Solution of Fractal Dynamic Pharmacokinetics Problem

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Abstract— *There are numerous uses for fractional differential equations in engineering, physics, and technology. A path for solving the fractal differential equations was examined, and its homogeneous form of* $D^{\alpha}y + cy = 0$ *was introduced. The notions of Riemann-Liouville fractional derivatives served as the foundation for the journey.. The solutions of the linear non-homogeneous fractal differential equations are given in detail.*

Keywords—: Riemann –Liouville Derivatives; Fractal Differential Equations; LT Laplace Transform.;

1- Introduction

In a message to Leibniz, L'Hopital posed the query that gave origin to fractional calculus, questioning him how could occur if n = 1/2 and about the nth variant of the linear function f (t) = t [26]. What would happen if n a fraction in general? and it was because of this correspondence between L'Hopital and Leibniz that the new field of mathematics known as fractional calculus was developed. [1].

Many scientists did not frequently have any knowledge of these fractional integrals and derivatives.

However, as result to their regular occurrence in several implementation in the areas of viscoelasticity, fluid mechanics, physics, biology, entropy theory, engineering, and image processing, Over the past few decades, many other science contexts have utilized these integrals and derivatives. [5,9,10,23] It is common knowledge that the integral and differential operators of fractional order are non-local operators. It serves as one of the factors that makes the theory of fractional calculus a great resource for analyzing memories and inherited characteristics of many physical phenomena. For instance, it was discovered that half-order derivatives and integrals were most effective for the formation of several electrochemical difficulties Compared to conventional models [15,16,17,18]. Fractional calculus theory application to entropy theory has grown both as a instrument and as a hot area of project [22,14] One-compartment models of pharmacokinetics (PK) have recently been used using fractional calculus, which enables the construction of fractional differential equations (FDEs)[4], When consuming a medicine, you do so to get a desired result, such as lessened symptoms.

something, whether it be a headache, an asthma attack alleviation, a drop in blood pressure, etc. A tablet's contents are absorbed and distributed when you consume one. where it remains until it is digested or excreted. the subject of PK, or pharmacokinetics, is the term used to describe this. Because of a mod-using data from clinical research to explain the component what the body does with the medicine and how it is dispersed there.

In 1937, T. Teorell released a two-part study titled Kinetics of distribution of substances administered to the body that gave rise to the field of pharmacokinetics. Pharmacokinetics was described using compartmental models in this study and in studies that came after it, effectively turning it into a straightforward application of the mathematics of linear differential equations [1]. Focusing on more physiological principles, such as clearance and volume, during the 1970s, led by Manchester University professor Malcolm Rowland and others, made the knowledge more immediately helpful to physicians. With this, the subject's presentation changed from a mathematical modeling standpoint, which many students in the discipline found difficult to understand, to a concept based mostly on a non-mathematical description. [1].

The proposed method for solving the linear of FODE with homogenous form of exponential function and comparison between the proposed method and the Laplace transform method was used and it has been shown to be encouraging and acceptable, with Basic concepts of fractal differential equations, and examples and programmed in MATLAB.

This research is organized into seven sections: The fundamental ideas of fractal differential equations are described in Section 2. The fundamental ideas of the Riemann-Liouville fractional integral and derivative are explained in Section 3. In Section 4, explained the derivative of fractal differential equations. Section 5 presented the proposed method for solving homogeneous fractal differential equation. Section 6 shows implementations and comparisons of proposed method, followed by the conclusion in Section 7.

2- BASIC CONCEPTS OF FRACTAL DIFFERENTIAL EQUATIONS

Definition 2.1 [17]: The complete gamma function Γ (t) plays a significant role in the theory of fractional calculus. The definition of Γ (t) provided by the Euler limit is comprehensive.

$$\Gamma(t) := \lim_{N\to\infty} \frac{N!Nt}{t(t+1)(t+2)\ldots(t+N)}, \quad t > 0.$$

However, the advantageous integral convert form is as follows:

$$\Gamma(t) := \int_0^\infty u^{t-1} e^{-u}\, du, \quad R(t) > 0.$$

Definition 2.2 [8][15]: on the interval (a, b) the function u(x) of the α the order left and right Riemann –Liouville integrals are known as:

$$ {}_aI_x^\alpha\ u(t) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{u(s)}{(x-s)^{1-\alpha}}\, ds,$$

$$ {}_xI_b^\alpha\ u(t) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{u(s)}{(x-s)^{1-\alpha}}\, ds,$$

They are sometimes referred to as left-sided and right-sided fractional integrals, respectively where $n-1<\alpha<n \in z^+$ are termed fractional integrals of the order α

Definition 2.3 [15][13]: The α th order left and right Riemann –Liouville derivative of function u(x) on the interval (a,b) are defined as follow:

$$RL\ {}^{D_{a,x}^\alpha}u(x) := \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-\tau)^{n-\alpha-1}\ u(\tau)d\tau,$$

$$RL\ {}^{D_{x,b}^\alpha}u(x) := \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^b (\tau-x)^{n-\alpha-1}\ u(\tau)d\tau,$$

where $n-1<\alpha<n \in z^+$.

3- Basic concepts of Riemann-Liouville fractional integral and derivative

1- If $F \in C[0,\infty)$,then the Riemann –Liouville fractional order integral has an additional crucial characteristic [15],[13]:

$$I^\alpha\big(I^\beta f(x)\big) = I^\beta\big(I^\alpha f(x)\big) = I^{\alpha+\beta} f(x),$$

where $\alpha > 0$ and $\beta > 0$.

$$= \frac{1}{\Gamma(\alpha+\beta)} \int_0^x (x-t)^{\alpha+\beta-1}\ f(t)\, dt.$$

2- Let us utilize notion the fractional derivative of order α a fractional derivative of order β [12],[16]

$$D^\alpha\left(D^\beta f(t)\right) = D^{\alpha+\beta} f(t).$$

3- For $\alpha > 0$, $t > 0$ [16],[18]

$$D^\alpha(I^\alpha f(t)) := f(t).$$

4- $D^\alpha\big(\lambda f(t) + \mu g(t)\big) = \lambda D^\alpha f(t) + \mu D^\alpha g(t)$ [16].

5- $D^\alpha\big(kf(t)\big) = kD^\alpha f(t), \quad \alpha > 0.$

6- $D^\alpha\big(f(t).g(t)\big) = [D^\alpha f(t)].g(t) + f(t)[D^\alpha g(t)]$[16].

7- $D^\alpha\big(f(t).g(t)\big) = D^\alpha f(t) + D^\alpha g(t), \quad \alpha \in R.$

Remark: $(\boldsymbol{\lambda x^\alpha})$

$$y(x) = L^{-1} \frac{s^{\alpha-\beta}}{s^\alpha-\lambda} = x^{\beta-1} \sum_{k=0}^\infty \frac{(\lambda x^\alpha)^k}{\Gamma(\alpha k+\beta)}, \quad (1)$$

4- THE DERIVATIVE OF FRACTAL DIFFERENTIAL EQUATIONS [16][13]

1- $D^\alpha(x^n) = \frac{\Gamma(n-1)}{\Gamma(n-\alpha+1)}\ x^{n-\alpha}$

2- $D^\alpha(\sin \alpha x) = a^\alpha \sin(ax + \frac{\pi}{2}\alpha)$

3- $D^\alpha(\cos \alpha x) = a^\alpha \cos(ax + \frac{\pi}{2}\alpha)$

4- $D^\alpha\ \left(e^{kx}\right) = k^\alpha\ e^{kx}$

5- $D^\alpha(C) = \frac{Cx^{-\alpha}}{\Gamma(1-\alpha)}$

5- THE PROPOSED METHOD FOR SOLVING HOMOGENEOUS

The first order differential equation of pharmacokinetics problem has the form

$$Dy - Ay = 0,\ \ y(0) = c. \quad (2)$$

Where the solution of (2) is

$y = C\, e^{bx}$,

clearly it is a unique solution for unique order derivative, so one can expanding equation to fractional order derivative. We have a solution for each fractional alpha-derivative.

The Proposed method for solving homogeneous linear fractal order differential equation with constant coefficients.

$D^\alpha y - Ay = 0, \quad y(0) = c$.

By letting, $y = ce^{bx}$

So $D^\alpha y - Ay = 0,$

$D^\alpha(c\, e^{bx}) - cAe^{bx} = 0$

$C\, b^\alpha e^{bx} - cAe^{bx} = 0$

$b^\alpha = A \implies \alpha \ln b = \ln A$

$\ln b = \frac{\ln A}{\alpha} \implies b = e^{\frac{\ln A}{\alpha}}$

One can note that If $\alpha = 1 \implies b = A,\ y = c\, e^{Ax}$ which is representing the ordinary differential equation.

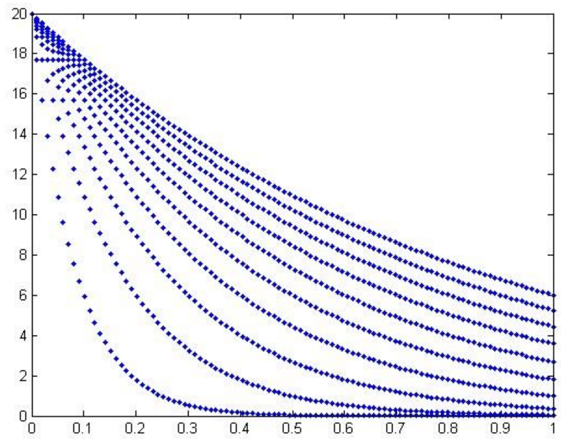

FIGURE1: APPEARS THE SOLUTION OF EQUATION FOR $\alpha = 0.1, 0.2, 0.3, 0.4, \ldots 1$ (ORDINARY CASE)

6- IMPLEMENTATIONS AND COMPARISONS OF PROPOSED METHOD

For clearing proposed method, an example was studied for many fractional derivatives as in Figure 1. For testing our method, the comparison with L.T method is studied as follows where A=0.3 referred to the dilation of computation of drug.

Example (1): to solve the fractal differential equation using LT

$D^{\alpha}y - Ay = 0, \qquad D^{\alpha-1}(0) = c$.

By using L.T.

$LD^{\alpha}y - Ay\ (s) = 0$,

$s^{\alpha}y(s) - c - A\frac{1}{s} = 0,$

$y(s) = \frac{c}{s^{\alpha}-A},$

$y(x) = cL^{-1}\frac{1}{s^{\alpha}-A},$ (3)

with comparison (3) with (4).

By using remark,

$y(x) = L^{-1}\frac{s^{\alpha-\beta}}{s^{\alpha}-\lambda} = x^{\beta-1}\sum_{k=0}^{\infty}\frac{(\lambda x^{\alpha})^k}{\Gamma(\alpha k+\beta)}$

$then\, s^{\alpha-\beta} = 1 \implies \alpha = \beta, \lambda = A.$

So $y(x) = cx^{\alpha-1}\sum_{k=0}^{\infty}\frac{(Ax^{\alpha})^k}{\Gamma(\alpha k+\alpha)},$

$\alpha = 1,$

$y(x) = c\sum_{k=0}^{\infty}\frac{(Ax)^k}{\Gamma(k+1)},$

$y = C\, e^{bx}$.

And it is a program solution for α.

7- Conclusions and Discussion

In this research, a novel approach has been effectively used to resolve a pharmacokinetics-related physics problem using a fractional model. It has been generalized to fractal differential equations in order to find the many solution, one for each fractional order derivative. The fractal order formula $0 < \alpha < 1$ showed us that there are several cases of the fractal equation according to the order, and this gives the possibility of choosing the most appropriate solution, especially in the problem of pharmacokinetics, as in the Figure 1. The comparison with L.T method is studied to clarify the power and reliability of proposed method as appears in Figure 1. There were no fractal equations previously for the subject or problem of pharmacokinetics, but rather an ordinary linear differential equation of the first order. In this paper, the order of the equation has been generalized into a fractal order, and it has been solved with a new hypothesis. Unfortunately, there is no similar problem for the purpose of comparing fractal solutions. In our fractal order, the ordinary order appeared in one of its cases. For the solutions of the hypothesized new drug fractal equation, it was solved using the Laplace transform in addition to our new solution, and the complexities of the Laplace method appeared in terms of the software solution, as well as the flexibility of our method in terms of the exact solution

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