Large (k,3)-arcs in PG(2,19) and the related linear codes

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Abstract— A (n,r)-arc $\mathcal{K}$ in a projective plane PG(2,q) is a set of n points such that some r, but no r+1 of them, are collinear. A (n,r)-arc $\mathcal{K}$ is called complete if it is not contain in a (n+1,r)-arc. A linear $[n,k,d]_q$-code $\mathcal{C}$ over a finite field $\mathbb{F}_q$ is a k-dimensional subspace of $\mathbb{F}_q^n$ with minimum hamming distance d and length n. A code with parameters $[g_q(k,d),k,d]_q$ with Griesmer bound $g_q(k,d)$, is called Griesmer code. The major aim of this research is to find large size for the complete (k,3)-arcs in the projective plane of order nineteen PG(2,19) using the method of secants distributions, and the disjoint union of arcs, as well as, adding and removing points to (from) particular conic respectively. Also, we find the Griesmer codes that correspond to each large complete (k,3)-arcs, k=29,30,31. We introduced 20 inequivalent (29,3)-arcs up to secant distribution, 10 of them are complete. Also, we introduced 8 inequivalent (30,3)-arcs up to secant distribution, 2 of them are complete. Moreover, we construct 3 inequivalent complete (31,3)-arcs up to secant distribution, and then find the corresponding linear codes to some of the (29,3)-arcs, (30,3)-arcs and (31,3)-arcs. In particular, we established 3 types of Griesmer codes, and we find the weight enumerator that correspond to each one of them.

Keywords— Finite projective plane; (k,n)-arcs; linear codes; Griesmer code.

I. INTRODUCTION

The aim of coding theory is to develop methods that enable the recipient of message to detect or even correct that occur while transmitting data. Many aspects of coding theory can be directly translated into geometry problems.

A linear $[n,k,d]_q$-code $\mathcal{C}$ over a finite field $\mathbb{F}_q$, called Galois field of order q, is subspace of dimension k of the n-dimensional vector space $V(n,q)=F_q^n$ such that any two distinct vectors in $\mathcal{C}$ differ in at least of d places. The elements of the code are called codewords. Also the parameters n,k and d are called length, dimension, and minimum distance of $\mathcal{C}$ [1]. An important problem in coding theory is that to optimize one of the parameters k,n,d for given value of the other two and fixed q.

The subject of arcs is not only interesting in its purely geometrical setting. An (n,r)-arcs have applications in coding theory, where they can be interpreted as a linear $[n,3,n-r]$ or $[n,3,n-r]_q$-code. It follows that every $[n,3,n-r]_q$-code is equivalent to (n,r)-arc in $PG(2,q)$ containing at least r collinear points. In [2], R. Hill studied the fundamentals of coding theory. In [3], Al-Zangana has been studied the geometry of the plane of order nineteen and its application to error-correcting codes. For more details about the relation between linear codes and arcs see [4], [5]. The aim of this research is to construct large complete (k,n)–arcs in $PG(2,19)$, and to find the corresponding linear codes. Also, we find the weight enumerators for some codes of large size, and determine which of these codes are Griesmer codes.

In this work, we study the structure of large $(k,3)$–arcs in $PG(2,q)$, $k=29,30,31$. We use our algorithm in Section IV, to construct such $(k,3)$–arcs. Also, we determine the complete and incomplete $(k,3)$–arcs, $k=29,30,31$. Moreover, we use large complete of these arcs to find Griesmer codes of big size.

In Section II, we introduce the cyclic projectivity that gives all points and lines of the plane $PG(2,19)$. Also, we give some points and lines of $PG(2,19)$.

In section III, we give some definitions and facts that are important in our study. In Section IV, we introduce our algorithm to find all the results that related to large complete $(k,3)$–arcs, $k=29,30,31$.

In Section V, we give our results on large complete $(k,3)$–arcs, $k=29,30,31$ in $PG(2,19)$ and the new related Griesmer codes. Furthermore, we find the weight enumerator that correspond to each Griesmer code.

II. THE PROJECTIVE PLANE OF ORDER NINETEEN

The projective plane of order nineteen has 381 points and 381 lines, 20 points on each line and 20 lines passing through each point.

Consider the homogeneous polynomial

$$f(X) = X^3 - 2X^2 - 2X - 2$$

in $\mathbb{F}_{19}[X]$, where $\mathbb{F}_{19}$ is the Galois field of order nineteen. The polynomial $f(X)$ is an irreducible over $\mathbb{F}_{19}$. The corresponding companion matrix of $f(X)$ is M_f, where

$$M_f = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 2 & 2 \end{bmatrix}.$$

The points of $PG(2{,}19)$ are generated by M_f as follows:

$$P_i = P_1 {M_f}^{i-1}, i = 1{,}2{,}\ldots{,}381\,, \qquad (1)$$

where $P_1 = (1, 0, 0)$, and $M_f^0 = I$ is the 3×3 identity matrix.

Moreover, M_f can be used to determine all lines of $PG(2{,}19)$ as follows: if ℓ_i is the line with coefficients a, b, c, we write

$$\ell_i = [a, b, c]; \text{ that is the line}$$

$$ax + by + cz = 0.$$

So, we get

$$\ell_i = \ell_1 {M_f}^{i-1}, i = 1{,}2{,}\ldots{,}381\,, \qquad (2)$$

where $\ell_1 = [1, 0, 0]$.

We find all points and lines of PG(2,19) by using the program in Gap [6].

Table I illustrates some points of PG(2,19).

Table I.

***i*: *P*$_i$**	***Coordinates of point***	***Description***
$1{:}P_1$	(1,0,0)	$P_1 = P_1 {M_f}^0 = P_1 I = P_1$
$2{:}P_2$	(0,1,0)	$P_2 = P_1 {M_f}^1$
$3{:}P_3$	(0,0,1)	$P_3 = P_1 {M_f}^2$
$4{:}P_4$	(1,1,1)	$P_4 = P_1 {M_f}^3$
$5{:}P_5$	(1,11,11)	$P_5 = P_1 {M_f}^4$
$\vdots$	$\vdots$	$\vdots$
$99{:}P_{99}$	(1,2,0)	$P_{99} = P_1 {M_f}^{98}$
$100{:}P_{100}$	(0,1,2)	$P_{100} = P_1 {M_f}^{99}$
$\vdots$	$\vdots$	$\vdots$
$221{:}P_{221}$	(1,0,3)	$P_{221} = P_1 {M_f}^{220}$
$222{:}P_{222}$	(1,17,1)	$P_{222} = P_1 {M_f}^{221}$
$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$
$380{:}P_{380}$	(0,1,6)	$P_{380} = P_1 {M_f}^{379}$
$381{:}P_{381}$	(1,1,9)	$P_{381} = P_1 {M_f}^{380}$

Table II illustrates some lines of PG(2,19).

Table II.

$\boldsymbol{\ell_i{:}\,ax + by + cz = 0}$	***Points on the line***	***Description***
$\ell_1{:}x = 0$	2, 3, 9, 18, 26, 74, 100, 132, 183, 195, 224, 235, 237, 268, 273, 303, 323, 337, 362, 380	$\ell_1 = \ell_1 {M_f}^0 = \ell_1 I = \ell_1$
$\ell_2{:}y = 0$	1, 3, 34 , 39, 69, 89, 103, 128, 146, 149, 150, 156, 165, 173, 221, 247, 279, 330, 342, 371	$\ell_2 = \ell_1 {M_f}^1$
$\ell_3{:}z = 0$	1,2, 8, 17, 25, 73, 99, 131, 182, 194, 223, 234, 236, 267, 272, 302, 322, 336, 361, 379	$\ell_3 = \ell_1 {M_f}^2$
$\ell_4{:}x + y + z = 0$	19, 51, 102, 114, 143, 154, 156, 187, 192, 222, 242, 256, 281, 299, 302, 303, 309, 318, 326, 374	$\ell_4 = \ell_1 {M_f}^3$
$\ell_5{:}x + 11y + 11z = 0$	31, 43, 72, 83, 85, 116, 121, 151, 171, 185, 210, 228, 231, 232, 238, 247, 255, 303, 329, 361	$\ell_5 = \ell_1 {M_f}^4$
$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$
$\ell_{380}{:}y + 6z = 0$	1, 10, 18, 66, 92, 124, 175, 187, 216, 227, 229, 260, 265, 295, 315, 329, 354, 372, 375, 376	$\ell_{380} = \ell_1 {M_f}^{379}$
$\ell_{381}{:}x + y + 9z = 0$	6, 11, 41, 61, 75, 100, 118, 121,	$\ell_{381} = \ell_1 {M_f}^{380}$

$\ell_i: ax + by + cz = 0$	*Points on the line*	*Description*
	122, 128, 137, 145, 193, 219, 251, 302, 314, 343, 354, 356	

In column 2 of Table II, we gives the indices of points that lie on the line

$$\ell_i: ax + by + cz = 0$$

in correspond first column.

Henceforth, we denote to $PG(2, q)$ by $\mathbb{P}_q^2$.

III. DEFINITIONS AND SOME BASIC RESULTS

Definition 3.1[7] A (k, n)-arc $\mathcal{K}$ in a projective plane $\mathbb{P}_q^2$ is a set of k points such that some line of the plane meets $\mathcal{K}$ in n points but such that no line meets $\mathcal{K}$ in more than n points, where $n \geq 2$.

Definition 3.2[8] A line ℓ of $\mathbb{P}_q^2$ is an μ-secant of a (k, n)-arc $\mathcal{K}$ if ℓ intersects $\mathcal{K}$ in μ points. Let τ_i be the total number of i-secants to $\mathcal{K}$. The number of i-secants to $\mathcal{K}$ through a point Q of $\mathbb{P}_q^2 \backslash \mathcal{K}$ is denoted by σ_i or $\sigma_i(Q)$. Moreover, a point Q of $\mathbb{P}_q^2 \backslash \mathcal{K}$ is called point of index zero if $\sigma_n(Q) = 0$.

A (k, n)-arc is complete if there is no $(k + 1, n)$-arc containing it. For more information on complete and in complete (k, n)-arcs, one can see [9].

Lemma 3.1[8] For a (k, n)-arc $\mathcal{K}$ in $\mathbb{P}_q^2$, the following equations hold:

$$\sum_{i=0}^{n} \tau_i = q^2 + q + 1; \qquad (3)$$

$$\sum_{i=1}^{n} i\tau_i = k(q + 1); \qquad (4)$$

$$\sum_{i=2}^{n} i(i - 1)\tau_i = k(k - 1); \qquad (5)$$

Definition 3.2[10] The points out of a (k, n)-arc $\mathcal{K}$ in $\mathbb{P}_q^2$ which passes through it $i\text{-}bisecant$ of $\mathcal{K}$ is called a point of index i.

Definition 3.3[11] For a (k, n)-arc $\mathcal{K}$, if the only i are for which $\tau_i \neq 0$ are $m_1 < m_2 < \cdots < m_r < n$, then $\mathcal{K}$ is of type $(m_1, m_2, \ldots, m_r, n)$.

Definition 3.4[11] A (k, n)-arc $\mathcal{K}$ in $\mathbb{P}_q^2$ of type $(m_1, m_2, \ldots, m_r, n)$ such that $\tau_i = t_i$, $i = m_1, m_2, \ldots, m_r, n$ is said to be has secant-distribution

$$({t_{m_1}}^{\tau_{m_1}}, {t_{m_2}}^{\tau_{m_2}}, \ldots, {t_{m_r}}^{\tau_{m_r}}, {t_n}^{\tau_n}).$$

Definition 3.5[12] A linear $[n, k, d]_q$-code (or projective linear $[n, k, d]_q$-code) $\mathcal{C}$ over a finite field $\mathbb{F}_q$ is a subspace of dimension k of the n-dimensional vector space $V(n, q) = \mathbb{F}_q^n$ such that any two distinct vectors in $\mathcal{C}$ differ in at least of d places. The elements of the code are called codewords. Also the parameters n, k and d are called length, dimension, and minimum distance of $\mathcal{C}$.

Definition 3.6[13] For any two codewords the Hamming distance between c_1 and, c_2 is denoted by $d(c_1, c_2)$ and it is defined to be the number of positions in which the corresponding coordinates differ. The minimum distance of $\mathcal{C}$ is

$$d(\mathcal{C}) = min\{d(c_1, c_2): c_1, c_2 \in \mathcal{C}, c_1 \neq c_2\}.$$

The error-correcting capability e is defined by

$$e = \lfloor (d - 1)/2 \rfloor.$$

A central problem in coding theory is that of optimizing one of the parameters n, k and d for given values of the other two and q-fixed. There are two versions introduced in [13], namely

1. Find $d_q(n, k)$, the largest value of d for which there exists an $[n, k, d]_q$-code.

2. Find $n_q(k, d)$, the smallest value of n for which there exists an $[n, k, d]_q$-code.

A code which achieves one of these two values is called d-optimal or n-optimal respectively. The well-known lower bound for $n_q(k, d)$ is the Griesmer bound [14], [15].

$$n_q(k, d) \geq g_q(k, d) = \sum_{j=0}^{k-1} \left\lceil \frac{d}{q^j} \right\rceil$$

($\lceil x \rceil$ denotes the smallest integer $\geq x$). A code with parameters $\left[g_q(k, d), k, d\right]_q$, is called Griesmer code.

Definition 3.7[12] The weight $w(x)$ of $x \in V(n, q)$ is

$$w(x) = d(x, 0);$$

that is, $w(x)$ is the number of non-zero elements in x .

Lemma 3.2[1] Let $\mathcal{C}$ be a code. Then

$$d(c_1, c_2) = w(c_1 - c_2)$$

for all $c_1, c_2 \in \mathcal{C}$.

Definition 3.7[16] Let $\mathcal{C}$ be a linear $[n, k, d]_q$ -code and w_i be the number of codewords of weight i in a code $\mathcal{C}$, the list w_i for $0 \leq i \leq n$ is called the weight distribution of $\mathcal{C}$.

Definition 3.8[12] Let $\mathcal{C}$ be a linear $[n, k, d]_q$–code. The weight enumerator of $\mathcal{C}$ is defined as the following polynomial

$$\mathcal{W}_{\mathcal{C}}(Z)=\sum_{w=0}^{n}A_wZ^w.$$

The homogeneous weight enumerator of $\mathcal{C}$ is defined as

$$\mathcal{W}_{\mathcal{C}}(X,Y)=\sum_{w=0}^{n}A_wX^{n-w}\,Y^w.$$

The weight enumerator and homogeneous weight enumerator can be written in another form, that is

$$\mathcal{W}_{\mathcal{C}}(Z)=\sum_{c\in\mathcal{C}}Z^{w(c)}$$

and

$$\mathcal{W}_{\mathcal{C}}(X,Y)=\sum_{c\in\mathcal{C}}X^{n-w(c)}\,Y^{w(c)}$$

where $w(c)$ is the weight of $c\in\mathcal{C}$.

Definition 3.9[1] A generator matrix of a linear $[n,k,d]_q$ -code $\mathcal{C}$ is $k\times n$ matrix over the finite field $\mathbb{F}_q$ whose rows from a basis of $\mathcal{C}$; it is denoted by G.

Definition 3.10[1] A linear $[n,k,d]_q$-code $\mathcal{C}$ over a finite field $\mathbb{F}_q$ is said to be maximum distance separable (MDS) code if d satisfies the following bound :

$$d(\mathcal{C})=n-k+1.$$

And if $d=n-k$, then the code is called almost maximum distance separable (AMDS).

Theorem 3.1[2] In $\mathbb{P}_q^2$, there exists a projective $[n,k,d]_q$-code if and only if there exists a $(n,n-d)$-arc .

For more information about linear codes, one can see [16].

In the next sections, we give all our results about large arcs of degree three over Galois field $\mathbb{F}_{19}$. Moreover, we introduced all the related codes with their weight enumerators.

IV. THE ALGORITHM USING TO CONSTRUCT LARGE COMPLET (K,3)-ARCS IN PG(2,19)

In this section, we illustrate the algorithm that used to construct large complete large (k,3)-arcs in projective plane $\mathbb{P}_{19}^2$, and determined the inequivalent (k,3)-arcs up to secant-distributions. Two $(k,3)$-arcs are said to be equivalent up secants distributions if they have the same secants distributions, otherwise, they are called inequivalent up secants distributions:

Algorithm: Finding all inequivalent $(k,3)$-arcs up secants distributions, k=29,30,31.

Step(1):

1. Finding the points and lines of $PG(2,19)$ by the formula defining in equation (1). In this case, we get 381 points and 381 lines as shown in Table I, and Table II.

2. Consider some conic in $\mathbb{P}_{19}^2$, say

$$C_1: X_1X_2+X_1X_3-2X_2X_3=0.$$

In fact, C forms a complete (20,2)-arc in $\mathbb{P}_{19}^2$.

3. There are 361 not on the conic C. Adding these points to C give us 361 (21,3)-arcs. Among these arcs, there are inequivalent (21,3)-arcs up secants, say

$$\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3,\cdots,\mathcal{A}_s.$$

4. Adding points of index zero to each (21,3)-arc

$$\mathcal{A}_i, i=1,2,3,,s$$

will give us (22,3)-arcs, and among these arcs we find all inequivalent (22,3)-arc up secants distributions, say

$$\mathcal{B}_1,\mathcal{B}_2,\mathcal{B}_3,\cdots,\mathcal{B}_t.$$

5. Repeat item (4) for the new arcs until we get some inequivalent (29,3)-arcs up secants distributions.

Step(2): To find new inequivalent $(29,3)$-arcs

1. Adding and removing some point to (from) the conic C_1 respectively to find new inequivalent (29,3)-arcs up secants distributions.

2. Construct (29,3)-arcs as disjoint union of $(k,2)$-arcs and $(k',3)$-arcs.

3. Among the (29,3)-arcs which are produced in item (2), chose the inequivalent (29,3)-arcs up secants distributions.

Step(3): To find inequivalent $(30,3)$-arcs:

1. Add some points which lie on tangents and external lines to C_1 to get (30,3)-arcs.

2. Construct (30,3)-arcs as disjoint union of $(k,2)$-arcs and $(k',3)$-arcs.

3. Among the (30,3)-arcs which are produced in item (2) and item (3), chose the inequivalent (30,3)-arcs up secants distributions.

Step(4): To find inequivalent $(31,3)$-arcs:

1. Consider all the incomplete (30,3)-arcs in $\mathbb{P}_{19}^2$ which are produced in Step(3) .

2. Adding one point of index zero to each incomplete (30,3)-arc in item (1), give us the inequivalent (31,3)-arcs up secants distributions.

Section V illustrates the application of our Algorithm.

V. LARGE COMPLET (K,3)-ARCS IN PG(2,19)

Definition 4.1 In projective plane $\mathbb{P}_{19}^2$, two $(k,3)$-arcs are said to be equivalent up secants distributions if they have the same secants distributions.

Theorem 5.1 In projective plane $\mathbb{P}_{19}^2$:
There are 20 inequivalent (29,3)-arcs up secants distributions, 10 of them are complete.

Proof: The points on the irreducible conic, say

$$C_1 : X_1X_2 + X_1X_3 - 2X_2X_3 = 0$$

form (20,2)-arc in $\mathbb{P}_{19}^2$. By adding the 8-set of points $S = \{5,8,30,60,88,101,128,321\}$ to C_1, we get (28,3)-arc $\mathcal{A}(28,3)$ with secants distributions

$$(121^{\tau_0}, 38^{\tau_1}, 144^{\tau_2}, 78^{\tau_3}).$$

Consequently, we have the following complete (29,3)-arcs:

(1) $\mathcal{A}_1(29,3) = \mathcal{A}(28,3) \cup \{230\}$ with secant distributions $(118^{\tau_0}, 35^{\tau_1}, 139^{\tau_2}, 89^{\tau_3})$.
(2) $\mathcal{A}_2(29,3) = \mathcal{A}(28,3) \cup \{284\}$ with secant distributions $(117^{\tau_0}, 38^{\tau_1}, 136^{\tau_2}, 90^{\tau_3})$.

On the other hand, if we added the 8-set of points $S = \{5,39,40,111,139,199,319,368\}$ to C_1, we get (28,3)-arc $\mathcal{B}(28,3)$ with secants distributions

$$(119^{\tau_0}, 44^{\tau_1}, 138^{\tau_2}, 80^{\tau_3}).$$

Moreover, we have the following complete (29,3)-arcs:

(1) $\mathcal{B}_1(29,3) = \mathcal{B}(28,3) \cup \{37\}$ with secants distributions $(115^{\tau_0}, 44^{\tau_1}, 130^{\tau_2}, 92^{\tau_3})$.
(2) $\mathcal{B}_2(29,3) = \mathcal{B}(28,3) \cup \{338\}$ with secants distributions $(114^{\tau_0}, 47^{\tau_1}, 127^{\tau_2}, 93^{\tau_3})$.

Adding the 8-set of points $S = \{5,8,30,60,88,101,128,321\}$ to C_1, we get (28,3)-arc $\mathcal{D}(28,3)$ with secants distributions

$$(116^{\tau_0}, 53^{\tau_1}, 129^{\tau_2}, 83^{\tau_3}).$$

Consequently, we have the following incomplete (29,3)-arcs:

(1) $\mathcal{D}_1(29,3) = \mathcal{D}(28,3) \cup \{243\}$ with secants distributions $(111^{\tau_0}, 56^{\tau_1}, 118^{\tau_2}, 96^{\tau_3})$.
(2) $\mathcal{D}_2(29,3) = \mathcal{D}(28,3) \cup \{251\}$ with secants distributions $(110^{\tau_0}, 59^{\tau_1}, 115^{\tau_2}, 97^{\tau_3})$.

Now, by removing the point 171 from the conic C_1, we get (19,2)-arc $\mathcal{E}$. Adding the 10-set of points $S = \{5,63,111,155,173,179,262,266,286,379\}$ to $\mathcal{E}$ gives the complete $\mathcal{E}(29,3) = \mathcal{E} \cup S$ with secants distributions

$$(109^{\tau_0}, 62^{\tau_1}, 112^{\tau_2}, 98^{\tau_3}).$$

Furthermore, if we add the following 9-set of points to the conic C_1, we get the following 5 complete (29,3)-arcs:

(1) $\mathcal{F}_1(29,3) = C_1 \cup \{5,8,52,88,101,247,284,321,348\}$ with secants distributions $(120^{\tau_0}, 29^{\tau_1}, 145^{\tau_2}, 87^{\tau_3})$.

(2) $\mathcal{F}_2(29,3) = C_1 \cup \{5,111,155,266,274,279,286,314,379\}$ with secants distributions $(116^{\tau_0}, 41^{\tau_1}, 133^{\tau_2}, 91^{\tau_3})$.

(3) $\mathcal{F}_3(29,3) = C_1 \cup \{6,8,15,55,69,75,111,213,222\}$ with secants distributions $(113^{\tau_0}, 50^{\tau_1}, 124^{\tau_2}, 94^{\tau_3})$.

(4) $\mathcal{F}_4(29,3) = C_1 \cup \{151,259,284,287,318,353,377,380,381\}$ with secants distributions $(112^{\tau_0}, 53^{\tau_1}, 121^{\tau_2}, 95^{\tau_3})$.

(5) $\mathcal{F}_5(29,3) = C_1 \cup \{5,8,32,60,168,183,320,338,355\}$ with secant distributions $(108^{\tau_0}, 65^{\tau_1}, 109^{\tau_2}, 99^{\tau_3})$.

The disjoint union of the (11,2)-arc

$$\mathcal{G}_1 = \{22,23,83,86,96,122,138,142,250,301,366\}$$

and the (10,2)-arc

$$\mathcal{G}_2 = \{50,111,170,210,217,244,253,258,377,378\}$$

and the (8,2)-arc

$$\mathcal{G}_3 = \{256,300,310,321,333,350,353,374\}$$

gives us new (29,3)-arc $\mathcal{G}(29,3) = \mathcal{G}_1 \uplus \mathcal{G}_2 \uplus \mathcal{G}_3$ with secants distributions $(107^{\tau_0}, 68^{\tau_1}, 106^{\tau_2}, 100^{\tau_3})$, where $\uplus$ denotes the disjoint union of sets.

The disjoint union the (11,2)-arc

$$\mathcal{J}_1 = \{9,22,23,50,83,96,111,142,217,253,321\}$$

and the (10,2)-arc

$$\mathcal{J}_2 = \{122,138,170,201,210,244,250,258,350,378\}$$

and the (8,3)-arc

$$\mathcal{J}_3 = \{256,300,301,310,333,337,353,377\}$$

gives us another new (29,3)-arc $\mathcal{J}(29,3) = \mathcal{I}_1 \uplus \mathcal{I}_2 \uplus \mathcal{I}_3$ with secants distributions $(105^{\tau_0}, 74^{\tau_1}, 100^{\tau_2}, 102^{\tau_3})$.

The disjoint union the (11,2)-arc

$$\mathcal{L}_1 = \{22,23,83,86,96,122,138,142,250,301,366\}$$

and the (9,2)-arc

$$\mathcal{L}_2 = \{111,170\ ,201, 210,217,244,253,358,310\}$$

and the (9,3)-arc

$$\mathcal{L}_3 = \{256,300,303,321,333,350,353,377,378\}$$

gives the new (29,3)-arc $\mathcal{L}(29,3) = \mathcal{L}_1 \uplus \mathcal{L}_2 \uplus \mathcal{L}_3$ with secants distributions $(104^{\tau_0}, 77^{\tau_1}, 97^{\tau_2}, 103^{\tau_3})$.

The disjoint union the (11,2)-arc

$$\mathcal{M}_1 = \{9,22,23,83,86,138,142,217,258,301,377\}$$

and the (9,2)-arc

$$\mathcal{M}_2 = \{96,111,122\ ,170,201, 244,250,350,378\}$$

and the (9,3)-arc

$$\mathcal{M}_3 = \{210,253,256,300,303,310,321,333,353\}$$

gives us the (29,3)-arc $\mathcal{M}(29,3) = \mathcal{M}_1 \uplus \mathcal{M}_2 \uplus \mathcal{M}_3$ with secants distributions $(103^{\tau_0}, 80^{\tau_1}, 94^{\tau_2}, 104^{\tau_3})$.

The disjoint union the (11,2)-arc

$$\mathcal{N}_1 = \{9,22,23,50,86,138,142,170,217,256,353\}$$

and the (9,2)-arc

$$\mathcal{N}_2 = \{96,111,122\ ,201, 244,250,258,321,378\}$$

and the (9,3)-arc

$$\mathcal{N}_3 = \{210,253,300,301,303,310,333,350,377\}$$

gives us the (29,3)-arc $\mathcal{N}(29,3) = \mathcal{N}_1 \uplus \mathcal{N}_2 \uplus \mathcal{N}_3$ with secants distributions $(102^{\tau_0}, 83^{\tau_1}, 91^{\tau_2}, 105^{\tau_3})$.

The disjoint union the (11,2)-arc

$$\mathcal{O}_1 = \{9,23,50,86,111,122,138,142,170,217,337\}$$

and the (8,2)-arc

$$\mathcal{O}_2 = \{96\ ,201, 210,244,250,253,258,310\}$$

and the (10,3)-arc

$$\mathcal{O}_3 = \{256,300,301,303,321,333,350,353,377,378\}$$

gives us the (29,3)-arc $\mathcal{O}(29,3) = \mathcal{O}_1 \uplus \mathcal{O}_2 \uplus \mathcal{O}_3$ with secants distributions $(101^{\tau_0}, 86^{\tau_1}, 88^{\tau_2}, 106^{\tau_3})$.

The disjoint union the (12,2)-arc

$$\mathcal{P}_1 = \{9,50,86,111,122,138,142,170,217,253,337,366\}$$

and the (8,2)-arc

$$\mathcal{P}_2 = \{96\ ,201, 210,244,250,256,258,310\}$$

and the (9,3)-arc

$$\mathcal{P}_3 = \{300,301,303,321,333,350,353,377,378\}$$

gives us the (29,3)-arc $\mathcal{P}(29,3) = \mathcal{P}_1 \uplus \mathcal{P}_2 \uplus \mathcal{P}_3$ with secants distributions $(100^{\tau_0}, 89^{\tau_1}, 85^{\tau_2}, 107^{\tau_3})$.

Finally, the disjoint union of the (9,2)-arc

$$\mathcal{Q}_1 = \{9,23,50,83,86,111,122,142,201\}$$

and the (10,2)-arc

$$\mathcal{Q}_2 = \{22,96\ ,138, 170,217,250,256,258,310,374\}$$

and the (10,3)-arc

$$\mathcal{Q}_3 = \{253,300,303,321,333,337,350,353,377,378\}$$

gives us the (29,3)-arc $\mathcal{Q}(29,3) = \mathcal{Q}_1 \uplus \mathcal{Q}_2 \uplus \mathcal{Q}_3$ with secants distributions $(99^{\tau_0}, 92^{\tau_1}, 82^{\tau_2}, 108^{\tau_3})$.

To summarize these results, we have the following table:

Table III.

Name of the arc	***Secant distributions***	***Completeness***
$\mathcal{A}_1(29,3)$	$(118^{\tau_0}, 35^{\tau_1}, 139^{\tau_2}, 89^{\tau_3})$	Complete
$\mathcal{A}_2(29,3)$	$(117^{\tau_0}, 38^{\tau_1}, 136^{\tau_2}, 90^{\tau_3})$	Complete
$\mathcal{B}_1(29,3)$	$(115^{\tau_0}, 44^{\tau_1}, 130^{\tau_2}, 92^{\tau_3})$	Complete
$\mathcal{B}_2(29,3)$	$(114^{\tau_0}, 47^{\tau_1}, 127^{\tau_2}, 93^{\tau_3})$	Complete
$\mathcal{D}_1(29,3)$	$(111^{\tau_0}, 56^{\tau_1}, 118^{\tau_2}, 96^{\tau_3})$	Incomplete
$\mathcal{D}_2(29,3)$	$(110^{\tau_0}, 59^{\tau_1}, 115^{\tau_2}, 97^{\tau_3})$	Incomplete
$\mathcal{E}(29,3)$	$(109^{\tau_0}, 62^{\tau_1}, 112^{\tau_2}, 98^{\tau_3})$	Complete
$\mathcal{F}_1(29,3)$	$(120^{\tau_0}, 29^{\tau_1}, 145^{\tau_2}, 87^{\tau_3})$	Complete
$\mathcal{F}_2(29,3)$	$(116^{\tau_0}, 41^{\tau_1}, 133^{\tau_2}, 91^{\tau_3})$	Complete
$\mathcal{F}_3(29,3)$	$(113^{\tau_0}, 50^{\tau_1}, 124^{\tau_2}, 94^{\tau_3})$	Complete
$\mathcal{F}_4(29,3)$	$(112^{\tau_0}, 53^{\tau_1}, 121^{\tau_2}, 95^{\tau_3})$	Complete
$\mathcal{F}_5(29,3)$	$(108^{\tau_0}, 65^{\tau_1}, 109^{\tau_2}, 99^{\tau_3})$	Complete
$\mathcal{G}(29,3)$	$(107^{\tau_0}, 68^{\tau_1}, 106^{\tau_2}, 100^{\tau_3})$	Incomplete

Name of the arc	*Secant distributions*	*Completeness*
$\mathcal{J}(29,3)$	$(105^{\tau_0},74^{\tau_1},100^{\tau_2},102^{\tau_3})$	Incomplete
$\mathcal{L}(29,3)$	$(104^{\tau_0},77^{\tau_1},97^{\tau_2},103^{\tau_3})$	Incomplete
$\mathcal{M}(29,3)$	$(103^{\tau_0},80^{\tau_1},94^{\tau_2},104^{\tau_3})$	Incomplete
$\mathcal{N}(29,3)$	$(102^{\tau_0},83^{\tau_1},91^{\tau_2},105^{\tau_3})$	Incomplete
$\mathcal{O}(29,3)$	$(101^{\tau_0},86^{\tau_1},88^{\tau_2},106^{\tau_3})$	Incomplete
$\mathcal{P}(29,3)$	$(100^{\tau_0},89^{\tau_1},85^{\tau_2},107^{\tau_3})$	Incomplete
$\mathcal{Q}(29,3)$	$(99^{\tau_0},92^{\tau_1},82^{\tau_2},108^{\tau_3})$	Incomplete

Theorem 5.2: In projective plane $\mathbb{P}_{19}^2$:

(A) There are 8 inequivalent (30,3)-arcs up secants distributions, two of them are complete.

(B) There are 3 inequivalent complete (31,3)-arcs up secants distributions.

Proof: (A) Let us consider the conic $C_1: X_1X_2 + X_1X_3 - 2X_2X_3 = 0$. The intersection of the two tangents to C_1, say

$$\ell_{120}: X_1 + 6X_2 + 7X_3 = 0 \text{ and } \ell_{129}: X_1 + 6X_2 + X_3 = 0$$

gives the point 17.

The intersection of the two tangents to C_1, say

$$\ell_{151}: X_1 - 8X_2 + X_3 = 0 \text{ and } \ell_{238}: X_1 + X_2 - 8X_3 = 0$$

gives the point 5.

Also, the intersection of the two tangents to C_1, say

$$\ell_{102}: X_1 + 9X_2 + 9X_3 = 0 \text{ and } \ell_{144}: X_1 + 7X_2 + 4X_3 = 0$$

gives the point 284.

The union $C_1 \cup \{5,17,284\}$ implies a (23,3)-arc $\mathcal{A}$ with secants distribution $(147^{\tau_0},35^{\tau_1},172^{\tau_2},27^{\tau_3})$.

Let us consider the three external lines to C_1, say

$$\ell_{145}: X_1 - 6X_2 + 9X_3 = 0, \quad \ell_{147}: X_1 - 3X_2 + X_3 = 0$$

and

$$\ell_{350}: X_1 - 3X_2 - 7X_3 = 0.$$

If we choose the points 189, 318, 381 on ℓ_{145}, and the points 307, 358 on ℓ_{147}, and the points 9, 254 on ℓ_{350}, then we get

$$\mathcal{A} \cup \{9,189,254,307,318,358,381\}$$

which forms a complete (30,3)-arc $\mathcal{K}_1$ with secants distributions

$$(110^{\tau_0},48^{\tau_1},117^{\tau_2},106^{\tau_3}).$$

On the other hand, consider the 4 tangent lines to C_1, say

$$\ell_{54}: X_1 + 7X_2 + 4X_3 = 0,\ \ell_{233}: X_1 + 7X_2 + 6X_3 = 0,$$
$$\ell_{311}: X_1 + 5X_2 - 3X_3 = 0,\ \ell_{38}: X_1 + 5X_2 + 9X_3 = 0.$$

Take the pairs of points {15,251}, {44,126}, {6,184} and {222,243} on the lines ℓ_{54}, ℓ_{233}, ℓ_{311} and ℓ_{38} respectively.

Then $C_1 \cup \{6,15,44,126,184,222,243,251\}$ give us a (28,3)-arc $\mathcal{B}$ with secants distribution

$$(117^{\tau_0},50^{\tau_1},132^{\tau_2},82^{\tau_3}).$$

Now, chose the points 8,121 which lie on the external line to C_1, say

$$\ell_{94}: X_1 - 4X_2 + 8X_3 = 0.$$

It follows that $\mathcal{B} \cup \{8,121\}$ forms a complete (30,3)-arc $\mathcal{K}_2$ with secants distributions

$$(106^{\tau_0},60^{\tau_1},105^{\tau_2},110^{\tau_3}).$$

Now, the sets of points

$\mathcal{D}_1 = \{22,23,83,86,96,122,138,142,250,301,366\}$,
$\mathcal{D}_2 = \{50,111,170,201,210,217,244,253,258,378\}$,
$\mathcal{D}_3 = \{256,300,310,321,333,350,353,374,377\}$

form disjoint 11-arc, 10-arc and (9,3)-arc with secants distributions

$$(216^{\tau_0},110^{\tau_1},55^{\tau_2}),\ (226^{\tau_0},110^{\tau_1},45^{\tau_2})$$

and

$(236^{\tau_0},111^{\tau_1},33^{\tau_2},1^{\tau_3})$ respectively.

The disjoint union $\mathcal{K}_3 =: \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3$ forms a (30,3)-arc with secants distributions $(104^{\tau_0},66^{\tau_1},99^{\tau_2},112^{\tau_3})$.

On the other hand, the sets of points, say

$\mathcal{E}_1 = \{22,23,50,83,96,111,122,142,250,253,256\}$

and

$\mathcal{E}_2 = \{138,170,201,210,217,244,258,300,303,353,378\}$

represent 11-arcs with same secants distributions

$$(216^{\tau_0},110^{\tau_1},55^{\tau_2}).$$

Moreover, $\mathcal{E}_3 = \{301,310,321,333,337,350,366,377\}$ forms (8,3)-arc with secants distributions

$$(247^{\tau_0}, 110^{\tau_1}, 22^{\tau_2}, 2^{\tau_3}).$$

The disjoint union $\mathcal{K}_4 =: \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3$ forms another (30,3)-arc with secants distributions $(102^{\tau_0}, 72^{\tau_1}, 93^{\tau_2}, 114^{\tau_3})$.

Both of the sets

$$\mathcal{F}_1 = \{22,23,50,83,86,96,122,142,210,250\}$$

and

$$\mathcal{F}_2 = \{111,138,170,201,217,253,258,301,310,350\}$$

represent 10-arcs with same secants distributions

$$(226^{\tau_0}, 110^{\tau_1}, 45^{\tau_2}).$$

However, the set of points

$$\mathcal{F}_3 = \{244,256,300,303,321,333,353,366,377,378\}$$

forms (10,3)-arc with secants distributions

$$(223^{\tau_0}, 119^{\tau_1}, 36^{\tau_2}, 3^{\tau_3}).$$

Consequently, the disjoint union $\mathcal{K}_5 =: \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$ forms a (30,3)-arc with secants distributions

$$(101^{\tau_0}, 75^{\tau_1}, 90^{\tau_2}, 115^{\tau_3}).$$

The set of points

$\mathcal{M}_1 = \{9,22,23,83,86,138,142,217,258,301,377\}$ forms 11-arc with secants distributions

$$(216^{\tau_0}, 110^{\tau_1}, 55^{\tau_2}).$$

Also, the set of points

$\mathcal{M}_2 = \{96,111,122,170,201,244, 250, 337\}$ forms 8-arc with secants distributions

$$(249^{\tau_0}, 104^{\tau_1}, 28^{\tau_2}).$$

Moreover,

$$\mathcal{M}_3 = \{210,253,256,300,303,310,321,333,350,353,378\}$$

forms (11,3)-arc with secants distributions $(209^{\tau_0}, 131^{\tau_1}, 34^{\tau_2}, 7^{\tau_3})$.

The disjoint union $\mathcal{K}_6 =: \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3$ forms a new (30,3)-arc with secants distributions

$$(100^{\tau_0}, 78^{\tau_1}, 87^{\tau_2}, 116^{\tau_3}).$$

The set of points

$\mathcal{N}_1 = \{9,22,23,50, 86,138,142,170,217,256,337\}$ forms 11-arc with secants distributions

$$(216^{\tau_0}, 110^{\tau_1}, 55^{\tau_2}),$$

and the set of points $\mathcal{N}_2 = \{96,111,122,201,244,250,321,333,350\}$ forms 9-arc with secants distributions

$$(237^{\tau_0}, 108^{\tau_1}, 36^{\tau_2}).$$

Moreover,

$\mathcal{N}_3 = \{210,253,258,300,301,303,310,353,377,378\}$ forms (10,3)-arc with secants distributions

$$(221^{\tau_0}, 125^{\tau_1}, 30^{\tau_2}, 5^{\tau_3}).$$

The disjoint union $\mathcal{K}_7 =: \mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3$ forms a (30,3)-arc with secants distributions $(99^{\tau_0}, 81^{\tau_1}, 84^{\tau_2}, 117^{\tau_3})$.

The set of points $\mathcal{R}_1 = \{9,23,50, 86,111,122,138,142,170, 217,337,366\}$ forms 12-arc with secants distributions

$$(207^{\tau_0}, 108^{\tau_1}, 66^{\tau_2}).$$

Also, the set of points

$\mathcal{R}_2 = \{96,201,210,244,250,253,258,310\}$ forms 8-arc with secants distributions

$$(249^{\tau_0}, 104^{\tau_1}, 28^{\tau_2}).$$

Moreover,

$\mathcal{R}_3 = \{256,300,301,303,321,333,350,353,377,378\}$ forms (10,3)-arc with secants distributions

$$(222^{\tau_0}, 122^{\tau_1}, 33^{\tau_2}, 4^{\tau_3}).$$

The disjoint union $\mathcal{K}_8 =: \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$ forms a new (30,3)-arc with secants distributions

$$(98^{\tau_0}, 84^{\tau_1}, 81^{\tau_2}, 118^{\tau_3}).$$

To summarize these results, we have the following table:

Table IV.

Name	***Secant distributions***	***Completeness***
$\mathcal{K}_1$	$(110^{\tau_0}, 48^{\tau_1}, 117^{\tau_2}, 106^{\tau_3})$	Complete
$\mathcal{K}_2$	$(106^{\tau_0}, 60^{\tau_1}, 105^{\tau_2}, 110^{\tau_3})$	Complete
$\mathcal{K}_3$	$(104^{\tau_0}, 66^{\tau_1}, 99^{\tau_2}, 112^{\tau_3})$	Incomplete
$\mathcal{K}_4$	$(102^{\tau_0}, 72^{\tau_1}, 93^{\tau_2}, 114^{\tau_3})$	Incomplete

Name	***Secant distributions***	***Completeness***
$\mathcal{K}_5$	$(101^{\tau_0}, 75^{\tau_1}, 90^{\tau_2}, 115^{\tau_3})$	Incomplete
$\mathcal{K}_6$	$(100^{\tau_0}, 78^{\tau_1}, 87^{\tau_2}, 116^{\tau_3})$	Incomplete
$\mathcal{K}_7$	$(99^{\tau_0}, 81^{\tau_1}, 84^{\tau_2}, 117^{\tau_3})$	Incomplete
$\mathcal{K}_8$	$(98^{\tau_0}, 84^{\tau_1}, 81^{\tau_2}, 118^{\tau_3})$	Incomplete

(B) All the arcs $\mathcal{K}_3$, $\mathcal{K}_4$, $\mathcal{K}_5$, $\mathcal{K}_6$, $\mathcal{K}_7$ and $\mathcal{K}_8$ in part (A) are incomplete. So, by adding one point of index zero to $\mathcal{K}_3$, $\mathcal{K}_4$, $\mathcal{K}_7$, we get three inequivalent complete $(31,3)$-arcs as shown in Table IV.

Table V.

Name	***Description***	***Completeness***
$\mathcal{S}_1$	$\mathcal{K}_3 \cup \{303\}$	Complete
$\mathcal{S}_2$	$\mathcal{K}_4 \cup \{86\}$	Complete
$\mathcal{S}_3$	$\mathcal{K}_7 \cup \{366\}$	Complete

In Table V, by adding the point of index zero to $\mathcal{K}_3$, say the point 303 which has the coordinates (0,1,18), we get a complete $(31,3)$-arcs $\mathcal{S}_1$ with secants distributions

$$(99^{\tau_0}, 71^{\tau_1}, 84^{\tau_2}, 127^{\tau_3}).$$

The point of $\mathcal{S}_1$ are:

(1,14,10), (1,2,15), (1,16,13), (1,2,10), (1,18,5), (1,8,12), (1,13,16), (1,8,18), (1,18,8), (1,5,18), (1,13,14), (1,12,16), (1,10,2), (1,8,17), (1,3,13), (1,12,8), (1,17,8), (1,3,15), (1,17,15), (1,12,18), (1,10,14), (0,1,18), (1,18,12), (1,13,3), (1,10,5), (1,16,12), (1,5,10), (1,14,13), (1,15,3), (1,15,17), (1,15,2).

On the other hand, if we add the point of index zero 86, say the point with coordinate (1,18,5), we get complete $(31,3)$-arcs $\mathcal{S}_2$ with secants distributions

$$(98^{\tau_0}, 74^{\tau_1}, 81^{\tau_2}, 128^{\tau_3}).$$

The point of $\mathcal{S}_2$ are:

(1,14,10), (1,2,15), (1,16 13), (1,2,10), (1,18,5), (1,8,12), (1,13,16), (1,8,18), (1,18,8), (1,5,18), (1,13,14), (1,12,16), (1,10,2), (1,8,17), (1,3,13), (1,12,8), (1,17,8), (1,3,15), (1,17,15), (1,12,18), (1,10,14), (0,1,18), (1,18,12), (1,13,3), (1,10,5), (0,1,4), (1,16,12), (1,5,10), (1,14,13), (1,15,17), (1,15,2).

Finally, if we add the point of index zero 366, say the point with coordinate (1,14,13), we get complete $(31,3)$-arcs $\mathcal{S}_3$ with secants distributions

$$(97^{\tau_0}, 77^{\tau_1}, 78^{\tau_2}, 129^{\tau_3}).$$

The point of $\mathcal{S}_3$ are:

(0,1,5), (1,14,10), (1,2,15), (1,16,13), (1,18,5), (1,8,12), (1,13,16), (1,8,18), (1,18,8), (1,5,18), (1,13,14), (1,12,16), (1,10,2), (1,8,17), (1,3,13), (1,12,8), (1,17,8), (1,3,15), (1,17,15), (1,12,18), (1,10,14), (0,1,18), (1,18,12), (1,13,3), (1,10,5), (0,1,4), (1,16,12), (1,5,10), (1,14,13), (1,15,17), (1,15,2).

Adding points of index zero to other incomplete (30,3)-arcs, $\mathcal{K}_5$, $\mathcal{K}_6$ and $\mathcal{K}_8$ dose not give any new type. So, the only inequivalent complete $(31,3)$-arcs are $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ as shown in Table III.

Theorem 5.3: Over the Galois field of order 19, there exists:

(I) Griesmer $[29,3,26]_{19}$-code $\mathcal{C}_1$ with weight enumerator

$$\mathcal{W}_{\mathcal{C}_1}(Z) = 1 + 1602\,Z^{26} + 2502\,Z^{27} + 630\,Z^{28} + 2124\,Z^{29}.$$

(II) Griesmer $[30,3,27]_{19}$-code $\mathcal{C}_2$ with weight enumerator

$$\mathcal{W}_{\mathcal{C}_2}(Z) = 1 + 1908\,Z^{27} + 2106\,Z^{28} + 864\,Z^{29} + 1980\,Z^{30}.$$

(III) Griesmer $[31,3,28]_{19}$-code $\mathcal{C}_3$ with weight enumerator

$$\mathcal{W}_{\mathcal{C}_3}(Z) = 1 + 2286\,Z^{28} + 1512\,Z^{29} + 1278\,Z^{30} + 1782\,Z^{31}.$$

Proof: According to the theorem (3.1) and (5.1) an $(n, n\text{ - }d)$-arc in $PG(k-1,q)$ is equivalent to a projective $[n,k,d]_q$- $code$. Now if $q = 19$, $k = 3$ and $n - d = r$; r is degree of arc, then there is an one to one correspondence between (n,r)- arc in $PG(k-1,q)$ and a $[n,k,n-r]_q$-$code$.

(I) The $(29,3)$-arc $\mathcal{A}_1(29,3)$ in the proof of Theorem 4.1 give rise to the projective linear $[30,3,26]_{19}$-code define by the generator matrix $\mathcal{G}_1$ of size 3×29 , where

$$\mathcal{G}_1^T=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\1&1&1\\1&11&11\\1&5&0\\1&15&11\\1&9&5\\1&5&9\\1&13&18\\1&3&16\\1&7&2\\1&10&9\\1&1&6\\1&6&4\\1&0&2\\1&18&13\\1&8&17\\1&16&17\\1&17&8\\1&16&14\\1&4&6\\1&3&12\\1&13&3\\1&14&16\\1&11&15\\1&2&7\\1&12&3\\1&14&15\end{bmatrix}$$

with $e=\lfloor (d-1)/2\rfloor = 12$. Since

$$n_{19}(3{,}26)\geq g_{19}(3{,}26)=\sum_{j=0}^{2}\left\lceil\frac{26}{19^j}\right\rceil = 29,$$

it follows that $\mathcal{C}_1$ is Griesmer $[29{,}3,26]_{19}$-code.

The number of codewords that have weight $26,\ 27, 28, 29$ in a code $\mathcal{C}_1$ are 1602, 2502, 630 and 2124 respectively. Consequently, the weight enumerator of $\mathcal{C}_1$ is

$$\mathcal{W}_{\mathcal{C}_1}(Z)=1+1602\,Z^{26}+2502\,Z^{27}+630\,Z^{28}+2124\,Z^{29}.$$

(II) The (30,3)-arc $\mathcal{K}_1$ in the proof of Theorem 5.2 give rise to the projective linear $[30{,}3,27]_{19}$-code define by the generator matrix $\mathcal{G}_2$ of size 3×30 , where

$$\mathcal{G}_2^T=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\1&1&1\\1&11&11\\0&1&5\\1&15&11\\1&9&5\\1&3&0\\1&5&9\\1&13&18\\1&7&2\\1&6&4\\1&18&13\\1&15&12\\1&8&17\\1&17&8\\1&7&8\\1&16&14\\1&16&5\\1&4&6\\1&3&12\\1&14&3\\1&2&16\\1&14&16\\1&11&15\\1&2&7\\1&12&3\\1&1&9\\1&18&15\end{bmatrix}$$

with $e=\lfloor (d-1)/2\rfloor = 13$. Since

$$n_{19}(3{,}27)\geq g_{19}(3{,}27)=\sum_{j=0}^{2}\left\lceil\frac{27}{19^j}\right\rceil = 30,$$

it follows that $\mathcal{C}_2$ is Griesmer $[30{,}3,27]_{19}$-code.

The number of codewords that have weight $27,\ 28, 29, 30$ in a code $\mathcal{C}_2$ are 1908, 2106, 864 and 1980 respectively. Consequently, the weight enumerator of $\mathcal{C}_2$ is

$$\mathcal{W}_{\mathcal{C}_2}(Z)=1+1908\,Z^{27}+2106\,Z^{28}+864\,Z^{29}+1980\,Z^{30}.$$

(III) The (31,3)-arc $\mathcal{S}_1$ in the proof of Theorem 5.2 give rise to the projective linear $[31{,}3,28]_{19}$-code define by the generator matrix $\mathcal{G}_3$ of size 3×31 , where

$$\mathcal{G}_3^T = \begin{bmatrix} 0 & 1 & 18 \\ 1 & 2 & 15 \\ 1 & 2 & 10 \\ 1 & 3 & 15 \\ 1 & 3 & 13 \\ 1 & 5 & 18 \\ 1 & 5 & 10 \\ 1 & 8 & 18 \\ 1 & 8 & 17 \\ 1 & 8 & 12 \\ 1 & 13 & 18 \\ 1 & 10 & 14 \\ 1 & 10 & 5 \\ 1 & 10 & 2 \\ 1 & 12 & 18 \\ 1 & 12 & 16 \\ 1 & 12 & 8 \\ 1 & 13 & 16 \\ 1 & 13 & 14 \\ 1 & 13 & 3 \\ 1 & 14 & 13 \\ 1 & 14 & 10 \\ 1 & 15 & 17 \\ 1 & 15 & 3 \\ 1 & 15 & 2 \\ 1 & 16 & 12 \\ 1 & 17 & 15 \\ 1 & 17 & 8 \\ 1 & 18 & 12 \\ 1 & 18 & 8 \\ 1 & 18 & 5 \end{bmatrix}$$

with $e = \lfloor (d-1)/2 \rfloor = 13$. Since

$$n_{19}(3{,}28) \geq g_{19}(3{,}28) = \sum_{j=0}^{2} \left\lceil \frac{28}{19^j} \right\rceil = 31,$$

it follows that $\mathcal{C}_3$ is Griesmer $[31{,}3, 28]_{19}$-code.

The number of codewords that have $28, 29, 30, 31$ in a code $\mathcal{C}_3$ are 2286, 1512, 1278 and 1782 respectively. So, weight enumerator of $\mathcal{C}_3$ is

$$\mathcal{W}_{\mathcal{C}_3}(Z) = 1 + 2286\, Z^{28} + 1512\, Z^{29} + 1278\, Z^{30} + 1782\, Z^{31}.$$

All the above results are summarized in the following table:

Table VI.

The code $\mathcal{C}_m$	***Weight enumerator of the code $\mathcal{W}_{\mathcal{C}_m}(Z)$***	***Description***
$\mathcal{C}_1$	$1 + 1602\, Z^{26} + 2502\, Z^{27} + 630\, Z^{28} + 2124\, Z^{29}$	Griesmer $[29{,}3, 26]_{19}$-code
$\mathcal{C}_2$	$1 + 1908\, Z^{27} + 2106\, Z^{28} + 864\, Z^{29} + 1980\, Z^{30}$	Griesmer $[30{,}3, 27]_{19}$-code
$\mathcal{C}_3$	$1 + 2286\, Z^{28} + 1512\, Z^{29} + 1278\, Z^{30} + 1782\, Z^{31}$	Griesmer $[31{,}3, 28]_{19}$-code

VI. CONCLUSION

The principle results of this paper are the following.

1. Construct all inequivalent (29,3)-arcs up secants distributions as subsets of the projective plane *PG*(2,19) using our algorithm describing in Section IV. Then we find which of them are complete. In fact, we find 20 inequivalent (29,3)-arcs up secants distributions, 10 of them are complete as shown in Table III.
2. Construct all inequivalent (30,3)-arcs up secants distributions as subsets of the projective plane *PG*(2,19) using our algorithm describing in Section IV. In fact, we find 8 inequivalent (30,3)-arcs up secants distributions, 2 of them are complete as shown in Table IV.
3. Construct all complete and inequivalent (31,3)-arcs up secants distributions as subsets of the projective plane *PG*(2,19) using our algorithm describing in Section IV. In fact, we find 3 inequivalent (31,3)-arcs up secants distributions, all of them are complete as shown in Table V
4. We give error-correcting projective $[n, k, d]_q$-*codes* that corresponding to each inequivalent (29,3)-arcs, (30,3)-arcs and (31,3)-arcs, and determine which of them are Griesmer code. In fact, we construct three different Griesmer codes of size 29, 30 and 31. Moreover, we find the weight enumerator that correspond to each one of them as shown Table VI.

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