PROPERTY IN G-NDS.

Some results are discussed for the dynamics of w-expansive homeomorphisms on compact metric spaces.

Theorem .3.1

Assume that a compact metric space is $(\mathbf{X}, d)$ and $\ell_n: \mathbf{X} \to \mathbf{X}$ be an uniform continuous maps $\forall n \in \mathbb{N}$, the sequence ℓ_n^m is w-expansive in g-nonautonomous discrete dynamical systems if and only if $(\ell_n^m)^{-1}$ is w-expansive in g-nonautonomous discrete dynamical systems.

***Proof**:* Suppose that $e > 0$ be an w-expansive constant for ℓ_n^m For a fixed $x \in X$; $\{y \in X: d(\ell_n^i(x), \ell_n^i((y)) \leq e, 0 \leq n < i \in \mathbb{N}\}$ is the set that has at most w elements, i.e. $\{y \in X: d((\ell_n^i)^{-1}(x), (\ell_n^i)^{-1}(y)) \leq e, 0 \leq n < i \in \mathbb{N}\}$ is the set that has at most w-elements. As a result, $(\ell_n^m)^{-1}$ is w-expansive. Similarly, the inverse could be proof demonstrated. □

Theorem 3.2

Assume that a compact metric space is $(\mathbf{X}, d)$ and $\ell_n: \mathbf{X} \to \mathbf{X}$ of an equicontinuous uniformly continuous maps

$\forall n \in \mathbb{N}$, For any positive integer k, the sequence ℓ_n^m is w-expansive $\Leftrightarrow (\ell_n^m)^k$ is w-expansive

Proof: Suppose that $e > 0$ is a w-expansive constant for $\ell_{n,m}$ Because the family of maps known as ℓ_n^m is equicontinuous.

For each $h \geq 0$ and $hk+ \leq j(h+1)k$;

Since ℓ_n^j is uniformly continuous on $\boldsymbol{X}$ and therefor $\exists\, e_j > 0$ meaning that

$d(x,y) \leq e_j$ implies $d\left(\ell_{hk+1}^j(x), \ell_{hk+1}^j(y)\right) \leq e.$

Since the maps $\ell_n: \boldsymbol{X} \to \boldsymbol{X}$, $n \in \mathbb{N}$ are equicontinuous, e_j is not dependent on h.

We take

$e' = \min\{e_j: hk + \mathbf{1} \leq j \leq (h+\mathbf{1})k\}$**.** As a result, for any $h \geq \mathbf{0}$,

$d(x,y) \leq e'$ implies $d\left(\ell_{hk+1}^j(x), \ell_{hk+1}^j(y)\right) \leq e.$

Let $(\ell_n^m)^k = g_n^m$, where $g_m = \ell_{(m-1)k+1}^{mk}$ and $g_n^m = g_m \circ g_{m-1} \circ \dots \circ g_n$. Not that $\ell_n^{mk} = g_n^m$. Thus ,

for any $h \geq 0$ and $hk \leq j \leq (h+1)k$,

$d(g_n^h(x), g_n^h(y) \leq e$ which implies $d(\ell_n^{hk}(x), \ell_n^{hk}(y)) \leq e'$ and hence we get that

$d(\ell_{hk+1}^j(\ell_n^{hk}(x)), \ell_{hk+1}^j(\ell_n^{hk}(y))) \leq e$ which impliesd$(\ell_n^j(x), \ell_n^j(y)) \leq e$. Since e is an w-expansive constant for ℓ_n^m ,

the set $\Lambda(x,e)$ has at most w element (where $w = m - n$).Therefore

$\{y \in X: d(g_n^i(x), g_n^i(y)) \leq e'; 0 \leq n < i\}$ has at most $w$element and

hence $(\ell_n^m)^k$ is w-expansive and has a constant of w-expansive $e' \geq 0$.

On the other hand, if $(\ell_n^m)^k$ is w-expansive which constant of w-expansiveness

$e > \mathbf{0}$, thus, then for any $x \in \boldsymbol{X}$, the set $\Lambda(x,e)$ has at most w element, where $g_m = \ell_{(m-1)k+1}^{mk}$. thus,

the set $\{y \in X: d(\ell_n^{ik}(x), \ell_n^{ik}(y)) \leq e, for\ 0 \leq n < i\} = \{d(\ell_n^j(x), \ell_n^j(y)) < e, 0 \leq n < j\}$

has at most w element.

Hence, $\ell_n^m$is w-expansive with a constant of w-expansiveness

$\boldsymbol{e} > \mathbf{0}$ in g-nonautonomous discrete dynamic systems.

Theorem 3.3

Assume that two non-autonomous systems are (X_1, d_1) and (X_2, d_1) meaning that $(\ell_n^m)_{\mathbf{0} \leq \boldsymbol{n} < \boldsymbol{m}}$ is an uniformly conjugate to g_n^m. If ℓ_n^m is w-expansive of g-nonautonomous discrete dynamical systems $\forall\, n \in \mathbb{N}$, then g_n^m is also.

Proof : Suppose that c > 0 be an w-expansiveness constant for ℓ_n^m

There exists a uniform homeomorphism $h : X_1 \to X_2$ because ℓ_n^m is uniformly conjugate to g_n^m

so that $h \circ \ell_n^m = g_n^m \circ \boldsymbol{h}$,

for each $0 \le n < m \in \mathbb{N}$. As result that, $\ell_m^n \circ h^{-1} = h^{-1} \circ g_m^n$,

for each $0 \le n < m \in \mathbb{N}$.

therefore for each $e > 0$, there is $e' > 0$ so that for $x, y \in X_2$,

$d_2(x, y) \le e'$ implies $d_1(h^{-1}(x), h^{-1}(y)) \le e$.

For a constant $x \in X_2$, the set

$S = \{y \in X_2 : d_2(g_n^m(x), g_n^m(y)) \le e', \forall\, 0 \le n < \boldsymbol{m} \in \mathbb{N}\} \subseteq$

$\{y \in X_2 : d_1(h^{-1}(g_n^m(x)), h^{-1}(g_n^m(y))) \le e', \forall\, 0 \le n < \boldsymbol{m} \in \mathbb{N}\}$

$= \{y \in X_2 : d_1(\ell_n^m(h^{-1}(x)), \ell_n^m(h^{-1}(y))) \le e,\ 0 \le n < \boldsymbol{m} \in \mathbb{N}\}$.

So because ℓ_n^m is w-expansive and with w-expansiveness constant $\boldsymbol{e} > \boldsymbol{0}$,

and thus, $\boldsymbol{S}$ has w of different points (where $w = m - n$) and hence g_n^m

is w-expansive of g-NDS with w-expansiveness constant $e' > 0$. ◻

Theorem 3.4

Assume that a metric space is $(\mathbf{X}, d)$ and $\ell_n : \mathbf{X} \to \mathbf{X}$ be an uniform continuous map $\forall n \in \mathbb{N}$, for every $w \in \mathbb{N}$, $\exists\, w = m - n$, there is an w-expansive homeomorphism, defined in g-nonautonomous discrete dynamical systems , that is not $(w-1)$-expansive, possesses the shadowing property in g-nonautonomous discrete dynamical systems permits an unlimited number of chain recurrent classes.

***Proof*:** Take into account an expansive homeomorphism g_n^m constructed in a compact metric space (M, d_0) and meets the shadowing property.

Furthermore, assume that it has an unlimited number of periodic points $\{p_k\}_{k\in\mathbb{N}}$, which we may assume belong to various orbits .

If E is an infinite enumerable set, then define **X** as the set $M \cup E$. A bijection $r : N \to E$ exists is present in this situation.

Suppose that $Q = \{(a, b, c) : a \in \{1, \ldots, m-1\};\ \boldsymbol{m} \in \mathbb{N},\ b \in \{k\}_{k\in\mathbb{N}},$

$c \in \{0, \ldots, \pi(pk) - 1\};\ k \in \mathbb{N}\}$

and take notice that a bijection $s : Q \to N$ exists. Take into account the bijection.

$q : Q \to E$ specified by $q(a, b, c) = r \circ s(a, b, c)$.

As a result, then for any point, $x \in E$ has the expression $x = q(a, b, c)$ for some

$(a, b, c) \in Q$.

$d : \mathbf{X} \times \mathbf{X} \to \mathbb{R}^+$ is the function defined by

$$d(x,y) = \begin{cases} 0, & x = y, \\ d_0(x,y), & x,y \in M, \\ \frac{1}{k} + d_0(x, g_n^c(p_b)), & x \in M, y = q(a,b,c), \\ \frac{1}{k} + d_0(g_n^c(p_b), y), & x = q(a,b,c), y \in M, \\ \frac{1}{k} + \frac{1}{w} + d_0(g_n^c(p_b), g_n^r(p_h)), & x = q(a,b,c), y = q(e,h,r), \\ & b \neq h \text{ or } c \neq r, \\ \frac{1}{k}. & x = q(a,b,c), y = q(e,b,c) \\ & , a \neq e. \end{cases}$$

Then to prove that ℓ_n^m is w-expansive where ($w = m - n, \forall n, m \in \mathbb{Z}$), an expansiveness of g_n^m assures the presence of a number greater than zero $\delta > 0$ so that if $d(g_n^k(x), g_n^k(y)) \leq \delta$, for every $0 \leq n < k \in \mathbb{Z}$,

then $x = y$. Suppose that $\{x_n, x_{n+1} \ldots, x_m, x_{m+1}, \ldots, x_{m+n}\}$ are $w + 1$ different points of X satisfyin:

$d\left(\ell_n^k(x_i), \ell_n^k(x_j)\right) \leq \delta,\ 0 \leq n < k \in \mathbb{N}$, For every pair $(i,j) \in \{n, \ldots, m+n\} \times \{n, \ldots, m+n\}$. And most one of these points belong to M, and at least m of them belong to E,

by $d(\ell_n^s(x), \ell_n^s(q(a,b,c)) \leq \delta, \forall\ 0 \leq n < s \in \mathbb{Z}$. We contend that at least two of these points are the following: $q(a,b,c)$ and $q(a,h,r)$ with $b \neq h$. Indeed, if this is not the case, then two of them are of the form

$q(a,b,c)$ & $q(a,b,r)$ with $c \neq r$. It follows that for each $0 \leq n < s \in \mathbb{Z}$ we have

$$d(g_n^s(g_n^c(p_b)), g_n^s(g_n^r(p_b)) =$$
$$d(\ell_n^s(q(a,b,c)), \ell_n^s(q(a,b,r)) - \frac{2}{k}$$
$$< d(\ell_n^s(q(a,b,c)), \ell_n^s(q(a,b,r))$$
$$\leq \delta.$$

This implies that $g_n^c(p_b) = g_n^r(p_b)$, This further suggests that $c = r$ and thus we a contradiction is obtained.

And for each $0 \leq n < s \in \mathbb{Z}$ the following holds

$$d(g_n^s(g_n^c(p_b)), g_n^s(g_n^r(p_h))$$
$$= d(\ell_n^s(q(a,b,c)), \ell_n^s(q(e,h,r)) - \frac{2}{k}$$
$$<$$
$$d(\ell_n^s(q(a,b,c)), \ell_n^s(q(e,h,r))$$
$$\leq \delta.$$

Therefore, $g_n^c(p_b) = g_n^r(p_h)$ and $p_b = p_h$, This is in conflict with the fact that $b \neq h$. It is significant to note that this may be done since.

Periodic points $\{p_k\}_{k \in \mathbb{N}}$ are present in M in an infinite number.

Now to prove that ℓ_n^m is non $(w-1)$-expansive :

For every $\delta > 0$, choose $k \in \mathbb{N}, \ni \frac{1}{k} < \delta$, and the set $\Lambda\left(p_k, \frac{1}{k}\right)$ contains w different points ($w = m - n$) hence, for every $\boldsymbol{i} \in \{\mathbf{1}, \ldots, \boldsymbol{m} - \mathbf{1}\}$ the point $q(a,b,0)$ belonging to $\Lambda\left(p_k, \frac{1}{k}\right)$ that's implies the $\Lambda(p_k, \delta)$ contains includes at least w or more different points, and ℓ_n^m is non $(w-1)$-expansive .To prove ℓ_n^m has the shadowing property in g-nonautonomous discrete dynamical systems,

Because g_n^m has the shadowing property, we may consider $\delta_g > 0$, for any $\varepsilon > 0$ so that every δ_g-pseudo orbit of g_n^m is $\frac{\varepsilon}{2}$ -shadowed. Choose L belongs to $\mathbb{N}$ so that $\frac{1}{L} < \min\{\frac{\varepsilon}{2}, \frac{\delta_g}{3}\}$

and let $\delta = \frac{1}{L}$. If $\{x_i\}_{i \in \mathbb{N}} \subset \mathbf{X}$ is a δ-pseudo orbit of ℓ Hence, either $\{x_i\}_{i \in \mathbb{N}}$, so one of the

orbits $\{q(a,b,c);\ c \in \{0, \ldots, \pi(p_k) - 1\},\ a \in \{1, \ldots, m - 1\},\ b \in \{1, \ldots, L - 1\}\}$, or $\{x_i\}_{i=0}^{\infty}$ contains none of these orbits' points.

We will concentrate on the second case because $\{x_i\}_{i=0}^{\infty}$ in the first case is visibly shadowed. Then, if

$x_i = q(a,b,c)$ then $b \geq L$.

Define a sequence $\{y_i\}_{i=n}^{m} \subset$ M by

$$y_i = \begin{cases} x_i, & x_i \in M, \\ g_n^c(p_b), & x_i = q(a,b,c). \end{cases}$$

The sequence $\{y_i\}_{i=0}^{\infty}$ is a δ_g-pseudo orbit for g_n^m because, for any $i \in \mathbb{N}$

holds the following:

$$d((g_n^m)^{i+1}(y), y_{i+1}) = d((\ell_n^m)^{i+1}(y), y_{i+1})$$

$$\leq d\left((\ell_n^m)^{i+1}(y), (\ell_n^m)^{i+1}(x)\right) + d((\ell_n^m)^{i+1}(x), x_{i+1})$$

$$+ d(x_{i+1}, y_{i+1}),$$

$$\leq \frac{1}{L} + \frac{1}{L} + \frac{1}{L}$$

$$\leq \delta_g .$$

Then there exists $x \in M$ so that

$$d((g_n^m)^i(x), y_i) < \frac{\varepsilon}{2}, i \in Z.$$

It follows that $\{x_i\}_{i=0}^{\infty}$ is ε-shadowed by x, because for all $i \in Z$

holds the following:

$$d((\ell_n^m)^i(x), x_i) \leq d((\ell_n^m)^i(x), y_i) + d(y_i, x_i)$$

$$\leq \frac{\varepsilon}{2} + \frac{1}{L}$$

$$\leq \varepsilon .$$

Because this may be done for any $\varepsilon > 0$, We may conclude that ℓ_n^m has the shadowing property in g-nonautonomous discrete dynamical systems.

To demonstrate that ℓ_n^m permits an unlimited number of chain-recurrent classes, consider the various periodic orbits in E belonging to many different chain-recurrent classes. Indeed, each point $q(a,b,c) \in E$ meets the following condition:

$$d(q(a,b,c), x) \geq \frac{1}{k}, x \in \mathbf{X} \setminus \{q(a,b,c)\}.$$

This means that if $0 < \varepsilon < \frac{1}{k}$ then the orbit of $q(a,b,c)$ cannot be linked to by ε-pseudo orbits with any other point of **X**.

This demonstrates that the chain recurrent class of $q(a,b,c)$ contains only its orbit.

[a.] Sample of a Table footnote. *(Table footnote)*

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