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Explicit Form of Harnack's Inequality for Poisson's Dirichlet Problem

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Abstract*— *The explicit form of Harnack's inequality for non-negative harmonic functions in the open ball $\boldsymbol{B_{\tilde{\rho}}(0)}$ plays an important role in harmonic analysis and elliptic partial differential equations. In this paper, we establish the explicit form of Harnack's inequality for Poisson's Dirichlet problem in the open ball $\boldsymbol{B_{\tilde{\rho}}(0)}$ with non-negative boundary data. We involve the Green's function for Laplacian operator to find it out. Harnack's inequality for positive harmonic functions can be followed from Harnack's inequality for Poisson's Dirichlet problem when the source function is set to be zero.

Keywords*— *Harmonic functions, Green's function, Harnack's inequality, Dirichlet Problem.

I. Introduction

It is well known that the Harnack's inequality for non-negative harmonic function u over the open ball $\boldsymbol{B_{\tilde{\rho}}(0)} = \{\zeta \in R^n : |\zeta| < \tilde{\rho}\}$ [7,6,13] is given by

$$\tilde{\rho}^{n-2}\frac{\tilde{\rho}-|\zeta|}{(\tilde{\rho}+|\zeta|)^{n-1}}u(0) \le u(\zeta) \le \tilde{\rho}^{n-2}\frac{\tilde{\rho}+|\zeta|}{(\tilde{\rho}-|\zeta|)^{n-1}}u(0), \quad (1)$$

for every $\zeta \in B_{\tilde{\rho}}(0)$ which established from Poisson's integral formula for harmonic functions.

The inequality in (1) leads for another version of Harnack's inequality that work in bounded connected domains [7,13], if Ω is a bounded connected domain in R^n and Ω' is a compact subdomain in Ω, then there exists a constant $M \ge 1$, such that

$$sup_{\Omega'} u \le inf_{\Omega'} u, \quad (2)$$

for every non-negative harmonic function u in Ω. The constant M in (2) depends only the geometry of Ω and Ω'.

Even though inequality (2) is more general than (1), the constant in (2) is not given in an explicit way as it appears in (1).

One advantage of (2) is to deduce the convergence principle for monotone harmonic functions [2], if $\{u_n\}$ is an increasing sequence of harmonic functions in a bounded domain Ω such that

1. $u_n(\zeta) \le u_{n+1}(\zeta)$, for every $\zeta \in \Omega$ and $n = 1,2,\dots$.
2. There exists a point $\zeta \in \Omega$, such that $|u_n(\zeta)| \le C$, for some positive constant C and every $n = 1,2,\dots$.

Then for any compact subdomain Ω' of Ω, the sequence $\{u_n\}$ converges uniformly to a harmonic function in Ω' .

A new version of (1) for an uncompact subdomain in the punctured unit open ball has been established in [1,12] with adding an extra condition for the radial movement of the points inside the subdomain.

Inequality (2) has been generalized for linear elliptic operators of the divergence form [5,4]. Also, a short and easy proof of version of Harnack's for the general form of linear elliptic operators appears in [9].

The Harnack's inequality has been published, for fractional Laplacian operators [8], discrete difference equations [10], and in [11] for Lipschitz domains.

1.1 New Result

In this work, we present the Harnack's inequality for the Dirichlet problem

$$\begin{cases} \Delta u = f & in \quad B_{\tilde{\rho}}(0) \\ u = g & on \quad \partial B_{\tilde{\rho}}(0)\,, \end{cases} \quad (3)$$

where $f \in C(B_{\tilde{\rho}}(0))$ *and* $0 \le g \in C(\partial B_{\tilde{\rho}}(0))$.

We obtain the inequality

Theorem 1: Let u be a solution of (3), then it follows that

$$\tilde{\rho}^{n-2}\left(\frac{\tilde{\rho}-|\zeta|}{(\tilde{\rho}+|\zeta|)^{n-1}}\right)\left(u(0) - ||f||_{L^\infty(B_{\tilde{\rho}}(0))}\left(\frac{\tilde{\rho}^2}{2n}\right)\right) - ||f||_{L^\infty(B_{\tilde{\rho}}(0))}\left(\frac{\tilde{\rho}^2-|\zeta|^2}{2n}\right) \le u(\zeta)$$
$$\le \tilde{\rho}^{n-2}\left(\frac{\tilde{\rho}+|\zeta|}{(\tilde{\rho}-|\zeta|)^{n-1}}\right)\left(u(0) - ||f||_{L^\infty(B_{\tilde{\rho}}(0))}\left(\frac{\tilde{\rho}^2}{2n}\right)\right) + ||f||_{L^\infty(B_{\tilde{\rho}}(0))}\left(\frac{\rho^2-|\zeta|^2}{2n}\right), \quad (4)$$

for every $\zeta \in B_{\tilde{\rho}}(0)$.

Remark 1: If $f = 0$ in (3), inequality (1) follows, where the non-negativity of the harmonic function u could be obtained from the maximum principle of harmonic functions [6].

The proof of theorem (1) depends on Green's representation of Dirichlet problem (3).

II. PRELIMINARIES

This section is devoted to giving the basic definitions and propositions that are related to harmonic functions as well as the Dirichlet problem (3).

2.1 Derivatives and L^{∞}-Norm

Let Ω be an open subset of R^n, $n \geq 2$. For a given function u, the notation $u \in C^k(\Omega), k \geq 1$, means the function u has all derivatives of order less than or equal to k, and all such derivatives are continuous in Ω. Likewise, $u \in C^k(\overline{\Omega})$, refers to the derivatives of u are extended to be defined continuously on the boundary of Ω, $\partial\Omega$. While the belonging $u \in C(\overline{\Omega})$ refers to the function u is extended to be continuous up to the boundary $\partial\Omega$.

A domain is an open connected subset of R^n. If Ω is a C^1-domain, it indicates the boundary $\partial\Omega$ of the domain Ω is locally a graph of C^1-function [6].

For a C^1-domain, Ω, and $u \in C^1(\Omega)$, the normal derivative of u at $\eta_0 \in \partial\Omega$ is defined to be

$$\frac{\partial u(\eta_0)}{\partial v(\eta_0)} = Du(\eta_0) \cdot v(\eta_0)\,, \tag{5}$$

where Du is the gradient vector of u, $v(\eta_0)$ is the outward unit normal vector at η_0, and $(\cdot)$ refers to the inner product in R^n.

The L^{∞}-Norm of a given continuous function u in an open set $\Omega \subseteq R^n, n \geq 1$, given as

$$||u||_{L^{\infty}(\Omega)} = sup_{\zeta\in\Omega}|u(\zeta)|. \tag{6}$$

It follows that $|u(\zeta)| \leq ||u||_{L^{\infty}(\Omega)}$, for all $\zeta \in \Omega$.

2.2 Harmonic Function and its Fundamental Solution

Let $\Omega \subseteq R^n$ be an open set, $n \geq 2$, and $u \in C^2(\Omega)$. The Laplacian operator, Δ, of u at $\zeta \in \Omega$ is defined by

$$\Delta u(\zeta) = \sum_{i=1}^{n} \frac{\partial^2 u(\zeta)}{\partial \zeta_i^2} = \frac{\partial^2 u(\zeta)}{\partial \zeta_1^2} + \frac{\partial^2 u(\zeta)}{\partial \zeta_2^2} + \cdots + \frac{\partial^2 u(\zeta)}{\partial \zeta_n^2}. \tag{7}$$

The function u is called harmonic in Ω if it Laplacian vanishes in Ω, that is,

$$\Delta u(\zeta) = 0, \tag{8}$$

for every $\zeta \in \Omega$.

To introduce a radial solution for Laplacian, fix a point $\eta \in R^n$ and define the fundamental solution [2] for Δ by the form

$$\Gamma(\zeta-\eta) = \Gamma(|\zeta-\eta|) = \begin{cases} \dfrac{1}{n(2-n)\,\omega_n}|\zeta-\eta|^{2-n}, & n \geq 3 \\ \dfrac{1}{2\pi}\log|\zeta-\eta|, & n = 2, \end{cases} \tag{9}$$

where $\omega_n = \frac{2\pi^{\frac{n}{2}}}{n\Gamma\left(\frac{n}{2}\right)}$ the volume of the unit ball in R^n.

The radial function, Γ, in (9) is a function of ζ defined in $R^n\backslash\{\eta\}$ which satisfies Laplace's equation (8) in $R^n\backslash\{\eta\}$. It can be interpreted in the sense of the distribution [3] as follows

$$\Delta\Gamma(\cdot-\eta) = \delta_\eta \qquad in\ R^n, \tag{10}$$

where δ_η is the Dirac delta distribution at η [3]. That is,

$$\begin{cases} \delta_\eta : C_0^{\infty}(R^n) \longrightarrow R \\ \delta_\eta(f) = f(\eta). \end{cases} \tag{11}$$

The space $C_0^{\infty}(R^n)$ is the set of all smooth functions with compact support in R^n [2].

2.3 Green's Function of Laplacian Δ

Green's function $\tilde{r}$ of Laplacian, Δ, in the ball $B_{\tilde{\rho}}(0)$ has been studied [2,6,5] and it is given by

$$\tilde{r}(\zeta,\eta) = \begin{cases} \dfrac{1}{2\pi}\left(\log|\zeta-\eta| - \log\left|\dfrac{\tilde{\rho}}{|\zeta|}\zeta - \dfrac{|\zeta|}{\tilde{\rho}}\eta\right|\right), & n = 2 \\ \dfrac{1}{(2-n)\omega_n}\left(|\zeta-\eta|^{2-n} - \left|\dfrac{\tilde{\rho}}{|\zeta|}\zeta - \dfrac{|\zeta|}{\tilde{\rho}}\eta\right|^{2-n}\right), & n \geq 3, \end{cases} \tag{12}$$

for all $\zeta, \eta \in \overline{B_{\tilde{\rho}}(0)}$, $\zeta \neq \eta, and\ \zeta \neq 0$.

For $\zeta = 0$ and $0 \neq \eta \in \overline{B_{\tilde{\rho}}(0)}$, the Green's function is given by

$$\tilde{r}(0,\eta) = \begin{cases} \frac{1}{2\pi}(\log|\eta| - \log\tilde{\rho}), & n = 2 \\ \frac{1}{(2-n)\,\omega_n}\left(|\eta|^{2-n} - \tilde{\rho}^{2-n}\right), & n \geq 3. \end{cases} \tag{13}$$

The prominent properties of Green's function are summarized in the following proposition.

Proposition 1 [6]**:** Let $\tilde{r}$ be the Green's function of Laplacian in $B_{\tilde{\rho}}(0)$, then the following hold

1. (Symmetry) $\tilde{r}(\zeta,\eta) = \tilde{r}(\eta,\zeta)$, for all $\zeta \neq \eta$ and $\zeta, \eta \in \overline{B_{\tilde{\rho}}(0)}$.

2. (Non-positivity) $\tilde{r}(\zeta,\eta) \leq 0$, for all $\zeta \neq \eta$ and $\zeta, \eta \in \overline{B_{\tilde{\rho}}(0)}$.

3. (Normal derivative)

$$\left.\frac{\partial\tilde{r}(\zeta,\eta)}{\partial v(\eta)}\right|_{\eta\in\partial B_{\tilde{\rho}}(0)} = \frac{\tilde{\rho}^2-|\zeta|^2}{n\,\omega_n\tilde{\rho}|\zeta-\eta|^n}. \tag{14}$$

Employing Green's function of Laplacian, the Dirichlet problem (3) has the following representation [2,6]

$$u(\zeta) = \int_{\partial B_{\tilde\rho}(0)} g(\eta)\, \frac{\partial \tilde r(\zeta,\eta)}{\partial v(\eta)}\, ds_\eta + \int_{B_{\tilde\rho}(0)} \tilde r(\zeta,\eta)\, f(\eta)\, d\eta, \qquad (15)$$

for every $\zeta \in B_{\tilde\rho}(0)$.

Involving (14), equation (15) becomes

$$u(\zeta) = \frac{1}{n\,\omega_n\,\tilde\rho} \int_{\partial B_{\tilde\rho}(0)} g(\eta)\left(\frac{\tilde\rho^2 - |\zeta|^2}{|\zeta-\eta|^n}\right) ds_\eta + \int_{B_{\tilde\rho}(0)} \tilde r(\zeta,\eta)\, f(\eta)\; d\eta, \qquad (16)$$

for every $\zeta \in B_{\tilde\rho}(0)$.

III. PROVING THE MAIN RESULT

This section contains the proof of theorem (1). The proof depends on the Green's function for Laplacian.

Lemma 1: Let $\tilde r$ be the Green's function that is defined in (12) and (15). Then for every $\zeta \in \overline{B_{\tilde\rho}(0)}$

$$\int_{B_{\tilde\rho}(0)} \tilde r(\zeta,\eta) d\eta = \frac{|\zeta|^2 - \tilde\rho^2}{2n}. \qquad (17)$$

Proof. Define the non-negative auxiliary function $v(\zeta) = \tilde\rho^2 - |\zeta|^2$ for $\zeta \in \overline{B_{\tilde\rho}(0)}$. Then the function v satisfies the following Dirichlet problem

$$\begin{cases} \Delta v(\zeta) = -2n\,, & \zeta \in B_{\tilde\rho}(0) \\ v(\zeta) = 0\,, & \zeta \in \partial B_{\tilde\rho}(0). \end{cases} \qquad (18)$$

Utilizing Green's representation formula (16), the function v becomes

$$v(\zeta) = -\int_{B_{\tilde\rho}(0)} \tilde r(\zeta,\eta)\,(2n)\, d\eta. \qquad (19)$$

Therefore, the integral of Green's function over the ball $B_{\tilde\rho}(0)$ follows, that is,

$$\int_{B_{\tilde\rho}(0)} \tilde r(\zeta,\eta) d\eta = \frac{|\zeta|^2 - \tilde\rho^2}{2n},$$

for every $\zeta \in \overline{B_{\tilde\rho}(0)}$.

Lemma 2: For the Dirichlet problem (3), the average of g on $\partial B_{\tilde\rho}(0)$ is given by the formula

$$⨍_{\partial B_{\tilde\rho}(0)} g(\eta)\, ds_\eta = u(0) - \int_{B_{\tilde\rho}(0)} \tilde r(0,\eta)\, f(\eta)\, d\eta\,. \qquad (20)$$

where,

$$⨍_{\partial B_{\tilde\rho}(0)} g(\eta)\, ds_\eta = \frac{1}{n\,\omega_n\,\tilde\rho^{n-1}} \int_{\partial B_{\tilde\rho}(0)} g(\eta)\, ds_\eta\,. \qquad (21)$$

Proof: From Green's representation formula for the Dirichlet problem (15), it follows that

$$u(0) = \int_{\partial B_{\tilde\rho}(0)} g(\eta) \frac{\partial \tilde r(0,\eta)}{\partial v(\eta)} ds_\eta + \int_{B_{\tilde\rho}(0)} \tilde r(0,\eta) f(\eta) d\eta. \qquad (22)$$

Involving equation (14) for the quantity $\frac{\partial \tilde r(0,\eta)}{\partial v(\eta)}$, the following equation is obtained

$$u(0) = \frac{1}{n\,\omega_n \tilde\rho} \int_{\partial B_{\tilde\rho}(0)} g(\eta) \frac{\tilde\rho^2}{|\eta|^n} ds_\eta + \int_{B_{\tilde\rho}(0)} \tilde r(0,\eta) f(\eta) d\eta.$$

Since $|\eta| = \tilde\rho$ in the surface of the ball $B_{\tilde\rho}(0)$, it follows that

$$u(0) = ⨍_{\partial B_{\tilde\rho}(0)} g(\eta)\, ds_\eta + \int_{B_{\tilde\rho}(0)} \tilde r(0,\eta)\, f(\eta)\, d\eta. \qquad (23)$$

On account of equation (23), the average of g on $\partial B_{\tilde\rho}(0)$ is represented by equation (20).

Lemma 3: Let f be the continuous function in Dirichlet problem (3) and $\tilde r$ be the Green's function of Laplacian, then for every $\zeta \in B_{\tilde\rho}(0)$ it follows that

$$\left|\int_{B_{\tilde\rho}(0)} \tilde r(\zeta,\eta) f(\eta) d\eta\right| \le ||f||_{L^\infty(B_{\tilde\rho}(0))} \left(\frac{\tilde\rho^2 - |\zeta|^2}{2n}\right). \qquad (24)$$

From the non-positivity of Green's function and supremum norm of f inside the ball $B_{\tilde\rho}(0)$, the following inequality follows

$$\left(||f||_{L^\infty(B_{\tilde\rho}(0))}\right)\int_{B_{\tilde\rho}(0)}\tilde r(\zeta,\eta)d\eta \le \int_{B_{\tilde\rho}(0)}\tilde r(\zeta,\eta)f(\eta)d\eta$$

$$\le \left(-||f||_{L^\infty(B_{\tilde\rho}(0))}\right)\int_{B_{\tilde\rho}(0)}\tilde r(\zeta,\eta)d\eta. \qquad (25)$$

Employing lemma (1) in (25), inequality (24) follows.

Remark 2: If $\zeta = 0$ in (24), the following follows

$$-||f||_{L^\infty(B_{\tilde\rho}(0))}\left(\frac{\tilde\rho^2}{2n}\right) \le \int_{B_{\tilde\rho}(0)}\tilde r(0,\eta)\, f(\eta)\, d\eta$$
$$\le ||f||_{L^\infty(B_{\tilde\rho}(0))}\left(\frac{\tilde\rho^2}{2n}\right). \qquad (26)$$

After setting lemmas (1), (2), and (3), the proof of theorem (1) can be given.

Proof of Theorem (1). For $\eta \in \partial B_{\tilde\rho}(0)$, we have that

$$\tilde\rho - |\zeta| \le |\zeta-\eta| \le \tilde\rho + |\zeta|.$$

Therefore,

$$\frac{1}{(\tilde\rho+|\zeta|)^n} \le \frac{1}{|\zeta-\eta|^n} \le \frac{1}{(\tilde\rho-|\zeta|)^n}. \qquad (27)$$

Combining (16) and (27) and the non-negativity of g give the following

$$\frac{1}{n\,\omega_n\,\tilde\rho}\int_{\partial B_{\tilde\rho}(0)} g(\eta)\frac{\tilde\rho-|\zeta|}{(\tilde\rho+|\zeta|)^{n-1}}ds_\eta + \int_{B_{\tilde\rho}(0)}\tilde r(\zeta,\eta)f(\eta)d\eta$$
$$\le u(\zeta)$$
$$\le \frac{1}{n\,\omega_n\,\tilde\rho}\int_{\partial B_{\tilde\rho}(0)} g(\eta)\frac{\tilde\rho+|\zeta|}{(\tilde\rho-|\zeta|)^{n-1}}ds_\eta$$
$$+\int_{B_{\tilde\rho}(0)}\tilde r(\zeta,\eta)f(\eta)d\eta. \qquad (28)$$

Using the average for the boundary integral, we obtain

$$\tilde\rho^{n-2}\ \frac{\tilde\rho-|\zeta|}{(\tilde\rho+|\zeta|)^{n-1}} ⨏_{\partial B_{\tilde\rho}(0)} g(\eta)ds_\eta + \int_{B_{\tilde\rho}(0)}\tilde r(\zeta,\eta)f(\eta)\,d\eta$$
$$\le\ u(\zeta)\tilde\rho^{n-2}\frac{\tilde\rho+|\zeta|}{(\tilde\rho-|\zeta|)^{n-1}} ⨏_{\partial B_{\tilde\rho}(0)} g(\eta)\,ds_\eta$$
$$+\int_{B_{\tilde\rho}(0)}\tilde r(\zeta,\eta)\,f(\eta)\ d\eta. \qquad (29)$$

Involving lemmas (2) and (3) and remark (2) we reach the main result (4).

IV. COUNCLUSIONS

The explicit form of Harnack's inequality (4) for the Dirichlet problem (3) is a generalization for a basic inequality (1). That is, (1) can be followed from (4) by setting the source function to be zero in $B_{\tilde\rho}(0)$. Green's function for the Laplace operator in the open ball $B_{\tilde\rho}(0)$plays the crucial part for obtaining inequality (4). The non-negativity of the boundary data g on $\partial B_{\tilde\rho}(0)$is needed for concluding equation (28).

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