Central graphs and Italian domination parameters

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Abstract *Let $G = (V, E)$ be a graph with $V = V(G)$ and $E = E(G)$. A function $f : V \rightarrow \{0,1,2\}$ is said to be an Italian dominating function on a graph G if every vertex u with $f(u) = 0$ is adjacent to at least one vertex v with $f(v) = 2$ or is adjacent to at least two vertices x, y with $f(x) = f(y) = 1$. The value $w(f) = \sum_{v\in V} f(v)$ denotes the weight of an Italian dominating function. The minimum weight taken over all Italian dominating functions of G is called Italian domination number and denoted by $\gamma_I(G)$.*

Two parameters related to Italian dominating function (IDF) are restrained Italian (RIDF) and total restrained (TRDF) dominating functions f , for which the set of vertices v with $f(v) = 0$, and simultaneously the set of vertices v with $f(v) > 0$ and the set of vertices v with $f(v) = 0$ induce subgraphs with no isolated vertex respectively. The central graph $C(G)$ of a graph G is the graph obtained by subdividing each edge of G exactly once and joining all the non-adjacent vertices of G.

In this work, we initiate the study of restrained (total restrained) Italian domination number of the central of any graph G. For a family of standard graphs G, we obtain the precise value of restrained (total restrained) Italian domination number for $C(G)$, indeed for any graph G, the sharp bounds are provided for $C(G)$, and for G corona of K_1 , $G \circ K_1$ we establish the precise value of these parameters for $C(G \circ K_1)$.

Keywords ***Italian domination, restrained (total restrained) Italian domination, Central graph.***

2020 Mathematics Subject Classification: 05C69.

1. INTRODUCTION

Let $G = (V, E)$ be a graph with $V = V(G)$ and $E = E(G)$ denote the set of vertices and the set of edges of G respectively. Here we consider G as a finite simple graph and use [15] as a reference for terminology and notation which are not explicitly defined here. For a vertex v, $N(v) = N_G(v)(N[v] = N_G[v])$ denotes the *open neighborhood* (*closed neighborhood*) of v respectively, while the subset S of V, the *open neighborhood* is $N(S) = \cup_{v\in S} N(v)$ and the *closed neighborhood* is $N[S] = N(S) \cup S$. For any subset S of V, $G[S]$ represents the subgraph induced by S in G. A vertex of degree zero in G is said an *isolated vertex*, while a vertex of degree one is called a *pendant vertex* or a *leaf* of G. The *maximum degree* (*minimum degree*) of G is denoted by $\Delta(G)$ $(\delta(G))$.

The concept of domination in graphs during the past four decades have found growth rapidly. These growthes may be due to their applications to both theoretical and real-world problems, such as strategy of defence of cities, facility location problems and etc. Among the domination- type parameters that have been studied, some of them are the Roman [2, 12], restrained Roman [10], Italian [1, 7], restrained Italian [11] and total restrained Italian [9] domination numbers in graphs.

Stewart [12], introduced the concept of Roman domination that was motivated by the article of Ian Stewart entitled "Defend the Roman Empire!", and then paper by Cockayne in [2]. So far, in-depth studies have been done on these parameters and there have been several sources in the literature.

A dominating set, abbreviated D-set of a graph G is a set S such that any vertex out of S is adjacent to at least one vertex of S, while for a graph G with no isolated vertex, a total dominating set, abbreviated TD-set, of a graph G is a set S of vertices of G such that every vertex of G is adjacent to a vertex in S. The domination number (the total domination number) of G, denoted by $\gamma(G)$ $(\gamma_t(G))$, is the minimum cardinality of a D-set (TD-set) of G. A D-set (TD-set) of G of cardinality $\gamma(G)$ $(\gamma_t(G))$is called a $\gamma(G)$-set $(\gamma_t(G)$_set). A subset $S \subseteq V(G)$ is called a restrained dominating set if the subgraph induced by $V(G) \setminus S$ has no isolated vertices.

The minimum cardinality of a restrained dominating sets of G is called restrained domination number, denoted by $(\gamma_r(G)$ [3, 4, 5, 6, 14].

Let $f: V(G) \to \{0,1,2\}$ be a function, we show $V_i^f = \{v \in V(G) \mid f(v) = i\}$ for each $i = 0,1,2$ (V_i may be replaced V_i^fif there is no ambiguity with respect to the function f).

We call $w(f) = f(V(G)) = \sum_{v \in V} f(v)$as the weight of f . For a graph $G = (V, E)$, a function $f: V \to \{0,1,2\}$ with the property that, if $v \in V_0$ for some $v \in V(G)$, then there exists a vertex $w \in N_{(v)}$ such that $w \in V_2$ is said to be a Roman dominating function (RD function) of G. The Roman domination number (RD number) of G, denoted by $\gamma_R(G)$ is the minimum weight taken over all RD functions for G [2, 8, 16].

An (IDF) is a function $f: V(G) \to \{0,1,2\}$ with the property that for every vertex $u \in V(G)$ with $f(u) = 0$, it follows that either there is a vertex $v \in N(u)$ with $f(v) = 2$ or there are at least two vertices $x, y \in N(u)$ with $f(x) = f(y) = 1$. The minimum weight of an ID functions of G is called Italian domination number (ID number) $\gamma_I(G)$ [1].

A (RIDF) is an ID function f with the property that $G[V_0]$ has no isolated vertex [11].

A (TRIDF) is an RIDF f with the property that the subgraph induced by $G[V_0]$ and $G[V_1 \cup V_2]$ have no isolated vertices [9]. On the other hand, in terms of the (IDF), a (TRIDF) is an (IDF) $f: V(G) \to \{0,1,2\}$ such that the subgraph induced by the set of vertices of weight 0 and the subgraph induced by the set of vertices of positive weight have no isolated vertices. A minimum weight of any TRIDF f is called a TRID number denoted by $\gamma_{trI}(G)$.

A central graph C(G) of a graph G is obtained by subdividing each edge of G exactly once and joining all the non-adjacent vertices of G.

Let C_n, $p_n$and K_n denote the cycle, path and complete graph with n vertices respectively. The $K_{m,n}$ refer to complete bipartite graph and $K_{1,n}$ to a star graph. $S_{p,q}$ is a tree with only two support vertices such that one of them has p leaves and the other has q leaves called a double star. For two graphs G and H the corona is defined as the graph obtained by making one copy of G and $|V(G)|$ copies of H and joining the i^{th}vertex of G to every vertex in the i^{th} copy of H.

This paper is orderly as follows. The exact value of *RIDF* of central of standard graphs are established in Section 2. In Section 4, we study the TRIDF for central standard graphs, the precise value of TRID are obtained. In Section 4 we investigate the sharp bound on the RID and TRID of $C(G)$ for any graph G, and these parameters are provided with the precise value for $C(G \circ K_1)$. Finally we will remark some conclusions and problems in Section 5.

2. RIDF OF CENTRAL OF SOME STANDARD GRAPHS

In this section RIDF 0f some families of graphs are investigated and are detected the smallest *RID* number of them. More f0rmally, we pr0ve the following results.

From Corollary 10 of [1] we have.

Observation 2.1 ([1] Corollary 10) For cycle C_n, $\gamma_I(C_n) \lceil \frac{n}{2} \rceil$.

Authors in [11] demonstrated that in the context of a cycle C_n for $3 \leq n \leq 8$ and $n \neq 6$, $\gamma_{rI}(C_n) = \{n-1, n\}$ and it is easy to see that $\gamma_{rI}(C_6) = 4$. Also in [13] researcher proved that $\gamma_{rI}(C_n)$ for $n \geq 9$.

Theorem 2.2 "([13] Proposition 5)" For cycle C_n, $(n \geq 9)$, $\gamma_{rI}(C_n) = \begin{cases} 2\left\lceil \frac{n}{3} \right\rceil + 1 & n \equiv 2 (\text{mod } 3) \\ 2\left\lceil \frac{n}{3} \right\rceil & \text{otherwise} \end{cases}$

Now we obtain the RID function of central of C_n.

Theorem 2.3 Let $C(C_n)$ be the central of the cycle C_n, $(n \geq 4)$. Then $\gamma_{rI}(C(Cn)) = n$.

Proof. Let $V(C_n) = \{v_1, v_2, \cdots, v_n\}$and $C(C_n)$be the central of C_n with set of vertices $V(C(C_n))= V(C_n) \cup \{u_{1,2}, u_{2,3}, \cdots, u_{n-1,n}, u_{n,1}\}$where $u_{i,i+1}$ is the new vertex subdivides the edge $e_i = v_i v_{i+1}$; $1 \leq i \leq n$, $(\text{mod } n)$. We labeling value 1 to the vertices $u_{i,i+1}$ $(1 \leq i \leq n)$and zero to the all vertices v_is. This assignments is an RIDF of $C(C_n)$. Then $\gamma_{rI}(C(C_n)) \leq n$. On the 0ther hand, in $C(C_n)$ the vertices $u_{i,i+1}$ and $u_{i-1,i}$ aren't adjacent. Since $u_{i,i+1}$ has two neighbors v_i and v_{i+1}, and $\deg(u_{i,i+1}) = 2$. Therefore $f(N[u_{i,i+1}]) \geq 2$ or $f(u_{i,i+1}) \geq 1$. This can be finished if we assign $f(u_{i,i+1}) = 1$ and $f(v_i) = 0$. Thus $f(C(C_n)) \geq n$ and so $\gamma_{rI}(C(C_n)) \geq n$. Therefore $\gamma_{rI}(C(C_n)) = n$.

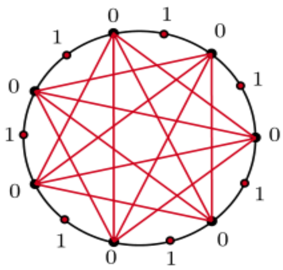

Figure 1: RIDF of $C(C_7)$

. The $\gamma_{rI}(P_n)$ for $n \geq 4$ it is resulted as follows, because for $1 \leq n \leq 3$ is clear.

Theorem 2.4 ([13] Proposition 5) For P_n, $(n \geq 4)$, $\gamma_{rI}(P_n) = \frac{2n+3+r}{3}$ where $n \equiv r (\text{mod } 3)$ for $r \in \{1, 2, 3\}$

Now we obtain the RID of central of P_n.

Theorem 2.5 For P_n, $(n \geq 3)$, $\gamma_{rI}(C(P_n)) = \begin{cases} 5 & \text{if } n \in \{3,4,5\} \\ n-1 & \text{if } n \geq 6 \end{cases}$.

Proof. Let P_n be a path with vertex set $V(P_n) = \{v_1, v_2, \cdots, v_n\}$. Let $C(P_n)$ be the central of P_n with vertex set $V(P_n) \cup \{u_{1,2}, u_{2,3}, \cdots, u_{n-1,n}\}$ where $u_{i,i+1}$ is the new vertex subdivides the edge $e = v_i v_{i+1}$ where $(1 \leq i \leq n-1)$. We bring up three situations.

1. For $n \in \{3, 4, 5\}$, it is not hard to see the proof.
2. Let n be even integer. We devote to v_i (where i is even and $2 \leq i \leq (n-2)$ value 2, devote to $u_{n,n-1}$ value 1 and otherwise value zero. Thus $\gamma_{rI}(C(P_n)) \leq n-1$. On the other hand, since $u_{i,i+1}$ has only two neighbors v_i and v_{i+1} in $C(P_n)$. Therefore for any RIDF f , $f(N[u_{i,i+1}]) \geq 2$ or $f(u_{i,i+1}) = 1$. then $f(v_1) + f(v_n) \geq 1$ in this case $w(f) \geq n$. This can be finished if we assign $f(u_{i,i+1}) = 0$, $f(u_{n,n-1})$ and $f(v_i) = 2$ for even integer $2 \leq i \leq n-2$ and $f(x) = 0$ for other vertex x. Then $w(f) \geq n-1$. Thus $\gamma_{rI}(C(P_n)) \geq n-1$ and then $\gamma_{rI}(C(C_n)) = n-1$ for even integer n.
3. Let $n \geq 7$ be an odd integer. We devote any vertex v_i (where i is even) value 2, and otherwise weight zero. Thus $\gamma_{rI}(C(P_n)) \leq n-1$. On the other hand, since $u_{i,i+1}$has only two neighbors v_i and v_{i+1} in $C(P_n)$. Therefore for any RIDF f , $f(N[u_{i,i+1}]) \geq 2$ or $f(u_{i,i+1}) = 1$. If $f(u_{i,i+1}) = 1$, then $f(v_1) + f(v_n) \geq 1$ in this case $w(f) \geq n$. But in the best case we consider $f(N[u_{i,i+1}]) \geq 2$ so that we devote value 2 to v_i for even i. This can be done with $f(v_i) = 2$ for even integer i and $f(x) = 0$ for other vertex x in $C(P_n)$. Thus $\gamma_{rI}(C(P_n)) \geq n-1$ and then $\gamma_{rI}(C(C_n)) = n-1$ for odd integer n.

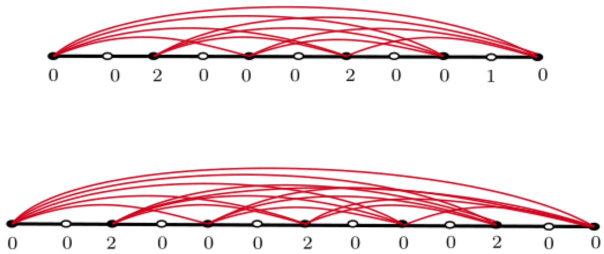

Figure 2: RIDF of $C(P_6)$ and $C(P_7)$

In [13] the author showed

Theorem 2.6 ([13] Proposition (11i)) For K_n *graph* if $n \geq 2$, we have $\gamma_{rI}(K_n) = 2$.

Theorem 2.7 For K_n *graph*, and $n \geq 4$ we have

$$\gamma_{rI}\left(C(K_n)\right) = \begin{cases} \frac{n}{2}\left(\frac{n+2}{2}\right) - 1 & \text{if n is even} \\ \frac{n-1}{2}\left(\frac{n+3}{2}\right) & \text{if n is odd} \end{cases}$$

Proof. Let K_n be a complete graph with vertex set $V(K_n) = \{v_1, v_2, \cdots, v_n\}$. Let $C(K_n)$ be the central of K_n, with set of vertices $V(K_n) \cup \{u_{i}, j : 1 \leq i < j \leq n\}$ where $u_{i,j}$ is the vertex that subdivides the edge $e = v_i v_j$. We prove by induction on n. If $n = 4$, then the $\gamma_{rI}(C(K_4)) = 5$, this is the basis of induction. Assume that theorem is true for $n \geq 4$. Now let $k = n + 1$ we have two cases.

If $n+1$ is even, then n is odd and according to the assumption of induction $\gamma_{rI}\left(C(K_n)\right) = \frac{n-1}{2}\left(\frac{n+3}{2}\right)$. For $\gamma_{rI}\left(C(K_{n+1})\right)$, we assign 0 to the vertex v_{n+1}, the 1 to $\{u_{n+1,1}, u_{n+1,3}, u_{n+1,4}, \cdots, u_{n+1,n}\}$, and 0, to $\{u_{n+1,2}, u_{n+1,6}, u_{n+1,8}, \cdots, u_{n+1,n-1}\}$, then the weight of these new assignments is $\frac{n+1}{2}$. Therefore $\gamma_{rI}\left(C(K_{n+1})\right) = \gamma_{rI}\left(C(K_n)\right) + \frac{n+1}{2} = \frac{n-1}{2}\left(\frac{n+3}{2}\right) + \frac{n+1}{2} = \frac{n+1}{2}\left(\frac{n+3}{2}\right) - 1$.

If $n+1$ is odd, then n is even and according to the assumption of induction $\gamma_{rI}\left(C(K_n)\right) = \frac{n}{2}\left(\frac{n+2}{2}\right) - 1$. For $\gamma_{rI}\left(C(K_{n+1})\right)$, we assign 0 to the vertex v_{n+1}, the 1 to $\{u_{n+1,1}, u_{n+1,3}, u_{n+1,4}, \cdots, u_{n+1,n}\}$, and 0, to $\{u_{n+1,2}, u_{n+1,6}, u_{n+1,8}, \cdots, u_{n+1,n-1}\}$, then the weight of these new assignments is $\frac{n+1}{2}$. Therefore $\gamma_{rI}\left(C(K_{n+1})\right) = \frac{n}{2}\left(\frac{n+2}{2}\right) - 1 + \frac{n+1}{2} = \frac{n-1}{2}\left(\frac{n+3}{2}\right)$.

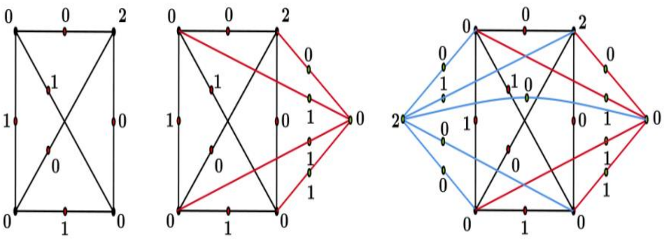

Figure 3: RIDF of $C(K_4)$, $C(K_5)$ and $C(K_6)$

In [13] showed for star graph $K_{1,n}$, $\gamma_{rI}\left(K_{1,n}\right) = n + 1$. Now we obtain RIDF of central graph of star, $C\left(K_{1,n}\right)$.

Theorem 2.8 For $K_{1,n}$ and $n \geq 3$, then $\gamma_{rI}(C\left(K_{1,n}\right)) = 5$.

Proof. Assume that $V(K_{1,n}) = \{v_0, v_1, \cdots, v_n\}$. Let $C(K_{1,n})$ be the central of $K_{1,n}$ with vertex set $V\left(C\left(K_{1,n}\right)\right) = V\left(K_{\{1,n\}}\right) \cup \{u_{0,1}, u_{0,2}, \cdots, u_{0,n}\}$ where $u_{0,i}$ is the new vertex corresponding to edge $e_i = v_0 v_i$. Thus $C(K_{1,n})$ obtain from $K_{1,n}$ by subdividing the edge $v_0 v_i$, by the vertex $u_{0,i}$ for $1 \leq i \leq n$. For any RIDF f, we have $f(v_0) + f\left(u_{0,i}\right) + f(v_i) \geq 2$, $\sum_{i=1}^{n} f(v\_i) \geq 2$, and if $f(v_0) > 0$ and $f(v_i) > 0$ for some i, then $f(u_{0,i}) > 0$. This denotes $w(f) \geq 5$. On the other hand, the assignment value 2 to v_0, v_1, value 1 to $u_{0,1}$ and 0 otherwise gives an RIDF of $C(K_{1,n})$ of weight at least 5. Therefore $\gamma_{rI}(C\left(K_{1,n}\right)) = 5$.

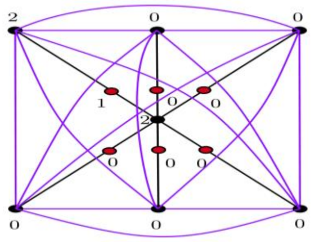

Figure 4: RID of $C(K_{1,6})$

In [13] it is seen that.

Theorem 2.9 ([13]Proposition (11 ii)) For $K_{m,n}$ if $m, n \geq 2$, we have $\gamma_{rI}(K_{m,n}) = 4$.

Now we obtain RID number of $C(K_{m,n})$.

Theorem 2.10 For $K_{m,n}$, if $m, n \geq 2$, then $\gamma_{rI}(C(K_{m,n})) = 3\min\{m,n\} + 2$.

Proof. Let M, N be two partite sets of graph $K_{m,n}$ and $m \geq n \geq 2$ where $M = \{v_1, v_2, \cdots, v_m\}$ and $N = \{w_1, w_2, \cdots, w_n\}$. Let $C(K_{m,n})$ be the central of $K_{m,n}$ with vertex set $V\left(C\left(K_{\{m,n\}}\right)\right) = V(K_{m,n}) \cup \{u_{\{i,j\}}: 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$, where $u_{i,j}$ is the vertex subdivides edge $e = v_i w_j$. We assign value 2 to any vertex v_i and 1 to the set of vertices $u_{i,1}$ for $(1 \leq i \leq m)$ and also assign value 2 to only one vertex w_j $(1 \leq j \leq n)$ say w_1, and zero to another vertices, deduce that $\gamma_{rI}\left(C\left(K_{m,n}\right)\right) \leq 3m + 2$. Other hand, for any RIDF f, we have $f(v_i) + f\left(u_{i,j}\right) \geq 2$ or $f\left(w_j\right) + f\left(u_{i,j}\right) \geq 2$ for any i and j, also we cannot assign zero to all vertices of one partite set. Thus assume without loss of generality that $f(w_1) = 2$ and $f(v_i) = 2$ for all i. For restrained, we should assign 1 to $u_{i,1}$ for all i. Therefore we at least have, $\sum_{i=1}^{m} f(v_i) \geq 2m$, $\sum_{i=1}^{m} f(u_{i,1}) \geq m$ and $f(w_1) = 2$. This denotes $w(f) \geq 3m + 2$ and the result is observed.

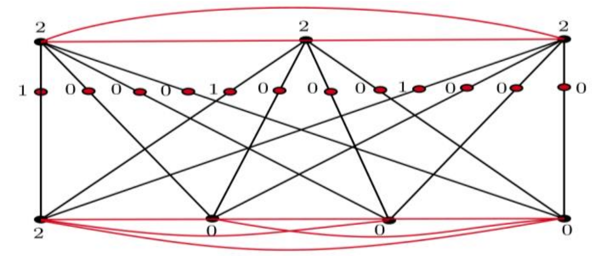

Figure 5: RID of $C(K_{3,4})$.

Theorem 2.11 If $S_{p,q}$ is a double star, then $\gamma_{rI}(C(G))(C(S_{p,q})) = 5$.

Proof. Let $V(S_{p,q}) = \{u_0, u_1, \cdots, u_p, v_0, v_1, \cdots, v_q\}$. Let $C(S_{p,q})$ be the central of $S_{p,q}$ with vertex set $V(S_{p,q}) \cup \{x_{0,1}, \cdots, x_{0,p}, y_{0,1}, \cdots, y_{0,q}\} \cup \{z_0\}$ where $\{x_{0,i}, y_{0,j}\}$ are the vertices that subdivide edges $e = u_0\, u_i, v_0 v_j$ respectively and z_0 is the vertex that subdivides $e = u_0 v_0$. As we see in bellow figure, $\{u_0, v_1, v_2, \cdots, v_p\}$ induces a clique of order $p+1$, also $\{v_0, u_1, u_2, \cdots, u_p\}$ induces a clique of order $q+1$ in $C(S_{p,q})$. Thus for any RIDF f on $C(S_{p,q})$, we have $f(u_0) + \sum_{i=0}^{p} f(v_i) \geq 2$ and $f(v_0) + \sum_{j=0}^{q} f(u_j) \geq 2$. If also $f(v_0)$ and $f(u_0)$ are positive, then $f(z_0)$ must be positive. Therefore $w(f) \geq 5$. Other hand, devoting 2 to v_0, u_0 value 1 to z_0 and zero to the other vertices , gives an RIDF of weight 5. Thus the result is observed.

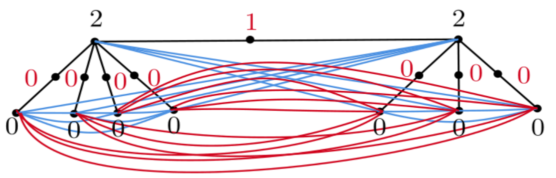

Figure 6: RID of $C(S_{4,3})$

3. TRID FUNCTION OF CENTRAL GRAPHS.

In this section the (TRID) of standard graphs and detection the smallest TRID number are studied. More formally, we prove the following results. More formally, we prove the following results. From Theorem, 2.5 and Figure 2, we have.

Proposition 3.1 For any path $P_n, (n \geq 6)$, $\gamma_{trI}(CP_n) = \begin{cases} n & \text{if n is even} \\ n-1 & \text{if n is old} \end{cases}$.

From Theorems, 2.5 and 3.1 we have.

Proposition 3.2 $\gamma_{trI}(C(P_n)) = \gamma_{rI}(C(P_n))$

It is famous, if we remove one edge from C_n the P_n is resulted. Furthermore, if we remove $u_{i,i+1}$ for some $1 \leq i \leq n-1$ from $C(C_n)$ the $C(P_n)$ is obtained.

Proposition 3.3 For any cycle $C_n, (n \geq 5)$, $\gamma_{trI}(C(C_n)) = \begin{cases} n & \text{if n is even} \\ n+2 & \text{if n is old} \end{cases}$.

Proof. Suppose that $V(C_n) = \{v_1, v_2, \cdots, v_n\}$ and $V(C(C_n)) = V(C_n) \cup \{u_{1,2}, u_{2,3}, \cdots u_{n,1}\}$. For any TRIDF f, $f(v_i) + f(u_{i,i+1}) + f(v_{i+1}) \geq 2$. Also if we assign positive weight to vertex $u_{i,i+1}$ then we should assign positive weight to one v_i or v_{i+1}. Thus it is suitable, we at least devote the value 2 to the vertices v_i is alternately. This shows that $w(f) \geq 2\left\lceil \frac{n}{2} \right\rceil$.

Therefore we have two situations.

1. Let n be an even integer. By devoting value 2 to v_i is for even i and 0 otherwise, we obtain a TRIDF of weight $n = 2\left\lceil \frac{n}{2} \right\rceil$ Hence $\gamma_{trI}(C(C_n)) = n$

Let n be an odd integer. In this case at least two successively vertices v_i and v_{i+1} should be assigned by value 2 and the vertex $u_{i,i+1}$ by value 1. This denotes, for any TRIDF g of $C(C_n)$ for n odd, $w(g) \geq 2\left\lceil \frac{n}{2} \right\rceil + 1 = n + 2$. Other hand, devoting 2 to vertices v_i is for odd i, value 1 to $u_{n,1}$ and zero otherwise, gives a TRIDF of weight $n + 2$. Hence $\gamma_{trI}(C(C_n)) = n + 2$ whenever n is odd.

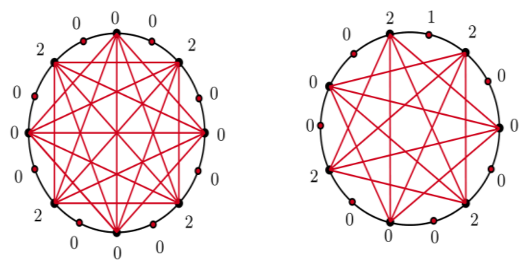

Figure 7: TRID of $C(C_8)$ and $C(C_7)$

From Theorem 3.3 we can have.

Corollary 3.4 If n is even. Then the $\gamma_{trI}(C(C_n)) = \gamma_{rI}(C(C_n))$

Corollary 3.5 If n is odd. Then the $\gamma_{trI}(C(C_n)) = \gamma_{rI}(C(C_n)) + 2$

Now we study TRID of $C(K_n), C(K_{m,n})$and $C(S_{p,q})$.

Theorem 3.6 For $K_n, n \geq 4$. Then the $\gamma_{trI}C(K_n) = \left(n + \frac{n(n-1)}{2}\right) - 1$.

Proof. We use the notation of Theorem 7.Let $V(C(K_n)) = \{v_1, \cdots, v_n\} \cup \{u_{i,j} : 1 \leq i < j \leq n\}$, where $u_{i,j}$ is the vertex subdivides edge v_iv_j. Any TRIDF f assign 0 to only one vertex v_i for $1 \leq i \leq n$ for totality. Hence at least one of vertices and at most $n - 2$ vertices of $\{u_{i,1}, u_{i,2}, \cdots, u_{i,i-1}, u_{i,i+1}, \cdots, u_{i,n}\}$ are assigned zero. Since $f(v_i) = 0$, it is easy to see that if $f(u_{i,j}) = 0$, then $f(v_j) = 2$ and if $f(u_{i,j}) > 0$, then $f(v_j) > 0$. Assume without loss of generality that $f(v_1) = 0 = f(u_{1,2}) = f(u_{1,3}) = \cdots = f(u_{1,k})$ for $1 \leq k \leq n-2$. Thus $f(v_2) = f(v_3) = \cdots = f(v_k) = 2$ and $f(x) \geq 1$ for otherwise and it is minimum, if $f(x) = 1$, (Figure 8). It can be seen with a simple calculation, $w(f) = \frac{n(n-1)}{2} - k + 2k + n - (k+1) = n + \frac{n(n-1)}{2} - 1$

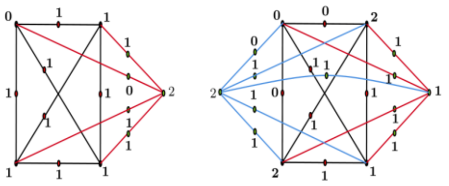

Figure 8: TRID of $C(K_5)$ and $C(K_6)$

By adapting of Theorem 2.8 We can obtain $\gamma_{trI}C(K_{1,n})$.

Theorem 3.7 For $K_{1,n}$ we have $\gamma_{trI}C(K_{1,n}) = 5$ for $n \geq 2$.

Proof. Because of $C(K_{1,n})$ is consisted a complete graph K_n which is induced by n leaves

$v_1, \cdots, v_n$ of $K_{1,n}$, one vertex of degree n which is the support vertex v of $K_{1,n}$ and n vertics $u_{0,1}, u_{0,2}, \cdots, u_{0,n}$ of degree 2 for which vertex $u_{0,j}$ subdivides the edge vv_j. For any TRIDF f of $C(K_{1,n})$, $\sum_{i=1}^{n} f(v_i) \geq 2, f(v) + f(u_{0,j}) \geq 2$ for any j. Assume without loss of generality that $f(v_1) = 2$. If $f(v) = 2$, then $f(u_{0,1}) \geq 1$ and if $f(v) = f(u_{0,1}) = 1$, then $f((u_{0,1}) \geq 1$ for $j \geq 2$. Anyway if $f(v_1) = 2$, then $f(v) + \sum_{j=2}^{n} f((u_{0,1}) \geq 3$. This shows that $w(f) \geq 5$. On the other hand, the function g with $g(v_1) = 2 = g(v), g(u_{0,1}) = 1$ gives us a TRIDF of weight 5. Therefore $\gamma_{trI}C(K_{1,n})) = 5$.

Theorem 3.8 For complete bipartite graph $K_{m,n}$, we have $\gamma_{trI}(C(K_{m,n})) = 3\min\{m,n\} + 2$, if and only if $K_{m,n} \neq K_{3,3}$.

See Theorem 2.10 and Figure 5.

In the proof of From the Theorems 2.10 and 3.8 the vertices assigned with positive value have no isolated vertex Therefore we have.

Proposition 3.9 $\gamma_{trI}\left(C(K_{m,n})\right) = \gamma_{rI}(K_{m,n})$

From Theorem 2.11 and Figure 6 we have

Proposition 3.10 For double star $S_{p,q}$, then $\gamma_{trI}(C(S_{p,q})) = 5$.

As an immediate of Theorem 2.11 and Proposition 3.10, we have.

Corollary 3.11 $\gamma_{trI}(S_{p,q}) = \gamma_{rI}(S_{p,q})$.

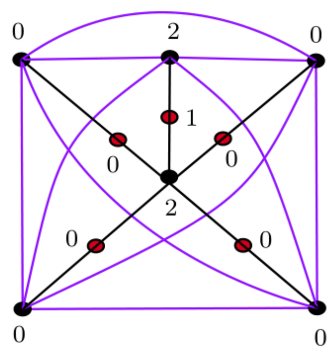

Figure 9: Total restrained Italian domination of $C(K_{1,5})$

4. RIDF AND TRIDF ON CENTRAL OF A GRAPH AND CENTRAL CORONA OF A GRAPH

In this section, we turn our attention on presenting sharp bounds on RID an TRID numbers for central of any graph and the exact value of central of the corona of any graph. For this, the size of graph G is the number of edges of G.

4.1 RIDF AND TRIDF ON CENTRAL OF A GRAPH

Let G be of order n and $\Delta(G) \leq n-2$. Then every vertex has a non-neighbor in G and then it makes a neighbor in $C(G)$.

Theorem 4.1.1 Suppose that G is a graph of order n, size m and $\Delta(G) \leq n-2$. If l is the number of pendant edges or the number vertices of G with degree 1, then $\gamma_{rI}(C(G)) \leq m+l$. This bound is sharp.

Proof. Let $u_{i,j}$ be the vertex in $C(G)$ that subdivides the edge $v_i v_j$. If $v_i v_j$ is a pendant edge we assign value 2 to $u_{i,j}$ and If $v_i v_j$ is a non-pendant edge we assign value 1 to $u_{i,j}$,. Now we devote 0 to all v_is and since $\Delta(G) \leq n-2$ any v_i is adjacent to a vertex v_k in $C(G)$, we obtain an RIDF of G of weight $2l+m-l=m+l$. Therefore $\gamma_{rI}(C(G)) \leq m+l$.
For seeing the sharpness, consider $G=C_n$, $G=P_4$ and use Theorems 2.3 and 2.5.

For TRID of any graph G we have the following sharp bound.

Theorem 4.1.2 Let G be a connected graph of order at least 4, minimum degree at least 2 and girth $k \geq 3$. Then $\gamma_{trI}(C(G)) \leq m+n-k+2$. This bound is sharp.

Proof. Let C_k be the cycle in G. Then by Theorem 14 $\gamma_{trI}(C(C_k)) \leq k+2$. Now we assign value 1 to the all vertices $V(C(G))-V(C(C_k))$. These assignments give a TRIDF on $C(G)$ with of weight at most $k+2+m+n-2k$. Therefore $\gamma_{trI}(C(G)) \leq m+n-2k+2+k=m+n-k+2$.
This bound is sharp for $G=C_n$ (n is odd) from Theorem 3.3.

4.2 Corona of K_1

In this subsection we investigate the RID and TRID function of corona of a graph with K_1.

4.2.1 *Let H be a graph of order n, size m and its dependence number is $\alpha(H)$. Let G be a corona H with K_1, that is $G=H \circ K_1$. Then*

- *a) $\gamma_{rI}(C(G)) \leq 4+n+m-\Delta(H)$. This bound is sharp for star $K_{1,n}$.*
- *b) $\gamma_{trI}(C(G)) \leq 2n+\alpha) + \binom{n-\alpha}{2}$. Equality holds if and only if the induced subgraph by the vertices out of the maximum independent set form a clique.*

Proof. *Let $V(H)=\{v_1,\dots,v_n\}$, $V(G)=V(H)\cup\{w_1,\dots,w_n\}$, and $V(C(G))=V(G)\cup\{u_{i,j}: 1 \leq i < j \leq n,\}\cup\{u_{k,k}: 1 \leq k \leq n\}$ where $u_{i,j}$ and $u_{k,k}$ subdivide the edges $e=v_iv_j$ and $e'=v_kw_k$ respectively.*
a) Let v_r be the vertex of maximum degree $deg(v_r)=\Delta(H)$. We assign 2 to the vertices v_r, w_r 1 to vertices $u_{k,k}$ ($1 \leq k \leq n$), $u_{i,j}$, ($1 \leq i < j \leq n$, with $i,j \neq r$) and 0 otherwise. These assignments give us an RIDF of weight $4+n+m-\Delta(H)$. Therefore the result is observed.

b) For any TRIDF f we have $f(w_k)+f(u_{k,k})+f(v_k) \geq 2$, and $f(v_i)+f(u_{i,j})+f(v_j) \geq 2$ if v_i and v_j are adjacent. For totality, if $f(u_{k,k})>0$, then at least one of $f(w_k)$ or $f(v_k)$ is positive and $f(u_{i,j})>0$, then at least one of $f(v_i)$ or $f(v_j)$ is positive, as well as we cannot have $v_i=v_j=0$ for any two vertices adjacency v_i, v_j and we cannot have $f(v_k)=f(w_k)=0$. For restrained, if v_i, v_j are adjacent and $f(v_i)$, $f(v_j)$ are positive, then $f(u_{i,j})>0$, also if $f(v_k)$, $f(w_k)$ are positive, then $f(u_{k,k})>0$. Now, without loss of generality, if $v_1, v_2, \cdots, v_\alpha$ are maximum independent vertices of H, then taking $f(v_i)=2$ for $i \geq \alpha+1$, $f(u_{k,k})=2$ and $f(w_k)=1$ for $i \leq \alpha$, $f(u_{i,j})=1$ for $\alpha+1 \leq i < j \leq n$ and 0 otherwise improve a TRIDF of weight $2(n-\alpha)+3\alpha+\binom{n-\alpha}{2}=2n+\alpha+\binom{n-\alpha}{2}$. Therefore $\gamma_{trI}(C(G))2n+\alpha)+\binom{n-\alpha}{2}$.
If equality holds and the two vertices v_i and v_j of weight 2 are not adjacent, then $w(f) \leq 2n+\alpha)+\binom{n-\alpha}{2}-1$, that is a contradiction, thus result holds.
Conversely, the set $\{v_{\alpha+1} \cdots v_n\}$ induces a clique. Since every vertex in the clique is devoted by value 2, hence each vertex $u_{i,j}$ for $\alpha+1 \leq i < j \leq n$ is assigned by value 1. The weight of other vertices are routine. Thus the equality holds.

Theorem 4.2.2 *Let P_n be a path of order n. Let G be a corona P_n, that is $G=P_n \circ K_1$. Then*

- *a) $\gamma_{rI}(C(G)) = \left\lfloor \frac{3n}{2} \right\rfloor$ for $n \geq 4$.*
- *b) $\gamma_{trI}(C(G)) = 2n$ for $n \geq 3$.*

Proof. *Assume that $V(P_n)=\{v_1,v_2,\dots,v_n\}$ and $V(G)=V(P_n)\cup\{w_1,w_2,\dots,w_n\}$. Let $u_{i,j} \in V(C(G))$ subdivides to $e=v_iv_j$ of P_n in G where $1 \leq i < j \leq n$ and $u_{k,k} \in V(C(G))$ subdivides edge $e=w_kv_k$. So $V(C(G))=V(G)\cup\{u_{\{i,j\}}: 1 \leq i<j \leq n\}\cup\{u_{\{k,k\}}: 1 \leq k \leq n\}$.*

- a) *For any RIDF f, $f(v_i)+f(u_{i,i+1})+f(v_{i+1}) \geq 1$, $f(v_i)+f(u_{i,i})+f(w_i) \geq 1$ and $f(v_{i+1})+f(u_{i+1,i+1})+f(w_{i+1}) \geq 1$. Also, if $f(v_i)=1$, then $(u_{i,i})+f(w_i) \geq 1$ and $f(u_{i,i+1})+f(v_{i+1}) \geq 1$. Thus we have $f(w_i)+f(u_{i,i})+f(v_i)+f(v_{i,i+1})+f(v_{i+1})+f(u_{i+1,i+1})+f(v_{i+1}) \geq 3$. Therefore $w(f) \geq \lfloor \frac{3n}{2} \rfloor$. This exhibits $\gamma_{rI}(C(G)) \geq \lfloor \frac{3n}{2} \rfloor$. Otherwise, devoting 2 to v_i for i even, devoting value 1 to $\{u_{k,k}\}$ for k odd and zero for otherwise present an RID of weight at most $\lfloor \frac{3n}{2} \rfloor$. That is, $\gamma_{rI}(C(G)) \leq \lfloor \frac{3n}{2} \rfloor$. Thus the proof is observed.*
- b) *Let f be a TRIDF. Then for totality of the vertices of positive weight and for position of the vertex of $u_{i,i}$, $f(v_i)+f(u_{i,i})+f(w_i) \geq 2$ for any $1 \leq i \leq n$. Therefore $w(f) \geq 2n$ and since f is any TRIDF, $\gamma_{trI}(C(G)) \geq 2n$. Otherwise, devoting 2 to v_i (for i is*

even) and to w_k (for k is odd) and zero otherwise. This shows that $\gamma_{trI}(C(G)) \leq 2n$. Thus the proof is completed.

For example see the graph $C(P_5 \circ K_1)$ with RID, TRID function and RID, TRID number.

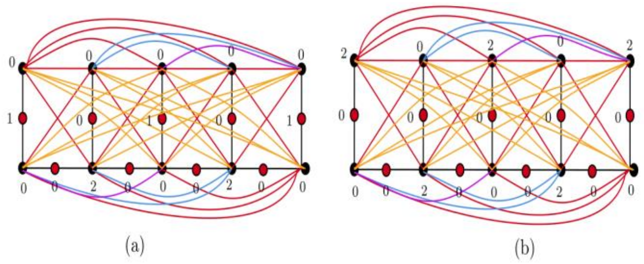

Figure 10: γ_{rI} of $C(P_5 \circ K_1)$ and γ_{trI} of $C(P_5 \circ K_1)$

Theorem 4.2.3 *Let C_n be a cycle of order $n \geq 4$. Let G be a corona C_n, that is $G = C_n \circ K_1$. Then $\gamma_{rI}(C(G)) = \left\lceil \frac{3n}{2} \right\rceil$.*

Proof. *Assume that $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $V(G) = V(C_n) \cup \{w_1, w_2, \ldots, w_n\}$. Let $u_{i,j} \in V(C(G))$ subdivides to $e = v_i v_j$ of C_n in G where $1 \leq i < j \leq n$ and $u_{\{k,k\}} \in V(C(G))$ subdivides edge $e = w_k v_k$. So $V(C(G)) = V(G) \cup \{u_{\{i,j\}}: 1 \leq i < j \leq n\} \cup \{u_{k,k}: 1 \leq k \leq n\}$. For any RIDF f, $f(v_i) + f(u_{i,i+1}) + f(v_{i+1}) \geq 1$, $f(v_i) + f(u_{i,i}) + f(w_i) \geq 1$ and $f(v_i) + f(u_{i,i}) + f(u_{i,i+1}) \geq 2$. Also, if $f(v_i) = 1$, then $f(u_{i,i}) + f(w_i) \geq 1$ and $f(u_{i,i+1}) + f(v_{i+1}) \geq 1$. For the RID property, if $f(v_i) > 0, f(v_{i+1}) > 0$ $(f(v_i) > 0, f(w_i) > 0)$, then $f(u_{i,i+1}) > 0$ $(f(u_{i,i}) > 0)$. Thus we have $f(w_i) + f(u_{i,i}) + f(v_i) + f(v_{i,i+1}) + f(v_{i+1}) + f(u_{i+1,i+1}) + f(v_{i+1}) \geq 3$. Therefore $w(f) \geq \left\lceil \frac{3n}{2} \right\rceil$. Since f is an any RIDF, $\gamma_{rI}(C(G)) \geq \left\lceil \frac{3n}{2} \right\rceil$. therwise, for even n, devoting 2 to v_i for i even, devoting 1 to $\{u_{k,k}\}$ (k odd) and zero for otherwise present an RID of weight at most $\left\lceil \frac{3n}{2} \right\rceil$. For odd n, devoting 2 to v_i for i even, devoting 1 to $u_{n,1}$ and $\{u_{k,k}\}$ (k odd), and zero for otherwise present an RID of weight at most $\left\lceil \frac{3n}{2} \right\rceil$. These improve that $\gamma_{rI}(C(G)) \leq \left\lceil \frac{3n}{2} \right\rceil$, see the Figure 11. Thus the proof is observed.*

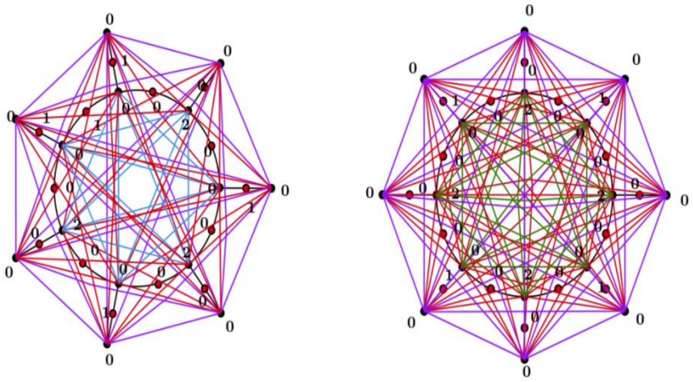

Figure 11: γ_{rI} $(C(C_7 \circ K_1))$ and γ_{rI} $(C(C_8 \circ K_1))$

Theorem 4.2.4 Let C_n be a cycle of order n. Let G be a corona C_n, that is $G = C_n \circ K_1$. Then

$$\gamma_{trI}(C(G)) = \begin{cases} 2n & \text{if n is even} \\ 2n+1 & \text{if n is odd} \end{cases}$$

Proof. Let f be a TRIDF. Then for totality of the vertices of positive weight and for position of the vertex of $u_{i,i}$, $f(v_i) + f(u_{i,i}) + f(w_i) \geq 2$ for any $1 \leq i \leq n$ and for the restrained property, if $f(v_i) > 0, f(v_{i+1}) > 0$ $(f(v_i) > 0, f(w_i) > 0)$, then $f(u_{i,i+1}) > 0$ $(f(u_{i,i}) > 0)$.

Therefore $w(f) \geq 2n$ for even n and $w(f) \geq 2n+1$ for odd n and since f is any TRIDF, $\gamma_{trI}(C(G)) \geq 2n$ for even n and $\gamma_{trI}(C(G)) \geq 2n+1$ for odd n. On the other hand devoting 2 to v_i (for i is odd) and to w_k (for k is even) and zero otherwise whenever n is even, and devoting value 2 to v_i (for i is odd) and to w_k (for k is even) and 1 to $u_{n,1}$ and zero otherwise whenever n is odd. These show that $\gamma_{trI}(C(G)) \leq 2n$ if n is even and $\gamma_{trI}(C(G)) \leq 2n+1$ if n is odd, see the Figure 12. Thus the proof holds.

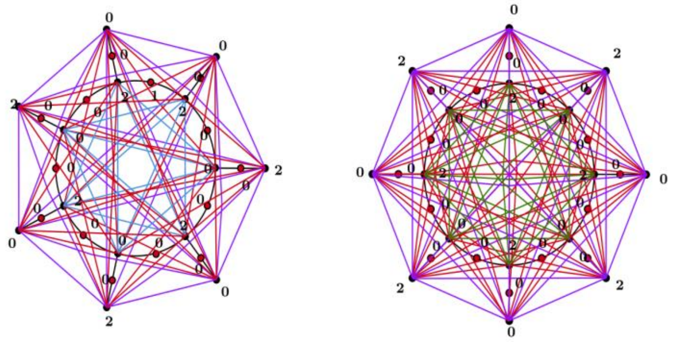

Figure 12: γ_{trI} of $C(C_7 \circ K_1)$ and $C(C_8 \circ K_1)$

Theorem 4.2.5 Let K_n be a complete graph of order n. Let G be a corona K_n, that is $G = K_n \circ K_1$. then

$$\gamma_{rI}(C(G)) = \begin{cases} \frac{n^2+4}{4} & \text{if n is even} \\ \frac{n^2+4-1}{4} & \text{if n is odd} \end{cases}.$$

Proof. Using notation of Theorem 7, suppose that $V(C(K\_n)) = V(K_n) \cup \{u_{i,j}: 1 \leq i < j \leq n\} \cup \{u_{k,k}: 1 \leq k \leq n\}$ where $u_{i,j}$ is the vertex that subdivides the edge $e = v_i v_j$ and $u_{k,k}$ is the vertex that subdivides the edge $e = v_k w_k$. We prove the theorem by induction methods on n.
If $n = 4$, then devoting value 2 to v_2 and v_4, value 1 to $u_{2,4}$, $u_{1,3}, u_{1,1}$ and $u_{3,3}$ and value 0 otherwise show that $\gamma_{rI}(C(K_4 \circ K_1)) = 8$. If $n = 5$, then devoting value 2 to v_2 and v_4, value 1 to $u_{2,4}, u_{1,3}, u_{1,1}, u_{3,3}$, $u_{1,5}$, $u_{3,5}$ and $u_{5,5}$ and value 0 otherwise show that $\gamma_{rI}(C(K_5 \circ K_1)) = 8 = 8 + 3 = 11$.
These can be the basis of induction. Assume that the result is true for $n \geq 5$. Now let $k = n + 1$. We bring up two situations.
If $n + 1$ is even, then n is odd and according to the assumption of induction $\gamma_{rI}(C(K_n \circ K_1)) = \frac{n^2+4n-1}{4}$. For obtaining $\gamma_{rI}(C(K_{n+1} \circ K_1))$, we assign value 2 to the vertices v_{n+1}, value 1 to vertices $u_{2i,n+1}$ for $1 \leq i \leq \frac{n-1}{2}$ and value 0, to w_{n+1}, $u_{n+1,n+1}$ and vertices $u_{2i+1,n+1}$ for $1 \leq i \leq \frac{n-1}{2}$, then the weight of these new assignments is $\frac{n+3}{2}$. Therefore $\gamma_{rI}(C(K_{n+1} \circ K_1)) = \gamma_{rI}(C(K_n \circ K_1)) + \frac{n+2}{2} = \frac{(n+1)^2+4-1}{4} = \frac{(n+1)^2+4(n+1)}{4}$.

If $n + 1$ is odd, then n is even and according to the assumption of induction $\gamma_{rI}(C(K_n \circ K_1)) = \frac{n^2+4n}{4}$. For obtaining $\gamma_{rI}(C(K_{n+1} \circ K_1))$, we assign value 0 to the vertices $v_{n+1}, w_{n+1}, u_{2i,n+1}$ for $1 \leq i \leq \frac{n}{2}$ and assign the value 1 to $u_{n+1,n+1}, u_{n+1,2i+1}$ for $1 \leq i \leq \frac{n}{2}$. Thus the weight of these new assignments is $\left\lfloor\frac{n+3}{2}\right\rfloor$. Therefore $\gamma_{rI}(C(K_{n+1} \circ K_1)) = \frac{n+3}{2} + \frac{n^2+4n}{4} + \frac{n+3}{2} = \frac{(n+1)^2+4(n+1)-1}{4}$

From Theorem 4.2.1 part (b) we have.

Theorem 4.2.6 Let $K_n$be a complete graph of order n. Let G be a corona K_n, that is $G = K_n \circ K_1$. then $\gamma_{trI}(C(G)) = m + n + 2$ where m is the size of K_n.
Proof. By Theorem 25 part (b), let $H = K_n$ then $\alpha(H) = 1$ and $\binom{n-1}{2} = \binom{n}{2} - (n-1) = m - (n-1)$. Therefor $2(n-\alpha) + 3\alpha + \binom{n-\alpha}{2} = 2(n-1) + 3 + m - (n-1) = m + n + 2$.

Next, we will check the $\gamma_{rI}(C(G))$ and $\gamma_{trI}(C(G))$where $G = K_{m,n} \circ K_1$.

Theorem 4.2.7 Let $K_{m,n}$ be a complete bipartite graph of order $m \times n$ and. Let G be a corona $K_{m,n}$, that is $G = K_{m,n} \circ K_1$. Then $\gamma_{rI}(C(G)) = \begin{cases} 3n + m + 1 & \text{if } n < m \\ 4n & \text{if } n = m \end{cases}$.

Proof. Let M, N be two partite sets of graph $K_{m,n}$ and $n \leq m$ where $M = \{v_1, v_2, \ldots, v_m\}$, $N = \{w_1, w_2, \ldots, w_n\}$ and $V(G) = V(K_{m,n}) \cup \{x_1, x_2, \ldots, x_m\} \cup \{y_1, y_2, \ldots, y_n\}$. Let $u_{i,j}$ be the vertex in C(G) subdivides $e = v_i w_j$ of $K_{m,n}$ in G where $1 \leq i \leq m$, $1 \leq j \leq n$ and $r_{k,k}$ be the vertex in C(G) subdivides edge $e = x_k v_k$ and $s_{l,l}$ be the vertex in C(G) subdivides edge $e = y_l w_l$. So $V(C(G)) = V(G) \cup \{u\_\{i,j\}: 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{r_{k,k}\}: 1 \leq k \leq m\} \cup \{s_{l,l}\}: 1 \leq l \leq n\}$.
For any RIDF f, and $n < m$ we have $f(r_{i,i}) + f(v_i) + f(u_{i,j}) + f(w_j) + f(s_{j,j}) \geq 3$
and $f(x_k) + f(r_{k,k}\}) + f(v_k) + f(w_j) + f(s_{j,j}) + f(y_j) \geq 3$. If $f(w_j) \in \{0,1\}$, then $f(u_{i,j}) + f(v_i) + f(r_{i,i}) + f(x_i) \geq 2$ and if $f(w_j) = 2$, then $f(N(v_i)) \geq 2$. Since in C(G) vertices v_is and w_is forms cliques K_m and K_n respectively and any v_i is not adjacent to w_i, $f(N(v_i)) \geq 2$ and $f(N(w_j)) \geq 2$. Also, if $f(v_i)$ and $f(w_j)$ are positive, then $f(u_{i,j})$ must be positive.
Therefore any RIDF f, $w(f) \geq \min\{n + mn + m, 2n + 2 + n + m - 1, 2n + 2m\}$. On the other hand, when $n < m$ assigning value 2 to the vertices w_is, and v_1, value 1 to $u_{1,j}$for $1 \leq j \leq n$, and $r_{k,k}$ for $k \neq 1$, value 0 otherwise guarantee an RIDF g of weight $2n + 2 + n + m - 1 = 3n + m + 1$. When and $n = m$ assigning value 2 to the vertices w_is, and x_is, value 0 otherwise guarantee an RIDF g of weight 4n. Therefor the result holds.
See the graphs (a) and (b) in the bellow figures.

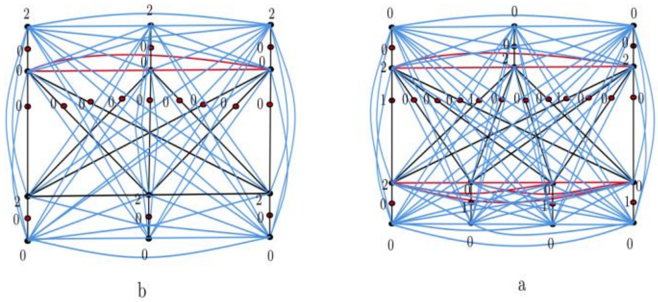

Figure 13: γ_{rI} of $C(K_{3,4} \circ K_1)$ and $C(K_{3,3} \circ K_1)$

Theorem 4.2.8 Let $K_{m,n}$ be a complete bipartite graph of order $n \times m$. Let G be a corona $K_{m,n}$, that is $G = K_{m,n} \circ K_1$. then $\gamma_{trI}(C(G)) = 2(n + m)$.

Proof. Using the notation of Theorem 4.2.7, for any TRIDF f of $C(G)$, $f(v_k) + f(x_k) + f(r_{k,k}) \geq 2$ and $f(w_l) + f(y_l) + f(s_{l,l}) \geq 2$. Also we should have $f(v_k) + f(x_k) \geq 1$ and $f(w_l) + f(y_l) \geq 1$. Therefore $w(f \geq 2n + 2m$.
Other hand , devoting value 2 to x_is and w_js give us a TRIDF g of weight $2n + 2m$. That is, $w(g) \leq 2n + 2m$. Since f and g are any TRIDF we deduce $\gamma_{trI}(C(G)) = 2(n + m)$.
As an immediate results from Theorems 4.2.7 and 4.2.8 we have.

Corollary 4.2.9 Let $K_{1,n}$ be a star and $G = K_{1,n} \circ K_1$. Then,

i. $\gamma_{rI}(C(G)) = n + 4$,
ii. $\gamma_{trI}(C(G)) = 2n + 2$.

From the proof of Theorems 4.2.7, 4.2.8 and using Corollary 4.2.9, with a simple verification can be showed that the following.

Corollary 4.2.10 Let $S_{p,q}$ be a double star of order $p + q + 2$ and $G = S_{p,q} \circ K_1$. then,

i. $\gamma_{rI}(C(G)) = 5 + p + q$.
ii. $\gamma_{trI}(C(G)) = 5 + 2(p + q)$.

5. Conclusions and problems

In Theorems 4.1.1, 4.1.2 we showed the inequalities $\gamma_{rI}(C(G)) \leq m + l$, $\gamma_{rI}(C(G)) \leq m + n - 1$ where n and m are the order and size of G respectively. Now we may have the questions, whether can find the conditions for which the equalities hold.
We studied the RID and TRIDF of central $G \circ K_1$ for any graph G in Subsection 4.2. One can investigate RID and TRIDF of central $G \circ K_2$ for any graph G and central $G \circ H$ for two arbitrary graphs G and H.

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