

# Countable Paracompactness of Nano Topological Space Modeled by Pre-open Sets Structures

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**Abstract**— A new class of paracompactness in the nano topological spaces, termed nano pre open sets, are presented and explored in this article. Since the nano topology contains at most five sets this leads us to establish new structures of nano open sets, nano open covers, and nano locally refinement covers. New requirement concepts are discussed like nano pre open covers, nano pre  $T_0, T_1, T_2$ , and nano pre-regular, nano pre-normal spaces, and countable nano pre-open paracompact. Many new theorems are stated and proved.

**Keywords**— Nano Topology, Paracompact Spaces, Regular Space, Normal Space, Pre-open, Separation Axioms, Countable Nano Pre-open Paracompact

## I. INTRODUCTION

Nano topologies proposed by Lellis Thivagar et al. [15],[16] and they are introduced in terms of a subset  $X$  of the universal set  $U$  defined by the lower and upper approximations of this subset. The pair  $(U, R)$  containing an equivalence relation  $R$  on  $U$  is called the space of approximation. In the same equivalence class, elements are called indistinguishable from each other.

Let  $X \subseteq U$ . The set of objects can be classified as  $X$  with respect to  $R$ , is the set  $L_R(X) = \bigcup_{x \in U} \{R(x) : R(x) \subseteq X\}$ , and it is called the lower approximation of  $X$  with respect to  $R$  such that  $R(x)$  is the equivalence class given by  $x$ .

The collection of objects that possibly classified as  $X$  with respect to  $R$  is called the upper approximation  $U_R(X)$  and it is equal to  $\bigcup_{x \in U} \{R(x) : R(x) \cap X \neq \emptyset\}$ , while the set of objects that cannot be classified as  $X$  or non- $X$  with respect to  $R$  and is denoted by  $B_R(X) = U_R(X) - L_R(X)$ , and it is called the boundary of  $X$  with respect to  $R$ .

This setting generates a nano topology  $\tau_R(X)$  on  $U$  containing nano open sets and their complements are called closed nano sets.  $(U, \tau_R(X))$  called a nano topological space.

Nano pre-neighborhoods defined by Sathishmohan et.al [13] nano pre interior, nano pre closure in nano topological spaces and investigated some properties related among them.

If  $A \subseteq U$ , a nano interior of  $A$  is the largest nano open set contained in  $A$  and is denoted by  $N_{Int}(A)$  while its nano closure is the smallest nano closed set containing  $A$  and denoted by  $N_{Cl}(A)$ . Since the nano topology has applications in some medical and physics models used to solve real-world

problems [7] we tried to modify these topologies by adding a new class of topologies to these topologies hoping to get new results applicable in medicine and life in the future.

One of the basic concepts of topology is the paracompact spaces. This topic generalizes both compact and metrizable spaces, many authors studied these spaces for more general spaces like topological linear spaces and infinite manifolds [1], [2], [3] and [8], they derived their characterization in terms of open cover that is refinement and locally finite, it is another open cover of the space that contains the open sets that intersect the neighbourhood of any element in the space finitely and these sets are contained at the same time any open cover of the space. Since the nano topology has at most five elements, this makes the study of paracompactness most difficult in this topology and this reason leads us to study these spaces via the pre-open sets structures derived in nano spaces.

Let  $X$  be a topological space, that is, a space containing open sets where the intersection of any finite set and any union of open sets is open. There are a lot of open sets. A collection of open sets whose union is  $X$  is referred to as a covering of  $X$ . The covering is referred to as countable if it is made up of a countable collection of open sets or as finite if it is made up of a finite collection of open sets; it is referred to as locally finite if every point in  $X$  is contained in some open set that only meets a small subset of the sets of the covering. If every open set of  $X$  is contained in some open set of  $U$ , the covering is referred to as a refinement of the covering  $U$ . When each countable covering has a locally finite refinement, the space  $X$  is said to be countably paracompact [6].

Dieudonne, J. [5] introduced the paracompact spaces, that contain the compact spaces and metric spaces as special cases. Dowker, C. H. [6] and Katetov, M. [9] generalized the notion of paracompactness to introduce the countably paracompact spaces.

## II. PRELIMINARIES

**Definition 2.1.[5]** Let  $\{k_i\}_{i \in \lambda}$  be a family of subsets of a topological space  $(X, T)$ . Then  $\{k_i\}_{i \in \lambda}$  is called a cover of  $X$  iff  $X = \cup_i k_i$  and it is said to be open cover if all the sets  $k_i$  are open.

**Definition 2.2.[5]** Let  $(X, T)$  be a topological space. An open cover  $\{k_i\}_{i \in \lambda}$  of  $X$  is called locally finite iff  $\forall x \in X, \exists$  a neighbourhood  $U$ , such that the set  $\{i \in \lambda : U \cap k_i \neq \emptyset\}$  is finite.

**Definition 2.3.[5]** If  $\mathcal{A}$  and  $\beta$  are two covers of a topological space  $(X, T)$ , then  $\beta$  refines  $\mathcal{A}$ , write  $\beta \subset \mathcal{A}$  iff each  $U \in \mathcal{A}$  is contained in some  $B \in \beta$  and  $\beta$  is a refinement of  $\mathcal{A}$ .

**Definition 2.4.[5]** A topological space  $(X, T)$  is called a paracompact, if it is hausdorff and any locally finite open refinement cover contained in every open cover of  $X$  has.

**Definition 2.5.[16]** Let  $U$  be the universe and  $X \subseteq U, R$  be an equivalence relation. Then the set  $\tau_R(X) = \{U, \emptyset, L_R(X), U_R(X), B_R(X)\}$  define a topology on  $U$  called the Nano topology on  $U$  with respect to  $X$ .  $(U, \tau_R(X))$  called a Nano topological space and it is denoted by  $NTS$  and its elements are called nano open sets.

**Definition 2.6.[13]** A subset  $A$  of a  $NTS (U, \tau_R(X))$  is called a nano per-open in  $U$ , if  $A \subseteq N - Int(N - Cl(A))$  and denoted by  $N_{PO}$ , its complement is called a nano pre-closed ( $N_{PC}$ )

**Definition 2.7.[13]** A subset  $A$  of a  $NTS (U, \tau_R(X))$  is called a nano-regular open in  $U$ , if  $N - Int(N - Cl(A)) = A$ , and it is denoted by  $N_{RO}$ . The complement of  $N_{RO}$  is called a nano-regular closed and it is denoted by  $N_{RC}$ .

**Definition 2.8.[12]** The union of all  $N_{PO}$  sets contained in  $A$  is called nano pre-interior of  $A$  and it is denoted by  $N_{PO} Int(A)$ .

**Definition 2.9.[16]** The intersection of  $N_{PC}$  sets containing  $A$  is called nano pre-closure of  $A$  and it is denoted by  $N_{PC} Cl(A)$ .

**Definition 2.10.[11]** A  $NTS (U, \tau_R(X))$  is called a nano- $T_0$  (or  $N - T_0$ ) space for  $a, b \in U$  and  $a \neq b$ , there exists a nano-open  $\mathcal{G} \ni a$  in  $H$  and  $b$  in not in  $\mathcal{G}$ .

**Definition 2.11.[11]** A  $NTS(U, \tau_R(X))$  is called a nono pre- $T_0$  (or  $N_{PO} - T_0$ ) space for  $a, b \in U$  and  $a \neq b$ ,  $\exists$  a  $N_{PO}$  set  $\mathcal{G} \ni a \in \mathcal{G}$  and  $b \notin \mathcal{G}$ .

**Example 2.12.** Let  $U = \{1, 2, 3\}$ ,  $\tau_R(X) = \{\emptyset, U, \{1\}, \{1, 2\}, \{2\}\}$ .  $N_{PO}(U, X) = \{\emptyset, U, \{1\}, \{2\}, \{1, 2\}\}$ .  $1 \neq 2 \Rightarrow 1 \in \{1\} \wedge 2 \notin \{1\}$ .  $1 \neq 3 \Rightarrow 1 \in \{1\} \wedge 3 \notin \{1\}$ .  $2 \neq 3 \Rightarrow 2 \in \{2\} \wedge 3 \notin \{2\}$ .  $\therefore (U, \tau_R(X))$  is a  $N_{PO} - T_0$  space.

**Definition 2.13.[10]** A  $NTS (U, \tau_R(X))$  is called a nano- $T_1$  (or  $N - T_1$ ) space for  $a, b \in U \ni a \neq b$ ,  $\exists$  a nano-open sets  $\mathcal{G}$  and  $\mathcal{H} \ni a \in \mathcal{G}, b \notin \mathcal{G}$  and  $b \in \mathcal{H}, a \notin \mathcal{H}$ .

**Definition 2.14.[11]** A  $NTS (U, \tau_R(X))$  is called a nano pre- $T_1$  (or  $N_{PO} - T_1$ ) space for  $a, b \in U$  and  $a \neq b$ , there exists a  $N_{PO}$  sets  $\mathcal{G}$  and  $\mathcal{H} \ni a \in \mathcal{G}, b \notin \mathcal{G}$  and  $b \in \mathcal{H}, a \notin \mathcal{H}$ .

**Definition 2.15.[14]** A  $NTS (U, \tau_R(X))$  is called a nano- $T_2$  (or  $N - T_2$ ) space if for  $a, b \in U$  and  $a \neq b$ ,  $\exists$  disjoint nano-open sets  $\mathcal{G}$  and  $\mathcal{H} \ni a \in \mathcal{G}$  and  $b \in \mathcal{H}$ .

**Definition 2.16.[14]** A  $NTS (U, \tau_R(X))$  is called a nano pre- $T_2$  a nano pre-open Hausdorff (or  $N_{PO} - T_2$ ) space for  $a, b \in U$  and  $a \neq b$ ,  $\exists$  disjoint  $N_{PO}$  sets  $\mathcal{G}$  and  $\mathcal{H} \ni a \in \mathcal{G}$  and  $b \in \mathcal{H}$ .

**Example 2.17.** Let  $U = \{\alpha, \beta, \delta\}$ ,  $U/R = \{\{\alpha\}, \{\beta\}, \{\beta, \delta\}\}$ ,  $X = \{\alpha, \delta\}$  and  $\tau_R(X) = \{\emptyset, U, \{\alpha\}, \{\beta, \delta\}\}$ . The nano closed set  $F = \{U, \emptyset, \{\beta, \delta\}, \{\alpha\}\}$ .  $N_{PO}(U, X) = \{\emptyset, U, \{\alpha\}, \{\beta\}, \{\delta\}, \{\alpha, \beta\}, \{\alpha, \delta\}, \{\beta, \delta\}\}$ .  $\alpha \neq \beta \Rightarrow \alpha \in \{1\alpha\} \wedge \beta \in \{\beta\} \Rightarrow \{1\alpha\} \cap \{1\beta\} = \emptyset$ .  $\alpha \neq \delta \Rightarrow \alpha \in \{1\alpha\} \wedge \delta \in \{\delta\} \Rightarrow \{1\alpha\} \cap \{1\delta\} = \emptyset$ .  $\beta \neq \delta \Rightarrow \beta \in \{1\beta\} \wedge \delta \in \{\delta\} \Rightarrow \{1\beta\} \cap \{1\delta\} = \emptyset$ .  $\therefore (U, \tau_R(X))$  is a  $N_{PO} - T_2$  space.

**Definition 2.18.[4]** A  $NTS (U, \tau_R(X))$  is said to be  $N_{PO}$ -regular space iff  $\forall N_{PC}, \forall p \notin N_{PO} \exists \mathcal{G}, \mathcal{H} \in N_{PO}(U, X), p \in \mathcal{G} \wedge N_{PC} \subseteq \mathcal{H}, \mathcal{G} \cap \mathcal{H} = \emptyset$ .

**Example 2.19.** Let

$U = \{\alpha, \beta, \delta\}$  and  $U/R = \{\{\alpha\}, \{\beta\}, \{\beta, \delta\}\}$ ,  $X = \{\alpha, \delta\}$ ,  $\tau_R(X) = \{\emptyset, U, \{\alpha\}, \{\beta, \delta\}\}$ . The nano closed set  $F = \{U, \emptyset, \{\beta, \delta\}, \{\alpha\}\}$ .  $N_{PO}(U, X) = \{\emptyset, U, \{\alpha\}, \{\beta\}, \{\delta\}, \{\alpha, \beta\}, \{\alpha, \delta\}, \{\beta, \delta\}\}$ .  $N_{PC}(U, X) = \{U, \emptyset, \{\beta, \delta\}, \{\alpha, \delta\}, \{\alpha, \beta\}, \{\delta\}, \{\beta\}, \{\alpha\}\}$ .  $\alpha \notin \{\beta, \delta\} \Rightarrow \alpha \in \{\alpha\} \wedge \{\beta, \delta\} \subseteq \{\beta, \delta\} \ni \{\alpha\} \cap \{\beta, \delta\} = \emptyset$ .  $\beta \notin \{\alpha\} \Rightarrow \beta \in \{\beta, \delta\} \wedge \{\alpha\} \subseteq \{\alpha\} \ni \{\beta, \delta\} \cap \{\alpha\} = \emptyset$ .  $\delta \notin \{\alpha\} \Rightarrow \delta \in \{\beta, \delta\} \wedge \{\alpha\} \subseteq \{\alpha\} \ni \{\beta, \delta\} \cap \{\alpha\} = \emptyset$ .  $\therefore (U, \tau_R(X))$  is a  $N_{PO}$ -regular space.

**Definition 2.20.[4]** A  $NTS (U, \tau_R(X))$  is said to be a  $N_{PO}$ -normal space iff  $\forall \mathcal{A}, \beta N_{PC}$  sets of  $U \ni \mathcal{A} \cap \beta = \emptyset$ . Then  $\mathcal{G}, \mathcal{H} \in N_{PO}, \mathcal{A} \subseteq \mathcal{G}$  and  $\beta \subseteq \mathcal{H}, \mathcal{G} \cap \mathcal{H} = \emptyset$ .

**Example 2.21.** Let  $U = \{\alpha, \beta, \delta\}$  and  $U/R = \{\{\alpha\}, \{\beta\}, \{\beta, \delta\}\}$ ,  $X = \{\alpha, \delta\}$ ,  $\tau_R(X) = \{\emptyset, U, \{\alpha\}, \{\beta, \delta\}\}$ .

Is  $(U, \tau_R(X))$  is a  $N_{\mathcal{P}O}$  –normal space. The nano closed sets  $F = \{U, \emptyset, \{\beta, \delta\}, \{\alpha\}\}$ .  
 $N_{\mathcal{P}O}(U, X) = \{\emptyset, U, \{\alpha\}, \{\beta\}, \{\delta\}, \{\alpha, \beta\}, \{\alpha, \delta\}, \{\beta, \delta\}\}$ .  
 $N_{\mathcal{P}C}(U, X) = \{U, \emptyset, \{\beta, \delta\}, \{\alpha, \delta\}, \{\alpha, \beta\}, \{\delta\}, \{\beta\}, \{\alpha\}\}$ .  
 $\{\alpha\} \subseteq \{\alpha\}$  and  $\{\beta, \delta\} \subseteq \{\beta, \delta\}$ ,  $\{\alpha\} \cap \{\beta, \delta\} = \emptyset$ .  
 $\{\beta\} \subseteq \{\beta\}$  and  $\{\alpha\} \subseteq \{\alpha\}$ ,  $\{\beta\} \cap \{\alpha\} = \emptyset$ .  
 $\{\delta\} \subseteq \{\delta\}$  and  $\{\beta\} \subseteq \{\beta\}$ ,  $\{\delta\} \cap \{\beta\} = \emptyset$ .  
 $\therefore (U, \tau_R(X))$  is a  $N_{\mathcal{P}O}$  –normal space.

### III. COUNTABLE $N_{\mathcal{P}O}$ –PARACOMPACTNESS

#### Definition 3.1

Let  $(U, \tau_R(X))$  be a NTS. A nano open cover of  $(U, \tau_R(X))$  is a family  $\{k_i \in \tau_R(X)\}$  of nano open subset of  $(U, \tau_R(X))$   
 $\ni U = \cup_i k_i$  and it is called a nano locally finite iff  $\forall x \in X, \exists$  a nano neighbourhood  $V$  such that the set  $\{i \in \lambda : U \cap k_i \neq \emptyset\}$  is finite.

**Definition 3.2** If  $\mathcal{A}$  is a nano open cover of the NTS  $(U, \tau_R(X))$  and  $\beta$  is a  $N_{\mathcal{P}O}$  cover of the NTS  $(U, \tau_R(X))$  we say that  $\beta$  refines  $\mathcal{A}$  and write  $\beta \subset \mathcal{A}$  iff each  $U \in \mathcal{A}$  is contained in some  $B \in \beta$ . Then we say that  $\beta$  is a  $N_{\mathcal{P}O}$  refinement of  $\mathcal{A}$ .

**Definition 3.3** A NTS  $(U, \tau_R(X))$  is called a nano pre-open paracompact if  $X$  is a nano pre-open Hausdorff (or  $N_{\mathcal{P}O} - T_2$ ) space and every open cover  $\mathcal{V}$  of  $X$  has a nano locally finite  $N_{\mathcal{P}O}$  refinement cover  $\mathcal{M}$  and it is denoted by  $N_{\mathcal{P}O}$  –Paracompact space.

**Example 3.4** Let  $U = \{e_1, e_2, e_3, e_4\}$  with  $U/R = \{\{e_1\}, \{e_3\}, \{e_2, e_4\}\}$  and  $X = \{e_1, e_2\}$ .  
 $\tau_R(X) = \{\emptyset, U, \{e_1\}, \{e_1, e_2, e_4\}, \{e_2, e_4\}\}$ . The nano closed set  $F = \{U, \emptyset, \{e_2, e_3, e_4\}, \{e_3\}, \{e_1, e_3\}\}$ .  $N_{\mathcal{P}O}(U, X) = \{\emptyset, U, \{e_1\}, \{e_2\}, \{e_4\}, \{e_1, e_2\}, \{e_1, e_4\}, \{e_2, e_4\}, \{e_1, e_2, e_3\}, \{e_1, e_2, e_4\}, \{e_1, e_3, e_4\}\}$ .

Let  $\mathcal{G} = \{e_2\}$  and  $\mathcal{H} = \{e_3\}$  then it is  $N_{\mathcal{P}O} - T_2$  space. Let  $\mathcal{A}$  is a nano open cover and  $\beta$  is a  $N_{\mathcal{P}O}$  of  $U$ , we say that  $\beta$  is a  $N_{\mathcal{P}O}$  refinement of  $\mathcal{A}$

$\mathcal{A}_1 = \{U, \{e_1\}\}$ ,  $\mathcal{A}_2 = \{U, \{e_1, e_2, d\}\}$ ,  $\mathcal{A}_3 = \{U, \{e_2, d\}\}$ ,  $\mathcal{A}_4 = \{U, \{e_1\}, \{e_1, e_2, d\}\}$ ,  $\mathcal{A}_5 = \{U, \{e_1\}, \{e_2, d\}\}$ ,  $\mathcal{A}_6 = \{U, \{e_1, e_2, d\}, \{e_2, d\}\}$ . If  $\forall B \in \beta \exists A \in \mathcal{A} \ni B \subseteq A$ . Let  $\beta_1 = \{\{e_2\}, \{e_1, e_3, d\}\}$ ,  $\beta_2 = \{\{e_2, d\}, \{e_1, e_3, d\}\}$ ,  $\beta_3 = \{\{d\}, \{e_1, e_2, e_3\}\}$ ,  $\beta_4 = \{\{e_1, e_2\}, \{e_1, e_3, d\}\}$ ,  $\beta_5 = \{\{e_1, d\}, \{e_1, e_2, e_3\}\}$ ,  $\beta_6 = \{\{e_1, e_2, d\}, \{e_1, e_3, d\}\}$ ,  $\beta_7 = \{\{e_2, d\}, \{e_1, e_2, e_3\}\}$ ,  $\beta_8 = \{\{e_1, e_2, d\}, \{e_1, e_2, e_3\}\}$ ,  $\beta_9 = \{U, \{e_1\}\}$ .

$\beta_2$  is a  $N_{\mathcal{P}O}$  refinement of  $\mathcal{A}_3, \mathcal{A}_5$  and  $\mathcal{A}_6$ ,  $\beta_6$  is a  $N_{\mathcal{P}O}$  refinement of  $\mathcal{A}_2, \mathcal{A}_4$ .  $\beta_7$  is a  $N_{\mathcal{P}O}$  refinement of  $\mathcal{A}_3, \mathcal{A}_6$ ,  $\beta_8$  is a  $N_{\mathcal{P}O}$  refinement of  $\mathcal{A}_2, \beta_9$  is a  $N_{\mathcal{P}O}$  refinement of  $\mathcal{A}_1$ , then  $(U, \tau_R(X))$  is a  $N_{\mathcal{P}O}$  –Paracompact.

**Definition 3.4** A NTS  $U$  is called Countable  $N_{\mathcal{P}O}$  –Paracompact ( $C N_{\mathcal{P}O}$  –Paracompact) if it is a  $N_{\mathcal{P}O}$  – Hausdorff (or  $N_{\mathcal{P}O} - T_0$ ) space and every countable open cover of  $\mathcal{V}$  has a nano local finite  $N_{\mathcal{P}O}$  refinement cover  $\mathcal{M}$ .

**Theorem 3.5:** For every  $N_{\mathcal{P}O} - T_2$  space, the following conditions are equivalent:

- (i) The nano space  $U$  is  $C N_{\mathcal{P}O}$  –Paracompact.
- (ii) For every countable open cover  $\{\mathcal{A}_i\}_{i=1}^{\infty}$  of  $U$  there exists a nano local finite  $N_{\mathcal{P}O}$  cover  $\{\beta_i\}_{i=1}^{\infty}$  of  $U$  such that  $\beta_i \subset \mathcal{A}_i$  for  $i = 1, 2, \dots$
- (iii) For every increasing sequence  $\mathcal{W}_1 \subset \mathcal{W}_2 \subset \dots$  in  $N_{\mathcal{P}O}$  subset of  $U$  satisfying  $\cup_{i=1}^{\infty} \mathcal{W}_i = U$  there exists sequence  $\mathcal{F}_1, \mathcal{F}_2, \dots$  in  $U$  such that  $\mathcal{F}_i \subset \mathcal{W}_i$  for  $i = 1, 2, \dots$  and  $\cup_{i=1}^{\infty} N Int(\mathcal{F}_i) = U$ .
- (iv) For every decreasing sequence  $\mathcal{F}_1 \supset \mathcal{F}_2 \supset \dots$  in  $N_{\mathcal{P}C} \ni \cap_{i=1}^{\infty} \mathcal{F}_i = \emptyset \exists$  a sequence  $\mathcal{W}_1, \mathcal{W}_2, \dots$  in  $N_{\mathcal{P}O}$ , such that  $\mathcal{F}_i \subset \mathcal{W}_i$ , for  $i = 1, 2, \dots$  and  $\cap_{i=1}^{\infty} N Cl(\mathcal{W}_i) = \emptyset$ .

**Proof:** To show that (i)  $\Rightarrow$  (ii)

it suffices to take a nano locally finite  $N_{\mathcal{P}O}$  refinement  $\mathcal{M}$  of the cover  $\{\mathcal{A}_i\}_{i=1}^{\infty}$  for every  $\beta \in \mathcal{M}$  choose a natural number  $\beta_i$  such that  $\beta \subset \mathcal{A}_{\beta_i}$ , and let  $\beta_j = \cup \{\beta : \beta_i = i\}$ .

We shall show that (ii)  $\Rightarrow$  (iii).

Put  $\{\mathcal{W}_i\}_{i=1}^{\infty}$  be an open cover which is countable of  $X$ ,  $\exists$  a locally finite cover  $\{\beta_i\}_{i=1}^{\infty}$  of  $X$ ,  $\ni \beta_i \subset \mathcal{W}_i$ ; for  $i = 1, 2, \dots$ . The sets  $\mathcal{F}_i = X \setminus \cup_{j>i} \beta_j \subset \cup_{j \leq i} \beta_j$ , are  $N_{\mathcal{P}C}$  and since  $\cup_{j \leq i} \beta_j \subset \cup_{j \leq i} \mathcal{W}_j = \mathcal{W}_i$  we have  $\mathcal{F}_i \subset \mathcal{W}_i$  for  $i = 1, 2, \dots$ . The family  $\{\beta_i\}_{i=1}^{\infty}$ , being a nano locally finite, each point  $x \in X$  has a nano neighbourhood contained in some  $\mathcal{F}_i$ , i.e.  $\cup_{i=1}^{\infty} N Int(\mathcal{F}_i) = X$ .

It follows easily that (iii) and (iv) are equivalent (From De Morgan's laws); hence, it suffices to prove that (iii)  $\Rightarrow$  (i). Let  $\{\mathcal{A}_i\}_{i=1}^{\infty}$  be countable open cover of  $X$ .

Let  $\mathcal{W}_1 \subset \mathcal{W}_2 \subset \dots$  in  $N_{\mathcal{P}O}$  be an increasing sequence,  $\ni \mathcal{W}_i = \cup_{j \leq i} \mathcal{A}_j$ , as  $\cup_{i=1}^{\infty} \mathcal{W}_i = X$ , there exists a sequence  $\mathcal{F}_1, \mathcal{F}_2, \dots$  in  $N_{\mathcal{P}C}(U, X)$  such that  $\mathcal{F}_i \subset \mathcal{W}_i$  and  $\cup_{i=1}^{\infty} N Int(\mathcal{F}_i) = X$ . The set  $\beta_i = \mathcal{A}_i \setminus \cup_{j<i} \mathcal{F}_j \subset \mathcal{A}_i$  is a  $N_{\mathcal{P}O}$  for  $i = 1, 2, \dots$ ; since  $\cup_{j<i} \mathcal{F}_j \subset \cup_{j<i} \mathcal{W}_j \subset \cup_{j<i} \mathcal{A}_j$ , we have  $\mathcal{A}_i \setminus \cup_{j<i} \mathcal{A}_j \subset \beta_i$  which implies  $C$  that the family  $\{\beta_i\}_{i=1}^{\infty}$  is a cover of  $X$ .

Every point  $x \in X$  has a nano neighbourhood of the form  $N Int(\mathcal{F}_j)$ ; this nano neighbourhood is disjoint from all sets  $\beta_i$  for  $i > j$  so that the cover  $\{\beta_i\}_{i=1}^{\infty}$  is a nano locally finite.

To show that (i)  $\Rightarrow$  (iv)

Suppose that  $X$  is countably  $N_{\mathcal{P}O}$  –Paracompact. Suppose that  $\{\mathcal{F}_i : i = 1, 2, \dots\}$  is a decreasing sequence of  $N_{\mathcal{P}C}$  subsets of  $X$  as in the condition in the theorem. Then  $\mathcal{M} = \{X - \mathcal{F}_i : i = 1, 2, \dots\}$  is an open cover of  $X$ . Let  $\mathcal{V}$  be a nano locally finite  $N_{\mathcal{P}O}$  refinement of  $\mathcal{M}$ . For each  $i = 1, 2, \dots$ , define the following:

$\mathcal{W}_i = \cup \{V \in \mathcal{V} : V \cap \mathcal{F}_i \neq \emptyset\}$ .

It is clear that  $\mathcal{F}_i \subset \mathcal{W}_i$  for each  $i$ . The  $N_{\mathcal{P}O}$  sets  $\mathcal{W}_i$  are decreasing, hence  $\mathcal{W}_1 \supset \mathcal{W}_2 \supset \dots$  since the  $N_{\mathcal{P}C}$  sets  $\mathcal{F}_i$  are

decreasing. To show that  $\bigcap_{i=1}^{\infty} NCl(\mathcal{W}_i) = \emptyset$ , let  $z \in X$ . The goal is to find  $\mathcal{W}_j$  such that  $z \notin NCl(\mathcal{W}_j)$ . Once  $\mathcal{W}_j$  is found, we will obtain a set  $V$  such that  $z \in V$  and  $V$  contains no points of  $\mathcal{W}_j$ .

Since  $\mathcal{V}$  is nano locally finite, there exists a  $N_{\mathcal{P}O}$  set  $V$  such that  $z \in V$  and  $V$  meets only finitely many sets in  $\mathcal{V}$ . Suppose that these finitely many  $N_{\mathcal{P}O}$  sets in  $\mathcal{V}$  are  $V_1, V_2, \dots, V_m$ . Observe that for each  $i = 1, 2, \dots, m$ , there is some  $j(i) \ni V_i \cap \mathcal{F}_{j(i)} = \emptyset$  (i.e.  $V_i \subset X - \mathcal{F}_{j(i)}$ ). This follows from the fact that  $\mathcal{V}$  is a  $N_{\mathcal{P}O}$  refinement  $\mathcal{M}$ . Let  $j$  be the maximum of all  $j(i)$  where  $i = 1, 2, \dots, m$ . Then  $V_i \cap \mathcal{F}_j = \emptyset$  for all  $i = 1, 2, \dots, m$ . It follows that the  $N_{\mathcal{P}O}$  set  $V$  contains no points of  $\mathcal{F}_j$ . Thus  $z \in NCl(\mathcal{W}_j)$ .

For the other direction, suppose that the nano space  $X$  satisfies the condition given in the theorem. Let  $\mathcal{U} = \{U_i : i = 1, 2, \dots\}$  be  $N_{\mathcal{P}O}$  a cover of  $X$ . For each  $i$ , define  $\mathcal{W}_i$  as follows:

$$\mathcal{F}_i = X - U_1 \cup U_2 \cup \dots \cup U_i.$$

Then the  $N_{\mathcal{P}C}$  sets  $\mathcal{F}_i$  form a decreasing sequence of  $N_{\mathcal{P}C}$  sets with empty intersection. Let  $\mathcal{W}_i$  be decreasing  $N_{\mathcal{P}O} \ni \bigcap_{i=1}^{\infty} NCl(\mathcal{W}_i) = \emptyset$  and  $\mathcal{F}_i \subset \mathcal{W}_i$  for each  $i$ . Let  $\mathcal{C}_i = X - \mathcal{W}_i$  for each  $i$ . Then  $\mathcal{C}_i \subset \bigcup_{j=1}^i U_j$ . Define  $V_1 = U_1$ . For each  $i \geq 2$ , define  $V_i = U_i - \bigcup_{j=1}^{i-1} \mathcal{C}_j$ . Clearly each  $V_i$  is  $N_{\mathcal{P}O}$  and  $V_i \subset U_i$ . It is straightforward to verify that  $\mathcal{V} = \{V_i : i = 1, 2, \dots\}$  is a cover of  $X$ .

We claim that  $\mathcal{V}$  is nano locally finite in  $X$ . Let  $z \in X$ . Choose the least  $i \ni z \notin NCl(\mathcal{W}_i)$ . Choose a  $N_{\mathcal{P}O}$  set  $\delta \ni z \in \delta$  and  $\delta \cap NCl(\mathcal{W}_i) = \emptyset$ . Then  $\delta \cap \mathcal{W}_i = \emptyset$  and  $\delta \subset \mathcal{C}_i$ . This means that  $\delta \cap V_k = \emptyset$  for all  $k \geq i + 1$ . Thus the  $N_{\mathcal{P}O}$  cover  $\mathcal{V}$  is a nano locally finite  $N_{\mathcal{P}O}$  refinement of  $\mathcal{U}$ .

**Corollary 3.6.** A  $N_{\mathcal{P}O}$ -normal space  $X$  is countably  $N_{\mathcal{P}O}$ -Paracompact if and only if for every decreasing sequence  $\mathcal{F}_1 \supset \mathcal{F}_2 \supset \dots$  in  $N_{\mathcal{P}O}(U, X)$ , such that  $\bigcap_{i=1}^{\infty} \mathcal{F}_i = \emptyset$ ,  $\exists$  a sequence  $\mathcal{W}_1, \mathcal{W}_2, \dots$  in  $N_{\mathcal{P}O}(U, X)$  such that  $\mathcal{F}_i \subset \mathcal{W}_i$  for  $i = 1, 2, \dots$  and  $\bigcap_{i=1}^{\infty} \mathcal{W}_i = \emptyset$ .

**Theorem 3.7.** For every  $N_{\mathcal{P}O} - T_1$  space  $X$  the following equivalent:

- (i) The nano space  $X$  is  $N_{\mathcal{P}O}$ -normal and countably  $N_{\mathcal{P}O}$ -Paracompact.
- (ii) For every countable open cover  $\{\mathcal{A}_i\}_{i=1}^{\infty}$  of the nano space  $X$ ,  $\exists$  a nano locally finite  $N_{\mathcal{P}O}$  cover  $\{\beta_i\}_{i=1}^{\infty}$  of  $X$ ,  $\exists Cl \beta_i \subset \mathcal{A}_i$  for  $i = 1, 2, \dots$ .
- (iii) For every countable open cover  $\{\mathcal{A}_i\}_{i=1}^{\infty}$  of  $X$ ,  $\exists$  a  $N_{\mathcal{P}C}$  cover for  $i = 1, 2, \dots$ .

**Proof:** To prove that (i)  $\Rightarrow$  (ii)

It suffices to observe that, by virtue of theorem 3.7,  $\exists$  a nano locally finite  $N_{\mathcal{P}O}$  cover  $\{\mathcal{W}_i\}_{i=1}^{\infty}$  of  $X$ ,  $\exists \mathcal{W}_i \subset \mathcal{A}_i$ ; for  $i = 1, 2, \dots$ ,

The implication (ii)  $\Rightarrow$  (iii) is obvious; we shall show that (iii)  $\Rightarrow$  (i). Consider a  $N_{\mathcal{P}O} - T_1$  space  $X$  which satisfies condition (iii).

Let us observe first that  $X$  is  $N_{\mathcal{P}O}$ -normal; indeed, if for a pair  $\mathcal{A}, \beta \in N_{\mathcal{P}O}(U, X)$  such that  $\mathcal{A} \cup \beta = X$  we let

$\mathcal{A}_1 = \mathcal{A}, \mathcal{A}_2 = \beta, \mathcal{A}_3 = \mathcal{A}_4 = \dots = \emptyset$  then (iii) yields closed subsets  $\mathcal{F}_1, \mathcal{F}_2, \dots \in N_{\mathcal{P}C}(U, X)$  such that  $\mathcal{F}_1 \subset \mathcal{A}, \mathcal{F}_2 \subset \beta$  and  $\mathcal{F}_1 \cup \mathcal{F}_2 = X$ . Now, it follows from De Morgan's laws that, for every sequence  $\mathcal{F}_1, \mathcal{F}_2, \dots \in N_{\mathcal{P}C}(U, X)$ ,  $\exists \bigcap_{i=1}^{\infty} \mathcal{F}_i = \emptyset$ ,  $\exists$ , a sequence  $\mathcal{W}_1, \mathcal{W}_2, \dots \in N_{\mathcal{P}O}(U, X)$ ,  $\exists \mathcal{F}_i \subset \mathcal{W}_i$ ; for  $i = 1, 2, \dots$  and  $\bigcap_{i=1}^{\infty} \mathcal{W}_i = \emptyset$ ; hence,  $X$  is countably  $N_{\mathcal{P}O}$ -Paracompact by Corollary 3.6.

[Besides a nano locally finite and point-finite families of  $N_{\mathcal{P}O}$  sets, one also considers  $N_{\mathcal{P}O}$  star-finite and  $N_{\mathcal{P}O}$  star-countable families: a family  $\{\mathcal{A}_i\}_{i=1}^{\infty}$  of  $N_{\mathcal{P}O}$  subsets of  $X$  is  $N_{\mathcal{P}O}$  star-finite ( $N_{\mathcal{P}O}$  star-countable) if for every  $s_0 \in S$ ,  $\{s \in S : \mathcal{A}_s \cap \mathcal{A}_{s_0} \neq \emptyset\}$  is finite (countable). Clearly, any  $N_{\mathcal{P}O}$  star-finite family is point-finite. Let us note that a  $N_{\mathcal{P}O}$  star-finite family of subsets of a  $NTS$  is not necessarily a nano locally finite; however, a  $N_{\mathcal{P}O}$  star-finite open cover of a  $NTS$  is a nano locally finite].

### Conclusions

In the work, I developed a new class of nano topological spaces using pre-open sets, and we investigated a few countable paracompactness in pre-open sets in nano topological spaces. The standard nano topological concept is different from our idea, although it nevertheless meets certain of its requirements, and proved that every nano pre-open Hausdorff space is countable nano pre-open paracompact space. Every nano pre-open- $T_1$  space is countable nano pre-open paracompact space.

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