Other Properties of the Class D(T)

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Abstract— *The class of D(T)- operators are equivalent to the class of quasi-normal operators. This paper discusses additional properties of this class of operators. Assuming that if the operator T is not far from normality and U serves as an interrupter, it follows that the operator U will be both unique and positive. Moreover, we explore other properties that merge when the operator T commutes with T^* T. In one of our main theorems, we demonstrate that the operator T in the class D(T) is also normal when it is invertible.*

Keywords: D(T)-operators, Hilbert space.

1. INTRODUCTION

Hilbert space is an inner product space which is complete with respect to the norm induced by its inner. Many authors have studied the operator theory on Hilbert space (for more details [1, 4, 7]). In 2021, in [3] studied the class of $D(T)-$ operators and presented the main properties of this type of class in a Hilbert space. The class $D(T)$ ={ $U \epsilon B(H), where\ U \neq 0, I$: $T^*TU = UT^*T$}. This paper aims to investigate several properties of the $D(T)$ class. The first section focuses on studying various properties of an operator $U \epsilon D(T)$ under specific conditions. In the second section, we derive additional properties of an operator T itself when it commutes with T^*T.

Consider H to be a complex separable Hilbert space and let $B(H)$ denote the algebra of all bounded linear operators $T: H \to H$ on the Hilbert space H with an inner product $<\cdot\ ,\ \cdot>$. A bounded linear operator T is considered positive if $< Tx, x > \geq 0$; T is called normal if $TT^* - T^*T = 0$ and a self-adjoint operator if $T^* - T = 0$.

The following example explains that $D(T) \neq \emptyset$.

we consider T^* , the adjoint of the unilateral shift $T: \ell_2 \to \ell_2$; recall that T has matrix entries

$$T_{jk=}\begin{cases} 0 & if \quad j \neq k+1 \\ 1 & if \quad j = k+1. \end{cases}$$

Example:

we consider T^* , the adjoint of the unilateral shift $T: \ell_2 \to \ell_2$; recall that T has matrix entries

And U is the diagonal operator with diagonal entries $\left\{\frac{n+1}{n+2} : n = 0,1,2,\cdots\right\}$,

Then

$$T^*TU = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{3}{4} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 \end{bmatrix} = UT^*T.$$

Thus, the operator $U \in D(T)$.

2. PROPERTIES OF THE CLASS D(T)

In this section, we will demonstrate several properties of the class $D(T)$.

If $T\epsilon B(H)$ and in the class of $D(T)-$ operators, we assume that T is relatively close to being a normal operator by considering the operator U in $D(T)$ as an interrupter of T, such that $T^*T = TUT^*$. Consequently, we will impose additional requirements on the operator U, demanding that it to be self-adjoint, positive, and unique.

We refer to the following lemma due to it is significance in proving Theorem (2-2).

Lemma (2-1): ([3])
deduce that $U_1 - U_2 = 0$, confirming the uniqueness of U.

Theorem (2-2):
$If\ U\ \in D(T), then\ U\ is\ a\ self\ adjoint\ operator.$

Proof:
Given $U \in D(T),\ we\ have\ TUT^* \in D(T^*)$ due to Theorem (2-1).

Furthermore, since $T^*T = TUT^*$ and it is self $-$ adjoint, each T must satisfy $TUT^* = T^*U^*T$.

Implying that U is a self-adjoint operator.

Theorem (2-3):
$If\ U\ \in D(T), then\ the\ operator\ U\ is\ positive\ on\ Ran(T^*).$

Proof:
We start by observing that

$$< UT^*x, T^*x >= < TUT^*x, x >$$
$$= < T^*Tx, x >$$
$$=< Tx, Tx > = \ \|Tx\|^2 \quad for\ all\ x \in H.$$

Consequently, we establish that $< UT^*x, T^*x > \geq 0$, indicating the positivity of $U\ on\ Ran(T^*\ )$.

Theorem (2-4):
If $T^* has\ a\ dense\ range\ on\ Ran(T^*), then\ U$ is unique

Proof:
Let $U_1\ and\ U_2$ be two operators in $D(T)$. Considering this, we have TU_1T^*=$T^*T = TU_2T^*$.

Therefore, $T(U_1 - U_2)T^*$=0.

Given that T^* has a dense range, and $\overline{Ran(T^*)} = N(T)^{\perp}$.

We conclude that $N(T) = \{0\}$, establishing that T is injective.

As a result, $(U_1 - U_2)T^*$=0. Using the fact that T^* has a dense range once more, we deduce that $U_1 - U_2 = 0$, confirming the uniqueness of U.

3. OTHER PROPERTIES OF D(T) WHEN T

commutes with T^* T.

We begin this section by exploring the relationship between this class and normality. An operator T in D(T) does not necessarily have to be normal, but the converse is true.

Theorem (3-1):
If $T\ is\ a\ normal\ operator, then\ T \in D(T)$.

Proof:
When T is normal, we have $(T^*T)T = (TT^*)T = T(T^*T)$. Therefor, $T\epsilon D(T)$.

Theorem (3-2):
If T is an invertible operator and $T \in D(T), then$:

1. $T\ is\ a\ normal\ operator.$
2. $(TT^*)T^* = T^*(T^*T)$
3. $T(T^*T) = (T^*T)T$
4. $T^{-1}(T^*T) = (T^*T)T^{-1}$

Proof:
1- Given $T \in D(T)$:

$(T^*T)T = T(T^*T)$

Multiplying both sides by T^{-1}yields: $(T^*T)TT^{-1} = (TT^*)TT^{-1}$

Hence $T^*T = TT^*$, making T normal.

It is evident that (1) implies (2), (3) and (4).

Now let $T = A + Bi$ represent the Cartesian form of any operator T, where $A = Re\,T = \frac{T+T^*}{2}$, and $B = Im\,T = \frac{T-T^*}{2i}$ denote the real and imaginary components of T. Berberian [1] introduced a theorem that demonstrates T being normal if and only if $AB = BA$. This result is further extended and generalized in the subsequent Theorem (3-3). In fact, we will establish that $T \in D(T)$ if $AB = BA$, although the converse does not hold true.

The proof of Theorem (3-3) is straightforward; thus, we choose to omit it.

Theorem (3-3):
If T is an operator in $B(H)$, then

1. If $AB = BA, then\ T \in D(T).$
2. If $T \in D(T), then\ ABT = TBA.$

Theorem (3-4):
If $T\ is\ an\ operator\ such\ that\ T^*T$ commutes with A and B, then $T\epsilon D(T)$.

Proof:

Given that T commutes with both A and B, we have: $T^*T(A+Bi)=(A+Bi)T^*T$ and $T^*TT=TT^*T$. Thus, it follows that $T \in D(T)$.

Theorem (3-5):

If $T\ is\ an\ operator\ such\ that\ \ TT^*$ commutes with $A\ and\ B, and\ TTT^*=T^*TT,$ then $T\epsilon\ D(T)$.

Proof:

Since $TT^*A=ATT^*\ \ and\ TT^*B=BTT^*,$ then

$$TT^*\left(\frac{T+T^*}{2}\right)=\left(\frac{T+T^*}{2}\right)TT^*$$

and

$$TT^*\left(\frac{T-T^*}{2i}\right)=\left(\frac{T-T^*}{2i}\right)TT^*$$

This gives: $TT^*T=TTT^*$.

Since $TTT^*=T^*TT$, we have:

$$TT^*T=TTT^*=T^*TT.$$
$$T^*TT=TT^*T.$$

Thus, $T\epsilon(T)$.

Furthermore, within the context of the $D(T)$ class, if an operator T exhibits invertibility, two results can be obtained.

Theorem (3-6):

If $T\in D(T)$ be an invertible operator. Then TT^* commutes with A and B.

Proof:

Since $T\in D(T)$, we have:

$$TT^*A=TT^*\left(\frac{T+T^*}{2}\right)$$
$$=\frac{TT^*T+TT^*T^*}{2}$$

Hence, $(TT^*)T^*=T^*(T^*T)$

$$\Rightarrow\ \ \frac{TT^*T+TT^*T^*}{2}\ \ =\frac{T^*TT+T^*TT^*}{2}$$
$$=\frac{T(TT^*)+(T^*T)T^*}{2}$$
$$=\left(\frac{T+T^*}{2}\right)TT^*$$
$$=ATT^*.$$

Similarly, $TT^*B=BTT^*$.

Theorem (3-7):

If T is an invertible operator and $\in D(T)$, then T^*T commutes with A and B.

Proof:

1)Since $T\ is\ an\ invertible\ operator\ in\ the\ class\ of\ operators\ \ D(T),\ we\ have$:

$$T^*TA=T^*T\left(\frac{T+T^*}{2}\right)\ \ =\frac{T^*TT+T^*TT^*}{2}$$
$$=\frac{TT^*T+T^{*^2}T}{2}$$
$$=\left(\frac{T+T^*}{2}\right)T^*T.$$

= AT^*T.

2) Similarly, $T^*TB=BT^*T$.

IV Conclusion

This paper discusses other properties of operators are called $D(T)$ −operator. The main results have been proved in this paper, which are:

1. $If\ U\ \in\ D(T), then\ U\ is\ a\ self\ adjoint\ operator.$.
2. The operator $U\ is\ positive\ on\ Ran(T^*)$.
3. If $T^*has\ a\ dense\ range\ on\ Ran(T^*),\ then\ U\ is\ unique.$
4. If $T\ is\ a\ normal\ operator, then\ T\in D(T)$.
5. If T is an invertible operator and $T\in D(T), then (TT^*)T^*=T^*(T^*T)\ and\ T(T^*T)=(T^*T)T.$

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