Approximate Solution of Linear and Nonlinear Partial Differential Equations Using Picard's Iterative Method

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Abstract— *This work discusses a new description of Picard's iterative method* (PIM) *to get the approximate or sometimes exact solution of (linear and nonlinear) partial differential equations ,Provided that by examples to illustrate the reliability and ease of this method.*

Keywords— *Picard's Iterative Method (PIM), linear partial differential equations (LPDEs), nonlinear Partial Differential Equations(NLPDEs).*

I. Introduction

Big interest has emerged in recent decades in partial differential equations (linear or nonlinear) because of their important role in various fields of sciences and engineering. For example, in physics, partial differential equations are used to describe many phenomena such as wave motion, pendulum motion, heat flow, chaotic systems. In chemistry, they are used to model chemical reactions, in economics, to find optimal investment strategies, in medicine, to model the growth and spread of some diseases, and in the field of scientific computing that has appeared recently, it requires simulating a physical event by solving the differential equation that describes its physical behavior numerically, such as weather forecasting, with this great expansion in the applications of partial differential equations which are the topic of our survey. Mathematicians faced some problems in solving some of these equations which necessitated the development of new methods to find an exact or approximate solution to them, like Adomian decomposition method [1], homotopy perturbation method [2], Variational iteration method [3] , semi- analytical iterative method [4], Runge-Kutta method [5] Picard's Iterative method [6] which is under study, etc.

In this paper we presented a new simplified iterative formula for solving partial differential equations based on (PIM) for solving ordinary differential equations.

II. Description of the Method (PIM)

In this section, we give outline and implement (PIM) for (LPDEs) and (NLPDEs) to obtain in a rapidly convergent series. The Picard's approximation series converge quickly.

Consider

$$f(t,x,u,D_t u,\dots,D_t^m u,D_x u,\dots,D_x^m u,\dots)=0\ , \tag{1}$$

where $D_t^m=\frac{\partial^m}{\partial t^m}$ and $D_x^m=\frac{\partial^m}{\partial x^m}$ with

$u(x,0)=g_1(x),D_t u(x,0)=g_2(x),\dots,\ D_t^{m-1}u(x,0)=g_m(x)$

Firstly, put equation (1) in the form:

$$D_t^m u=h(t,x,u,D_t u,\dots,D_t^{m-1}u,D_x u,\dots,D_x^m u,\dots) \tag{2}$$

By integrating eq. (2) over the interval (0,t) *m* times, we get:

$$u_{n+1}(x,t)=\sum_{j=0}^{m-1}D_t^j u_0(x,t)\frac{t^j}{j!}+L^{-1}h(t,x,u_n,D_t u_n,\dots,D_t^{m-1}u_n,D_x u_n,\dots,D_x^m u_n,\dots)\ , \tag{3}$$

where L^{-1} is the integral operator of m times from 0 to *t* and $D_t^j u_0(x,t)=D_t^j u(x,0)=g_j(x)$.

III. Applications of PIM:

LPDEs and NLPDEs are solved here using the (PIM)

Example 1: the non-homogeneous linear diffusion equation [7]

$$u_t=u_{xx}-u\ ,0<x<\pi\ ,t>0, \tag{4}$$

with

$u(x,0)=\sin x$

And

$u(0,t)=0,\ \ u(\pi,t)=0$

Using eq.(3) to write the iterations of eq.(4) as follows:

$u_{n+1}(x,t) = u_0(x,t) + L_t^{-1}(\,D_x^2\,u_n - u_n)$

$u_1(x,t) = u_0(x,t) + L_t^{-1}(\,D_x^2\,u_0 - u_0)$

$= sinx + \ L_t^{-1}(\,-2sinx)$

$= sinx\,(1-2t)$

$u_2(x,t) = u_0(x,t) + L_t^{-1}(\,D_x^2\,u_1 - u_1)$

$= sinx + L_t^{-1}[-2sinx(1-2t)]$

$= sinx(1-2t+2t^2)$

$u_3(x,t) = u_0(x,t) + L_t^{-1}(\,D_x^2\,u_2 - u_2)$

$= sinx + L_t^{-1}[-2sinx(1-2t+2t^2)]$

$= sinx(1-2t+2t^2-\frac{4t^3}{3})$,

and so on .the solution is

$u_n(x,t) = sinx(1-2t+\frac{(2t)^2}{2!}-\frac{(2t)^3}{3!}+\cdots)$

And the closed of it is

$u(x,t) = e^{-2t}sinx$

Example 2:

The Newell- Whitehead- Segel equation (NWSE) [8] is one of the diffusion equations and is written as:

$u_t = ku_{xx} + au - bu^q$,

where a, b and $k \in \mathbb{R}$, $k > 0$, and q $\in \mathbb{Z}^+$.

We illustrate the following (NWSE) for (a=b=k=1, q=2)

$u_t = u_{xx} + u - u^2$, (5)

with

$u(x,0) = \frac{1}{(1+e^{x/\sqrt{6}})^2}$

The Picard′s iterative form of eq. (5) is:

$u_{n+1}(x,t) = u_0(x,t) + L_t^{-1}(\,D_x^2u_n + u_n - u_n^2)$

The first iterative is:

$u_1(x,t) = u_0(x,t) + \ L_t^{-1}(\,D_x^2u_0 + u_0 - u_0^2)$

$= \frac{1}{(1+e^{x/\sqrt{6}})^2} + L_t^{-1}\left[\frac{e^{x/\sqrt{6}}\left(2e^{x/\sqrt{6}}-1\right)}{3\left(1+e^{x/\sqrt{6}}\right)^4} + \frac{1}{\left(1+e^{x/\sqrt{6}}\right)^2} - \frac{1}{\left(1+e^{x/\sqrt{6}}\right)^4}\right]$

$= \frac{1}{(1+e^{x/\sqrt{6}})^2} + \left[\frac{e^{x/\sqrt{6}}\left(2e^{x/\sqrt{6}}-1\right)}{3\left(1+e^{x/\sqrt{6}}\right)^4} + \frac{1}{\left(1+e^{x/\sqrt{6}}\right)^2} - \frac{1}{\left(1+e^{x/\sqrt{6}}\right)^4}\right]t$

then

$u_1(x,t) = \frac{1}{\left(1+e^{x/\sqrt{6}}\right)^2} + \frac{5e^{x/\sqrt{6}}}{3\left(1+e^{x/\sqrt{6}}\right)^3}t$

The second iterative is:

$u_2(x,t) = u_0(x,t) + \ L_t^{-1}(\,D_x^2u_1 + u_1 - u_1^2)$

$= \frac{1}{\left(1+e^{x/\sqrt{6}}\right)^2} + L_t^{-1}\left[\frac{e^{x/\sqrt{6}}\left(2e^{x/\sqrt{6}}-1\right)}{3\left(1+e^{x/\sqrt{6}}\right)^4} + \frac{5e^{x/\sqrt{6}}\left(1-7e^{x/\sqrt{6}}+4\left(e^{x/\sqrt{6}}\right)^2\right)}{18\left(1+e^{x/\sqrt{6}}\right)^5}t + \frac{1}{\left(1+e^{x/\sqrt{6}}\right)^2} + \frac{5e^{x/\sqrt{6}}}{3\left(1+e^{x/\sqrt{6}}\right)^3}t - \left(\frac{1}{\left(1+e^{x/\sqrt{6}}\right)^2} + \frac{5e^{x/\sqrt{6}}}{3\left(1+e^{x/\sqrt{6}}\right)^3}t\right)^2\right]$,

then

$u_2(x,t) = \frac{1}{(1+e^{x/\sqrt{6}})^2} + \frac{5e^{x/\sqrt{6}}}{3\left(1+e^{x/\sqrt{6}}\right)^3}t + \frac{25e^{x/\sqrt{6}}\left(-1+2e^{x/\sqrt{6}}\right)}{18\left(1+e^{x/\sqrt{6}}\right)^4}\frac{t^2}{2}$

And so on .The approximate solution is

$u(x,t) = \frac{1}{\left(1+e^{x/\sqrt{6}}\right)^2} + \frac{5e^{x/\sqrt{6}}}{3\left(1+e^{x/\sqrt{6}}\right)^3}t + \frac{25e^{x/\sqrt{6}}\left(-1+2e^{x/\sqrt{6}}\right)}{18\left(1+e^{x/\sqrt{6}}\right)^4}\frac{t^2}{2} + \cdots$

Example 3: Consider the KdV equation [9]

$u_t - 6uu_x + u_{xxx} = 0$, (6)

with $u(x,0) = \frac{2}{(x-3)^2}$

The Picard′s iterative form of eq. (6) is:

$u_{n+1}(x,t) = u_0(x,t) + L_t^{-1}(6uD_xu_n - D_x^3u_n)$

The first iterative is:

$u_1(x,t) = u_0(x,t) + \ L_t^{-1}(6uD_xu_0 - D_x^3u_0)$

$u_1(x,t) = \frac{2}{(x-3)^2} + L_t^{-1}\,(6\frac{2}{(x-3)^2}\cdot\frac{-4}{(x-3)^3} + \frac{48}{(x-3)^5})$

Then,

$u_1(x,t) = \frac{2}{(x-3)^2}$,

and

$u_2(x,t) = u_0(x,t) + L_t^{-1}(6uD_xu_1 - D_x^3u_1)$

$= \frac{2}{(x-3)^2}$

Hence, in iteration steps, we get

$u(x,t) = \frac{2}{(x-3)^2}$

Which is the exact solution of equation (6).

Example 4: Authors in [10] gave the non-linear homogeneous Klein Gordon equation as

$u_{tt} - u_{xx} + u^2 = x^2t^2$, (7)

with

$u(x,0) = 0\,, u_t(x,0) = x$

The Picard′s iterative form of eq. (7) is:

$u_{n+1}(x,t) = \sum_{j=0}^{1} D_t^j u_0(x,t)\frac{t^j}{j!} + L_{tt}^{-1}\,(x^2t^2 + D_x^2u_n - u_n^2)$

$u_1(x,t) = \sum_{j=0}^{1} D_t^j u_0(x,t)\frac{t^j}{j!} + L_{tt}^{-1}\,(x^2t^2 + D_x^2u_0 - u_0^2)$

$u_1(x,t) = xt + \frac{x^2t^4}{12}$

The second iteration is

$u_2(x,t) = \sum_{j=0}^{1} D_t^j u_0(x,t)\frac{t^j}{j!} + L_{tt}^{-1}\,(x^2t^2 + D_x^2 u_1 - u_1^2)$

$u_2(x,t) = xt + L_{tt}^{-1}\,(x^2t^2 + \frac{t^4}{6} - x^2t^2 - \frac{x^3t^5}{6} - \frac{x^4t^8}{144})$,

then

$u_2(x,t) = xt + \frac{t^8}{180} - \frac{x^3t^7}{252} - \frac{x^4t^{10}}{12960}$

And so on.

Conclusion

In this paper, (PIM) was used to solve LPDEs. and NLPDEs. The results obtained by this method are accurate in that they are close to the exact solution, and sometimes they are equal to this solution. It is also noted that this method is faster in obtaining results than many other iterative methods, is more effective and easy to use manually, and gives highly accurate results.

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