Six Points Implicit Block Method with Extra Derivative for Solving Second Order Ordinary Differential Equations

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Abstract—*The construction of the six-point block methods with extra derivatives for solving* $\boldsymbol{y'' = f(x, y, y')}$ *directly is presented in this paper. The suggested block methods concurrently approximate the problem's solution at six points and are formulated using the Hermite Interpolating Polynomial. The block methods do not reduce the equation to a first-order system of initial value problems (IVPs); instead, they directly obtain the numerical solutions. Additionally, the order and zero-stability of the suggested techniques are examined. We present numerical results and draw comparisons with other block methods that are currently in use. The results demonstrate how effective the suggested approaches are at solving second-order IVPs in general.*

Keywords—Block methods, six points , second order IVPs , Extra derivative

I. INTRODUCTION

For the purpose of directly solving general second order initial value problems (IVPs) without reducing the IVPs to a system of first order IVPs, a lot of research has been done on the block method. [5] Block linear multistep methods for solving second order IVPs were developed using the collocation technique. [8] solved general second order IVPs using the two-point block method. [9] An effective hybrid block method with seven points was created for the direct solution of second order general IVPs.[12] created a single-step, order five hybrid block method for solving second-order ordinary differential equations (ODEs) directly.[13] created an effective Falkner-type method of order two and three for solving the same kind of problems. [14] Block method was derived to solve second order IVPs with variable step size code. [15] The order four diagonal block method was introduced to solve the boundary value issues of the second order with Robin boundary conditions. In contrast, [16] developed an improved two-step hybrid block method for directly integrating the general second order IVPs that was formulated in variable step-size mode. The extra derivative was not incorporated into the method formulation in the majority of the previously cited work. The purpose of including the additional derivative in the formula for the technique allows for the acquisition of extremely accurate numerical solutions. In addition, a significant amount of computational effort was required to derive the majority of block methods in the literature, some of which used linear operators and others which were derived using collocation and interpolation techniques. Compared to the collocation and linear operator approaches, the integration technique we used to derive the methods in this paper was significantly simpler Only the function on the right-hand side of the integration was subjected to Newton interpolation in earlier block method work that was derived using integration techniques. In order to incorporate the additional derivatives of the problems that needed to be solved into the formula, Hermite interpolation was used in place of the function in this study. In this case, general second order ODEs (Equation 1) can be solved directly using block methods with an additional derivative. In this chapter, block method with extra derivative is derived for directly solving the initial value problems (IVPs) of general second order.

$$y'' = f(x, y, y'), y(\mathfrak{a}) = y_0, y'(a) = y_0', \mathfrak{a} \le x \le \mathfrak{b} \quad (1)$$

The second derivative with respect to x can be written as.

$$y''' = f'(x, y, y') = f_x + y' f_y + f f_{y'} = g(x, y, y'),$$

At this work, a new six points second derivative implicit block methodical was derived. The new methodical is derived using Hermite Interpolating Polynomial $\mathcal{P}$, which can be defined by

$$\mathcal{P}(x) = \sum_{i=0}^{n} \sum_{k=0}^{m_{i-1}} f_i^{(\mathcal{K})} L_{i,\mathcal{K}}(x) \quad (2)$$

Where $f_j = f(x_j)$, $x_j = a + hj$, $j = 0,1,\dots,n$ *and* $h = \frac{\mathfrak{b}-\mathfrak{a}}{\mathfrak{n}}$ n is positive integer.

$L_{j,\mathcal{K}}(x)$ is the definition of the generalized Lagrange polynomial .

$$L_{j,m_j}(x) = l_{j,m_j}(x) \ , j = 0,1,\dots,n,$$

$j = 0,1,\dots,n$, $\mathcal{K} = 0,1,\dots,m_j$ *and recursively for* $k = m_j - 2$; $m_j - 3,\dots,0$

$$L_{j,m_j}(x) = l_{j,m_j}(x) - \sum_{v=\mathcal{K}+1}^{m_{j-1}} l_{j,\mathcal{K}}^{(v)}(x_j) L_{j,v}(x)$$

II. Derivation of the New Method

In this section, we derive six points implicit block method for solving the initial value problems of the general second order . The values of $y_{n+1}, y_{n+2}, y_{n+3} y_{n+4}, y_{n+5}, y_{n+6}, y'_{n+1}, y'_{n+2}, y'_{n+3}, y'_{n+4}, y'_{n+5}$, and $y'_{n+6}$are calculated. For the evaluation of the first point y_{n+1}, y'_{n+1} at x_{n+1}integrating (1) over the interval $[x_n, x_{n+1}]$ gives:

$$\int_{x_n}^{x_{n+1}} y''\, dx = \int_{x_n}^{x_{n+1}} f(x,y,y')dx \qquad (3)$$

And

$$\int_{x_n}^{x_{n+1}} \int_{x_n}^{x} y'\, dxdx = \int_{x_n}^{x_{n+1}} \int_{x_n}^{x} f(x,y,y')\, dxdx \qquad (4)$$

Eq's (5) & (6) can be obtained by rewriting Eq's (3) & (4) as $x_{n+1} = x_n + h$.

$$y'(x_{n+1}) = y'(x_n) + \int_{x_n}^{x_{n+1}} f(x,y,y')dx \qquad (5)$$

$$y(x_{n+1}) = y(x_n) + hy'(x_n) + \int_{x_n}^{x_{n+1}} (x_{n+1} - x)f(x,y,y')dx \qquad (6)$$

The Hermite Interpolating Polynomial in , which is defined by p_6 as follows(7), will then be used to replace $f(x,y,y')$ in Eq's(5) & (6)

$$\mathcal{P}_6(x) = f_n L_{0,0}(x) + f_{n+1}L_{1,0}(x) + f_{n+2}L_{2,0}(x) + f_{n+3}L_{3,0}(x) + f_{n+4}L_{4,0}(x) + f_{n+5}L_{5,0}(x) + f_{n+6}L_{6,0}(x) + g_n L_{0,1}(x) + g_{n+1}L_{1,1}(x) + g_{n+2}L_{2,1}(x) + g_{n+3}L_{3,1}(x) + g_{n+4}L_{4,1}(x) + g_{n+5}L_{5,1}(x) + g_{n+6}L_{6,1}(x) \qquad (7)$$

Let $x = x_{n+6} + \delta\, h$ *and* $\delta = \frac{x - x_{n+6}}{h}$ (8)

If we change integration in(5)and(6) from -6 to -5 and replace $dx = h\, d\delta$, we get:

$$y'(x_{n+1}) = y'(x_n) + \int_{-6}^{-5} [\, f_n L_{0,0}(\delta) + f_{n+1}L_{1,0}(\delta) + f_{n+2}L_{2,0}(\delta) + f_{n+3}L_{3,0}(\delta) + f_{n+4}L_{4,0}(\delta) + f_{n+5}L_{5,0}(\delta) + f_{n+6}L_{6,0}(\delta) + g_n L_{0,1}(\delta) + g_{n+1}L_{1,1}(\delta) + g_{n+2}L_{2,1}(\delta) + g_{n+3}L_{3,1}(\delta) + g_{n+4}L_{4,1}(\delta) + g_{n+5}L_{5,1}(\delta) + g_{n+6}L_{6,1}(\delta)\,]\, h\, d\delta. \qquad (9)$$

$$y(x_{n+1}) = y(x_n) + hy'(x_n) + \int_{-6}^{-5} [\, (-5h - h\delta)(f_n L_{0,0}(\delta) + f_{n+1}L_{1,0}(\delta) + f_{n+2}L_{2,0}(\delta) + f_{n+3}L_{3,0}(\delta) + f_{n+4}L_{4,0}(\delta) + f_{n+5}L_{5,0}(\delta) + f_{n+6}L_{6,0}(\delta) + g_n L_{0,1}(\delta) + g_{n+1}L_{1,1}(\delta) + g_{n+2}L_{2,1}(\delta) + g_{n+3}L_{3,1}(\delta) + g_{n+4}L_{4,1}(\delta) + g_{n+5}L_{5,1}(\delta) + g_{n+6}L_{6,1}(\delta))\,]\, h\, d\delta \qquad (10)$$

Where

$$L_{0,0}(\delta) = \frac{(5+\delta)^2(4+\delta)^2(3+\delta)^2(2+\delta)^2(1+\delta)^2(\delta)^2}{518400}\left(1 + \frac{49(6+\delta)}{10}\right)$$

$$L_{0,1}(\delta) = \frac{\hbar(6+\delta)(5+\delta)^2(4+\delta)^2(3+\delta)^2(2+\delta)^2(1+\delta)^2(\delta)^2}{518400}$$

$$L_{1,0}(\delta) = \frac{(6+\delta)^2(4+\delta)^2(3+\delta)^2(2+\delta)^2(1+\delta)^2(\delta)^2}{14400}\left(1 + \frac{77(5+\delta)}{30}\right)$$

$$L_{1,1}(\delta) = \frac{\hbar(5+\delta)(6+\delta)^2(4+\delta)^2(3+\delta)^2(2+\delta)^2(1+\delta)^2(\delta)^2}{14400}$$

$$L_{2,0}(\delta) = \frac{(6+\delta)^2(5+\delta)^2(3+\delta)^2(2+\delta)^2(1+\delta)^2(\delta)^2}{2304}\left(1 + \frac{7(4+\delta)}{6}\right)$$

$$L_{2,1}(\delta) = \frac{\hbar(4+\delta)(6+\delta)^2(5+\delta)^2(3+\delta)^2(2+\delta)^2(1+\delta)^2(\delta)^2}{2304}$$

$$L_{3,0}(\delta) = \frac{(6+\delta)^2(5+\delta)^2(4+\delta)^2(2+\delta)^2(1+\delta)^2(\delta)^2}{1296}$$

$$L_{3,1}(\delta) = \frac{\hbar(3+\delta)(6+\delta)^2(5+\delta)^2(4+\delta)^2(2+\delta)^2(1+\delta)^2(\delta)^2}{1296}$$

$$L_{4,0}(\delta) = \frac{(6+\delta)^2(5+\delta)^2(4+\delta)^2(3+\delta)^2(1+\delta)^2(\delta)^2}{2304}\left(1 - \frac{7(2+\delta)}{6}\right)$$

$$L_{4,1}(\delta) = \frac{\hbar(2+\delta)(6+\delta)^2(5+\delta)^2(4+\delta)^2(3+\delta)^2(1+\delta)^2(\delta)^2}{2304}$$

$$L_{5,0}(\delta) = \frac{(6+\delta)^2(5+\delta)^2(4+\delta)^2(3+\delta)^2(2+\delta)^2(\delta)^2}{14400}\left(1 - \frac{77(1+\delta)}{30}\right)$$

$$L_{5,1}(\delta) = \frac{\hbar(1+\delta)(6+\delta)^2(5+\delta)^2(4+\delta)^2(3+\delta)^2(2+\delta)^2(\delta)^2}{14400}$$

$$L_{6,0}(\delta) = \frac{(6+\delta)^2(5+\delta)^2(4+\delta)^2(3+\delta)^2(2+\delta)^2(1+\delta)^2}{518400}\left(1 - \frac{49(\delta)}{10}\right)$$

$$L_{6,1}(\delta) = \frac{\hbar(\delta)(6+\delta)^2(5+\delta)^2(4+\delta)^2(3+\delta)^2(2+\delta)^2(1+\delta)^2}{518400}$$

Evaluating the integrals in (9) and (10) using MAPLE produces the first formula of the six- point implicit block method as follows :

$$y'_{n+1} = y'_n + \frac{h}{622702080000}(199368819177\, f_n - 68951829552\, f_{n+1} - 380416470375\, f_{n+2} + 300642304000\, f_{n+3} + 457138998375\, f_{n+4} + 110327270448\, f_{n+5} + 4592987927\, f_{n+6}) + \frac{h^2}{62270208000}(1784098013\, g_n - 33488665488 g_{n+1} - 71514207675\, g_{n+2} - 77935000000\, g_{n+3} - 31646886075\, g_{n+4} - 3963034512\, g_{n+5} - 90441763\, g_{n+6}) \qquad (11)$$

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{1868106240000}(469983596501\, f_n - 402081396000\, f_{n+1} - 709748678625\, f_{n+2} + 534196176000\, f_{n+3} + 831579916875\, f_{n+4} + 201705150624\, f_{n+5} + 8418354625\, f_{n+6}) + \frac{h^3}{186810624000} * (3829846679\, g_n - 54808104960\, g_{n+1} - 128008032975\, g_{n+2} - 141368944000\, g_{n+3} - 57728681325\, g_{n+4} - 7251690816\, g_{n+5} - 165820915\, g_{n+6}) \qquad (12)$$

Eq(13) and (14) is the result of integrating Eq(1) once over $[\, x_{n+1}, x_{n+2}\,]$ and again over the approximate solutions ofy_{n+2} and y'_{n+2} .

$$\int_{x_{n+1}}^{x_{n+2}} y''\, dx = \int_{x_{n+1}}^{x_{n+2}} f(x,y,y')dx \qquad (13)$$

$$\int_{x_{n+1}}^{x_{n+2}} \int_{x_{n+1}}^{x} y'\, dxdx = \int_{x_{n+1}}^{x_{n+2}} \int_{x_{n+1}}^{x} f(x,y,y')\, dxdx \qquad (14)$$

Taking $x_{n+2} = x_{n+1} + h$ and substituting into (13) and (14) we have :

$$y'(x_{n+2}) = y'(x_{n+1}) + \int_{x_{n+1}}^{x_{n+2}} f(x,y,y')dx \qquad (15)$$

$$y(x_{n+2}) = y(x_{n+1}) + hy'(x_{n+1}) + \int_{x_{n+1}}^{x_{n+2}} (x_{n+2} - x)f(x,y,y')dx \qquad (16)$$

Then, in (15) and (16) will be replaced by Hermite interpolating polynomial in (7). Also, by replacing (8) letting and changing the limit of integration from -5 to -4 in (15) and (16) we obtain:

$$y'(x_{n+2}) = y'(x_{n+1}) + \int_{-5}^{-4} [\, f_n L_{0,0}(\delta) + f_{n+1}L_{1,0}(\delta) + f_{n+2}L_{2,0}(\delta) + f_{n+3}L_{3,0}(\delta) + f_{n+4}L_{4,0}(\delta) + f_{n+5}L_{5,0}(\delta) + f_{n+6}L_{6,0}(\delta) + g_n L_{0,1}(\delta) + g_{n+1}L_{1,1}(\delta) + g_{n+2}L_{2,1}(\delta) + g_{n+3}L_{3,1}(\delta) + g_{n+4}L_{4,1}(\delta) + g_{n+5}L_{5,1}(\delta) + g_{n+6}L_{6,1}(\delta)\,]\, h\, d\delta. \qquad (17)$$

$$y(x_{n+2}) = y(x_{n+1}) + hy'(x_{n+1}) + \int_{-5}^{-4} [\, (-4h - h\delta)(f_n L_{0,0}(\delta) + f_{n+1}L_{1,0}(\delta) + f_{n+2}L_{2,0}(\delta) + f_{n+3}L_{3,0}(\delta) + f_{n+4}L_{4,0}(\delta) + f_{n+5}L_{5,0}(\delta) + f_{n+6}L_{6,0}(\delta) + g_n L_{0,1}(\delta) + g_{n+1}L_{1,1}(\delta) + g_{n+2}L_{2,1}(\delta) + g_{n+3}L_{3,1}(\delta) + g_{n+4}L_{4,1}(\delta) + g_{n+5}L_{5,1}(\delta) + g_{n+6}L_{6,1}(\delta))\,]\, h\, d\delta \qquad (18)$$

Evaluating the integrals in (17) and (18) using MAPLE produces the second formula of the six- point implicit block method as follows :

$$y'_{n+2} = y'_{n+1} + \frac{h}{622702080000}(1263709975\, f_n + 249886338096\, f_{n+1} + 263409462375\, f_{n+2} + 54636544000\, f_{n+3} + 43943465625\, f_{n+4} + 9204200400\, f_{n+5} + 358359529\, f_{n+6} + \frac{h^2}{20756736000}(7499105\, g_n + 1120692528\, g_{n+1} - 5002116375\, g_{n+2} - 2761387200\, g_{n+3} - 938225175\, g_{n+4} - 107689680\, g_{n+5} - 2333087\, g_{n+6}). \qquad (19)$$

$$y_{n+2} = y_{n+1} + hy'_{n+1} + \frac{h^2}{934053120000}(1111128376\, f_n + 280224697560\, f_{n+1} + 99601924875\, f_{n+2} + 42307608000\, f_{n+3} + 35883048000\, f_{n+4} + 7600779624\, f_{n+5} + 297373565\, f_{n+6}) + \frac{h^3}{93405312000}(19684969\, g_n + 3322065240\, g_{n+1} - 10756569600\, g_{n+2} - 6696548000\, g_{n+3} - 2312543025\, g_{n+4} - 267240456\, g_{n+5} - 5811410\, g_{n+6}). \qquad (20)$$

Eq(21) and (22) is the result of integrating Eq(1) once over $[\, x_{n+2}, x_{n+3}\,]$ and again over the approximate solutions of y_{n+3} and y'_{n+3} .

$$\int_{x_{n+2}}^{x_{n+3}} y''\, dx = \int_{x_{n+2}}^{x_{n+3}} f(x,y,y')dx \qquad (21)$$

$$\int_{x_{n+2}}^{x_{n+3}} \int_{x_{n+2}}^{x} y'\, dxdx = \int_{x_{n+2}}^{x_{n+3}} \int_{x_{n+2}}^{x} f(x,y,y')\, dxdx \qquad (22)$$

Taking $x_{n+3} = x_{n+2} + h$ and substituting into (21) and (22) we have :

$$y'(x_{n+3}) = y'(x_{n+2}) + \int_{x_{n+2}}^{x_{n+3}} f(x,y,y')dx \qquad (23)$$

$$y(x_{n+3}) = y(x_{n+2}) + hy'(x_{n+2}) + \int_{x_{n+2}}^{x_{n+3}} (x_{n+3} - x)f(x,y,y')dx \qquad (24)$$

Then, in (23) and (24) will be replaced by Hermite interpolating polynomial in (7). Also, by replacing (8) letting and changing the limit of integration from -4 to -3 in (23) and (24) we obtain :

$$y'(x_{n+3}) = y'(x_{n+2}) + \int_{-4}^{-3} [\, f_n L_{0,0}(\delta) + f_{n+1}L_{1,0}(\delta) + f_{n+2}L_{2,0}(\delta) + f_{n+3}L_{3,0}(\delta) + f_{n+4}L_{4,0}(\delta) + f_{n+5}L_{5,0}(\delta) + f_{n+6}L_{6,0}(\delta) + g_n L_{0,1}(\delta) + g_{n+1}L_{1,1}(\delta) + g_{n+2}L_{2,1}(\delta) + g_{n+3}L_{3,1}(\delta) + g_{n+4}L_{4,1}(\delta) + g_{n+5}L_{5,1}(\delta) + g_{n+6}L_{6,1}(\delta)\,]\, h\, d\delta. \qquad (25)$$

$$y(x_{n+3}) = y(x_{n+2}) + hy'(x_{n+2}) + \int_{-4}^{-3} [\, (-3h - h\delta)(f_n L_{0,0}(\delta) + f_{n+1}L_{1,0}(\delta) + f_{n+2}L_{2,0}(\delta) + f_{n+3}L_{3,0}(\delta) + f_{n+4}L_{4,0}(\delta) + f_{n+5}L_{5,0}(\delta) + f_{n+6}L_{6,0}(\delta) + g_n L_{0,1}(\delta) + g_{n+1}L_{1,1}(\delta) + g_{n+2}L_{2,1}(\delta) + g_{n+3}L_{3,1}(\delta) + g_{n+4}L_{4,1}(\delta) + g_{n+5}L_{5,1}(\delta) + g_{n+6}L_{6,1}(\delta))]\, h\, d\delta \qquad (26)$$

Evaluating the integrals in (25) and (26) using MAPLE produces the third formula of the six- point implicit block method as follows :

$$y'_{n+3} = y'_{n+2} + \frac{h}{622702080000}(200505833\, f_n + 7511373264\, f_{n+1} + 284375657625\, f_{n+2} + 300642304000\, f_{n+3} + 25860150375\, f_{n+4} + 3974367792\, f_{n+5} + 137721111\, f_{n+6}) + \frac{h^2}{62270208000}(3777757\, g_n + 230157936\, g_{n+1} + 5433977925\, g_{n+2} - 8687307200\, g_{n+3} - 1401356475\, g_{n+4} - 133910928\, g_{n+5} - 2655779\, g_{n+6}). \qquad (27)$$

$$y_{n+3} = y_{n+2} + hy'_{n+2} + \frac{h^2}{1868106240000}(321063755\, f_n + 12326273472\, f_{n+1} + 616706419125\, f_{n+2} + 260619216000\, f_{n+3} + 37861459125\, f_{n+4} + 6008349120\, f_{n+5} + 210339403\, f_{n+6}) + \frac{h^3}{186810624000}(6037445\, g_n + 374302368\, g_{n+1} + 10010400975\, g_{n+2} - 11184947200\, g_{n+3} - 2087463375\, g_{n+4} - 203179680\, g_{n+5} - 4060357\, g_{n+6}). \qquad (28)$$

Eq(.29) and (30) is the result of integrating Eq(1) once over $[\, x_{n+3}, x_{n+4}\,]$ and again over the approximate solutions of y_{n+4} and y'_{n+4} .

$$\int_{x_{n+3}}^{x_{n+4}} y''\, dx = \int_{x_{n+3}}^{x_{n+4}} f(x,y,y')dx \qquad (29)$$

$$\int_{x_{n+3}}^{x_{n+4}} \int_{x_{n+3}}^{x} y'\, dxdx = \int_{x_{n+3}}^{x_{n+4}} \int_{x_{n+3}}^{x} f(x,y,y')\, dxdx \qquad (30)$$

Taking $x_{n+4} = x_{n+3} + h$ and substituting into (29) and (30) we have :

$$y'(x_{n+4}) = y'(x_{n+3}) + \int_{x_{n+3}}^{x_{n+4}} f(x,y,y')dx \qquad (31)$$

$$y(x_{n+4}) = y(x_{n+3}) + hy'(x_{n+3}) + \int_{x_{n+3}}^{x_{n+4}} (x_{n+4} - x)f(x,y,y')dx \qquad (32)$$

Then, in (31) and (32) will be replaced by Hermite interpolating polynomial in (7). Also, by replacing (8) letting and changing the limit of integration from -3 to -2 in (31) and (32) we obtain :

$$y'(x_{n+4}) = y'(x_{n+3}) + \int_{-3}^{-2}[\, f_n L_{0,0}(\delta) + f_{n+1}L_{1,0}(\delta) + f_{n+2}L_{2,0}(\delta) + f_{n+3}L_{3,0}(\delta) + f_{n+4}L_{4,0}(\delta) + f_{n+5}L_{5,0}(\delta) + f_{n+6}L_{6,0}(\delta) + g_n L_{0,1}(\delta) + g_{n+1}L_{1,1}(\delta) + g_{n+2}L_{2,1}(\delta) + g_{n+3}L_{3,1}(\delta) + g_{n+4}L_{4,1}(\delta) + g_{n+5}L_{5,1}(\delta) + g_{n+6}L_{6,1}(\delta)\,]\, h\, d\delta. \quad (33)$$

$$y(x_{n+4}) = y(x_{n+3}) + hy'(x_{n+3}) + \int_{-3}^{-2}[\, (-2h - h\delta)(f_n L_{0,0}(\delta) + f_{n+1}L_{1,0}(\delta) + f_{n+2}L_{2,0}(\delta) + f_{n+3}L_{3,0}(\delta) + f_{n+4}L_{4,0}(\delta) + f_{n+5}L_{5,0}(\delta) + f_{n+6}L_{6,0}(\delta) + g_n L_{0,1}(\delta) + g_{n+1}L_{1,1}(\delta) + g_{n+2}L_{2,1}(\delta) + g_{n+3}L_{3,1}(\delta) + g_{n+4}L_{4,1}(\delta) + g_{n+5}L_{5,1}(\delta) + g_{n+6}L_{6,1}(\delta))]\, h\, d\delta \quad (34)$$

Evaluating the integrals in (33) and (34) using MAPLE produces the 4th formula of the six- point implicit block method as follows :

$$y'_{n+4} = y'_{n+3} + \frac{h}{622702080000}(137721111\, f_n + 3974367792\, f_{n+1} + 25860150375\, f_{n+2} + 300642304000\, f_{n+3} + 284375657625\, f_{n+4} + 7511373264\, f_{n+5} + 200505833\, f_{n+6}) + \frac{h^2}{62270208000}(2655779\, g_n + 133910928\, g_{n+1} + 1401356475\, g_{n+2} + 8687307200\, g_{n+3} - 5433977925\, g_{n+4} - 230157936\, g_{n+5} - 3777757\, g_{n+6}). \quad (35)$$

$$y_{n+4} = y_{n+3} + hy'_{n+3} + \frac{h^2}{934053120000}(101411965\, f_n + 2957377128\, f_{n+1} + 19859496000\, f_{n+2} + 320653848000\, f_{n+3} + 118210276875\, f_{n+4} + 5103923160\, f_{n+5} + 140226872\, f_{n+6}) + \frac{h^3}{93405312000}(1953490\, g_n + 99276552\, g_{n+1} + 1058303025\, g_{n+2} + 7438487200\, g_{n+3} - 3145766400\, g_{n+4} - 158085720\, g_5 - 2647913\, g_{n+6}). \quad (36)$$

Eq(37) and (38) is the result of integrating Eq(1) once over $[\, x_{n+4}, x_{n+5}\,]$ and again over the approximate solutions of y_{n+5} and y'_{n+5} .

$$\int_{x_{n+4}}^{x_{n+5}} y''\, dx = \int_{x_{n+4}}^{x_{n+5}} f(x,y,y')dx \quad (37)$$

$$\int_{x_{n+4}}^{x_{n+5}}\int_{x_{n+4}}^{x} y'\, dxdx = \int_{x_{n+4}}^{x_{n+5}}\int_{x_{n+4}}^{x} f(x,y,y')\, dxdx \quad (38)$$

Taking $x_{n+5} = x_{n+4} + h$ and substituting into (37) and (38)we have :

$$y'(x_{n+5}) = y'(x_{n+4}) + \int_{x_{n+4}}^{x_{n+5}} f(x,y,y')dx \quad (39)$$

$$y(x_{n+5}) = y(x_{n+4}) + hy'(x_{n+4}) + \int_{x_{n+4}}^{x_{n+5}}(x_{n+5} - x)f(x,y,y')dx \quad (40)$$

Then, in (39) and (40) will be replaced by Hermite interpolating polynomial in (7). Also, by replacing (8) letting and changing the limit of integration from -2 to -1 in (39) and (40) we obtain :

$$y'(x_{n+5}) = y'(x_{n+4}) + \int_{-2}^{-1}[\, f_n L_{0,0}(\delta) + f_{n+1}L_{1,0}(\delta) + f_{n+2}L_{2,0}(\delta) + f_{n+3}L_{3,0}(\delta) + f_{n+4}L_{4,0}(\delta) + f_{n+5}L_{5,0}(\delta) + f_{n+6}L_{6,0}(\delta) + g_n L_{0,1}(\delta) + g_{n+1}L_{1,1}(\delta) + g_{n+2}L_{2,1}(\delta) + g_{n+3}L_{3,1}(\delta) + g_{n+4}L_{4,1}(\delta) + g_{n+5}L_{5,1}(\delta) + g_{n+6}L_{6,1}(\delta)\,]\, h\, d\delta. \quad (41)$$

$$y(x_{n+5}) = y(x_{n+4}) + hy'(x_{n+4}) + \int_{-2}^{-1}[\, (-h - h\delta)(f_n L_{0,0}(\delta) + f_{n+1}L_{1,0}(\delta) + f_{n+2}L_{2,0}(\delta) + f_{n+3}L_{3,0}(\delta) + f_{n+4}L_{4,0}(\delta) + f_{n+5}L_{5,0}(\delta) + f_{n+6}L_{6,0}(\delta) + g_n L_{0,1}(\delta) + g_{n+1}L_{1,1}(\delta) + g_{n+2}L_{2,1}(\delta) + g_{n+3}L_{3,1}(\delta) + g_{n+4}L_{4,1}(\delta) + g_{n+5}L_{5,1}(\delta) + g_{n+6}L_{6,1}(\delta))]\, h\, d\delta \quad (42)$$

Evaluating the integrals in (41) and (42) , by using MAPLE produces the 5th formula of the six- point implicit block method as follows :

$$y'_{n+5} = y'_{n+4} + \frac{h}{622702080000}(358359529\, f_n + 9204200400\, f_{n+1} + 43943465625\, f_{n+2} + 54636544000\, f_{n+3} + 263409462375\, f_{n+4} + 249886338096\, f_{n+5} + 1263709975\, f_{n+6}) + \frac{h^2}{20756736000}(+2333087\, g_n + 107689680\, g_{n+1} + 938225175\, g_{n+2} + 2761387200\, g_{n+3} + 5002116375\, g_{n+4} - 1120692528\, g_{n+5} - 7499105\, g_{n+6}). \quad (43)$$

$$y_{n+5} = y_{n+4} + hy'_{n+4} + \frac{h^2}{1868106240000}(480331457\, f_n + 12411041952\, f_{n+1} + 60064300875\, f_{n+2} + 79294416000\, f_{n+3} + 591024537375\, f_{n+4} + 189209619168\, f_{n+5} + 1568873173\, f_{n+6}) + \frac{h^3}{186810624000}(9374963\, g_n + 434726208\, g_{n+1} + 3818940525\, g_{n+2} + 11459388800\, g_{n+3} + 23505908175\, g_{n+4} - 3442102272\, g_{n+5} - 28122007\, g_{n+6}). \quad (44)$$

Eq's (45) and (46) is the result of integrating Eq(1) once over $[\, x_{n+5}, x_{n+6}\,]$ and again over the approximate solutions of y_{n+6} and y'_{n+6} .

$$\int_{x_{n+5}}^{x_{n+6}} y''\, dx = \int_{x_{n+5}}^{x_{n+6}} f(x,y,y')dx \quad (45)$$

$$\int_{x_{n+4}}^{x_{n+5}}\int_{x_{n+4}}^{x} y'\, dxdx = \int_{x_{n+4}}^{x_{n+5}}\int_{x_{n+4}}^{x} f(x,y,y')\, dxdx \quad (46)$$

Taking $x_{n+5} = x_{n+4} + h$ and substituting into (45) and (46) we have :

$$y'(x_{n+6}) = y'(x_{n+5}) + \int_{x_{n+5}}^{x_{n+6}} f(x,y,y')dx \quad (47)$$

$$y(x_{n+6}) = y(x_{n+5}) + hy'(x_{n+5}) + \int_{x_{n+5}}^{x_{n+6}}(x_{n+6} - x)f(x,y,y')dx \quad (48)$$

Then, in (47) and (48) will be replaced by Hermite interpolating polynomial in (7). Also, by replacing (8) letting and changing the limit of integration from -1 to 0 in (47) and (48) we obtain :

$$y'(x_{n+6}) = y'(x_{n+5}) + \int_{-1}^{0}[\, f_n L_{0,0}(\delta) + f_{n+1}L_{1,0}(\delta) + f_{n+2}L_{2,0}(\delta) + f_{n+3}L_{3,0}(\delta) + f_{n+4}L_{4,0}(\delta) + f_{n+5}L_{5,0}(\delta) + f_{n+6}L_{6,0}(\delta) + g_n L_{0,1}(\delta) + g_{n+1}L_{1,1}(\delta) + g_{n+2}L_{2,1}(\delta) +$$

$$g_{n+3}L_{3,1}(\delta) + g_{n+4}L_{4,1}(\delta) + g_{n+5}L_{5,1}(\delta) + g_{n+6}L_{6,1}(\delta)\,]\,h\,d\delta. \quad (49)$$

$$y(x_{n+6}) = y(x_{n+5}) + hy'(x_{n+5}) + \int_{-1}^{0}[\,(-h\delta\,)(f_nL_{0,0}(\delta) + f_{n+1}L_{1,0}(\delta) + f_{n+2}L_{2,0}(\delta) + f_{n+3}L_{3,0}(\delta) + f_{n+4}L_{4,0}(\delta) + f_{n+5}L_{5,0}(\delta) + f_{n+6}L_{6,0}(\delta) + g_nL_{0,1}(\delta) + g_{n+1}L_{1,1}(\delta) + g_{n+2}L_{2,1}(\delta) + g_{n+3}L_{3,1}(\delta) + g_{n+4}L_{4,1}(\delta) + g_{n+5}L_{5,1}(\delta) + g_{n+6}L_{6,1}(\delta))]\,h\,d\delta \quad (50)$$

Evaluating the integrals in (49) and (50) , by using MAPLE produces the 6th formula of the six- point implicit block method as follows :

$$y'_{n+6} = y'_{n+5} + \frac{h}{622702080000}(4592987927\,f_n + 110327270448\,f_{n+1} + 457138998375\,f_{n+2} + 300642304000\,f_{n+3} - 380416470375\,f_{n+4} - 68951829552\,f_{n+5} + 199368819177\,f_{n+6}) + \frac{h^2}{62270208000}(90441763\,g_n + 3963034512\,g_{n+1} + 31646886075\,g_{n+2} + 77935000000\,g_{n+3} + 71514207675\,g_{n+4} + 33488665488\,g_{n+5} - 1784098013\,g_{n+6}) . \quad (51)$$

$$y_{n+6} = y_{n+5} + hy'_{n+5} + \frac{h^2}{934053120000}(2680304578\,f_n + 64638330360\,f_{n+1} + 269918539125\,f_{n+2} + 183865368000\,f_{n+3} - 215750366250\,f_{n+4} + 97612953672\,f_{n+5} + 64061430515\,f_{n+6}) + \frac{h^3}{93405312000}(52752187\,g_n + 2318706360\,g_{n+1} + 18605988450\,g_{n+2} + 46218028000\,g_{n+3} + 43267295025\,g_{n+4} + 22828945752\,g_{n+5} - 761223680\,g_{n+6}) . \quad (52)$$

III. ORDER AND ERROR CONSTANT OF SIX POINTS IMPLICIT BLOCK METHOD

In this portion, defines the order of the six-point block approaches that were developed in this article. As stated by [3] and [1],the local truncation error associated with the normalized form of the proposed method can be defend as the linear difference operator Equation (53)

$$\zeta[y(x);h] = \sum_{i=0}^{k}\alpha_i\,y(x+ih) - \sum_{i=0}^{k}h\,\beta_i y'(x+ih) - \sum_{i=0}^{k}h^2\,\gamma_i y''(x+ih) - \sum_{i=0}^{k}h^3\delta_i y'''(x+ih) \quad (53)$$

Assuming that $y(x)$is sufficiently differentiable, expanding (53) as a Taylor series expansion about the point yields the formula:

$$\zeta[y(x);h] = c_0y(x) + c_1hy'(x) + \cdots + c_ph^py^{(p)}(x) + \cdots$$

The constant coefficients and are supplied as follows: $c_0 = \sum_{i=0}^{k}\alpha_i$, $c_1 = \sum_{i=0}^{k}(i\alpha_i - \beta_i)$, ... $c_p = \sum_{i=0}^{k}(\frac{i^{\mathcal{P}}}{\mathcal{P}!}\alpha_i - \frac{i^{\mathcal{P}-1}}{(\mathcal{P}-1)!}\beta_i - \frac{i^{p-2}}{(\mathcal{P}-2)!}\gamma_i - \frac{i^{p-3}}{(\mathcal{P}-3)!}\delta_i)$, $\mathcal{P} = 2,3,\ldots$ (54)

According to [2], the new approach has p order if $c_0 = c_1 = c_2 = \cdots = c_p = 0$, $c_{p+1} \neq 0$ Therefore,$c_{p+1}h^{p+1}y^{(p+1)}(x)$ is the largest local truncation error at the position x_n, where c_{p+1} is the error constant . The formulas for the new six-points implicit block methodical are given in paragraphs (11) ,(12),(19),(20),(27),(28),(35) ,(36), (43) , (44) , (51) and (52) they are organized into the matrix's as follows:

$$\alpha\mathcal{y}_v = h\beta\mathcal{y}'_v + h^2\gamma\,\mathcal{f}_v + h^3\delta\mathcal{g}_v$$

where the coefficients with the v-vector$\mathcal{y}_v, \mathcal{y}'_v, \mathcal{f}_v$ and $\mathcal{g}_v, \alpha, \beta$, γ $and\delta$, are defined as:

$$\alpha = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&-1&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&-1&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&-1&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&-1&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&-1&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&-1&1 \end{bmatrix}$$

$$\beta = \begin{bmatrix} 0&0&0&0&0&1&-1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&-1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&-1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&-1&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&-1&0\\ 0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&-1\\ 0&0&0&0&0&0&0&0&0&0&1&0 \end{bmatrix}$$

γ=

$$\delta=\begin{bmatrix}
0&0&0&0&0&\frac{199368819177}{622702080000}&\frac{-68951829552}{622702080000}&\frac{-380416470375}{622702080000}&\frac{300642304000}{622702080000}&\frac{457138998375}{622702080000}&\frac{110327270448}{622702080000}&\frac{4592987927}{622702080000}\\
0&0&0&0&0&\frac{469983596501}{1868106240000}&\frac{-402081396000}{1868106240000}&\frac{-709748678625}{1868106240000}&\frac{534196176000}{1868106240000}&\frac{831579916875}{1868106240000}&\frac{201705150624}{1868106240000}&\frac{8418354625}{1868106240000}\\
0&0&0&0&0&\frac{1263709975}{622702080000}&\frac{249886338096}{622702080000}&\frac{263409462375}{622702080000}&\frac{54636544000}{622702080000}&\frac{43943465625}{622702080000}&\frac{9204200400}{622702080000}&\frac{358359529}{622702080000}\\
0&0&0&0&0&\frac{1111128376}{934053120000}&\frac{280224697560}{934053120000}&\frac{99601924875}{934053120000}&\frac{42307608000}{934053120000}&\frac{35883048000}{934053120000}&\frac{7600779624}{934053120000}&\frac{297373565}{934053120000}\\
0&0&0&0&0&\frac{200505833}{622702080000}&\frac{7511373264}{622702080000}&\frac{284375657625}{622702080000}&\frac{300642304000}{622702080000}&\frac{25860150375}{622702080000}&\frac{3974367792}{622702080000}&\frac{137721111}{622702080000}\\
0&0&0&0&0&\frac{321063755}{1868106240000}&\frac{12326273472}{1868106240000}&\frac{616706419125}{1868106240000}&\frac{260619216000}{1868106240000}&\frac{37861459125}{1868106240000}&\frac{6008349120}{1868106240000}&\frac{210339403}{1868106240000}\\
0&0&0&0&0&\frac{137721111}{622702080000}&\frac{3974367792}{622702080000}&\frac{25860150375}{622702080000}&\frac{300642304000}{622702080000}&\frac{284375657625}{622702080000}&\frac{7511373264}{622702080000}&\frac{200505833}{622702080000}\\
0&0&0&0&0&\frac{101411965}{934053120000}&\frac{2957377128}{934053120000}&\frac{19859496000}{934053120000}&\frac{320653848000}{934053120000}&\frac{118210276875}{934053120000}&\frac{5103923160}{934053120000}&\frac{140226872}{934053120000}\\
0&0&0&0&0&\frac{358359529}{622702080000}&\frac{9204200400}{622702080000}&\frac{43943465625}{622702080000}&\frac{54636544000}{622702080000}&\frac{263409462375}{622702080000}&\frac{249886338096}{622702080000}&\frac{1263709975}{622702080000}\\
0&0&0&0&0&\frac{480331457}{1868106240000}&\frac{12411041952}{1868106240000}&\frac{60064300875}{1868106240000}&\frac{79294416000}{1868106240000}&\frac{591024537375}{1868106240000}&\frac{189209619168}{1868106240000}&\frac{1568873173}{1868106240000}\\
0&0&0&0&0&\frac{4592987927}{622702080000}&\frac{110327270448}{622702080000}&\frac{457138998375}{622702080000}&\frac{300642304000}{622702080000}&\frac{-380416470375}{622702080000}&\frac{-68951829552}{622702080000}&\frac{199368819177}{622702080000}\\
0&0&0&0&0&\frac{2680304578}{934053120000}&\frac{64638330360}{934053120000}&\frac{269918539125}{934053120000}&\frac{183865368000}{934053120000}&\frac{-215750366250}{934053120000}&\frac{97612953672}{934053120000}&\frac{64061430515}{934053120000}
\end{bmatrix}$$

$$\begin{bmatrix}
0&0&0&0&0&\frac{1784098013}{62270208000}&\frac{-33488665488}{62270208000}&\frac{-71514207675}{62270208000}&\frac{-77935000000}{62270208000}&\frac{-31646886075}{62270208000}&\frac{-3963034512}{62270208000}&\frac{-90441763}{62270208000}\\
0&0&0&0&0&\frac{3829846679}{186810624000}&\frac{-54808104960}{186810624000}&\frac{-128008032975}{186810624000}&\frac{-141368944000}{186810624000}&\frac{-57728681325}{186810624000}&\frac{-7251690816}{186810624000}&\frac{-165820915}{186810624000}\\
0&0&0&0&0&\frac{7499105}{20756736000}&\frac{1120692528}{20756736000}&\frac{-5002116375}{20756736000}&\frac{-2761387200}{20756736000}&\frac{-938225175}{20756736000}&\frac{-107689680}{20756736000}&\frac{-2333087}{20756736000}\\
0&0&0&0&0&\frac{19684969}{93405312000}&\frac{3322065240}{93405312000}&\frac{-10756569600}{93405312000}&\frac{-6696548000}{93405312000}&\frac{-2312543025}{93405312000}&\frac{-267240456}{93405312000}&\frac{-5811410}{93405312000}\\
0&0&0&0&0&\frac{3777757}{62270208000}&\frac{230157936}{62270208000}&\frac{5433977925}{62270208000}&\frac{-8687307200}{62270208000}&\frac{-1401356475}{62270208000}&\frac{-133910928}{62270208000}&\frac{-2655779}{62270208000}\\
0&0&0&0&0&\frac{6037445}{186810624000}&\frac{374302368}{186810624000}&\frac{10010400975}{186810624000}&\frac{-11184947200}{186810624000}&\frac{-2087463375}{186810624000}&\frac{-203179680}{186810624000}&\frac{-4060357}{186810624000}\\
0&0&0&0&0&\frac{2655779}{62270208000}&\frac{133910928}{62270208000}&\frac{1401356475}{62270208000}&\frac{8687307200}{62270208000}&\frac{-5433977925}{62270208000}&\frac{-230157936}{62270208000}&\frac{-3777757}{62270208000}\\
0&0&0&0&0&\frac{1953490}{93405312000}&\frac{99276552}{93405312000}&\frac{1058303025}{93405312000}&\frac{7438487200}{93405312000}&\frac{-3145766400}{93405312000}&\frac{-158085720}{93405312000}&\frac{-2647913}{93405312000}\\
0&0&0&0&0&\frac{2333087}{20756736000}&\frac{107689680}{20756736000}&\frac{938225175}{20756736000}&\frac{2761387200}{20756736000}&\frac{5002116375}{20756736000}&\frac{-1120692528}{20756736000}&\frac{-7499105}{20756736000}\\
0&0&0&0&0&\frac{9374963}{186810624000}&\frac{434726208}{186810624000}&\frac{3818940525}{186810624000}&\frac{11459388800}{186810624000}&\frac{23505908175}{186810624000}&\frac{-3442102272}{186810624000}&\frac{-28122007}{186810624000}\\
0&0&0&0&0&\frac{90441763}{62270208000}&\frac{3963034512}{62270208000}&\frac{31646886075}{62270208000}&\frac{77935000000}{62270208000}&\frac{71514207675}{62270208000}&\frac{33488665488}{62270208000}&\frac{-1784098013}{62270208000}\\
0&0&0&0&0&\frac{52752187}{93405312000}&\frac{2318706360}{93405312000}&\frac{18605988450}{93405312000}&\frac{46218028000}{93405312000}&\frac{43267295025}{93405312000}&\frac{22828945752}{93405312000}&\frac{-761223680}{93405312000}
\end{bmatrix}$$

$$\mathcal{y}_v=\begin{bmatrix}y_{n-5}\\y_{n-4}\\y_{n-3}\\y_{n-2}\\y_{n-1}\\y_n\\y_{n+1}\\y_{n+2}\\y_{n+3}\\y_{n+4}\\y_{n+5}\\y_{n+6}\end{bmatrix},\mathcal{y}'_v=\begin{bmatrix}y'_{n-5}\\y'_{n-4}\\y'_{n-3}\\y'_{n-2}\\y'_{n-1}\\y'_n\\y'_{n+1}\\y'_{n+2}\\y'_{n+3}\\y'_{n+4}\\y'_{n+5}\\y'_{n+6}\end{bmatrix},\mathcal{f}_v=\begin{bmatrix}f_{n-5}\\f_{n-4}\\f_{n-3}\\f_{n-2}\\f_{n-1}\\f_n\\f_{n+1}\\f_{n+2}\\f_{n+3}\\f_{n+4}\\f_{n+5}\\f_{n+6}\end{bmatrix},\qquad \mathcal{g}_v=\begin{bmatrix}g_{n-5}\\g_{n-4}\\g_{n-3}\\g_{n-2}\\g_{n-1}\\g_n\\g_{n+1}\\g_{n+2}\\g_{n+3}\\g_{n+4}\\g_{n+5}\\g_{n+6}\end{bmatrix}$$

$$For\ p=0 \quad then\ \ c_0=\sum_{i=0}^{k}\alpha_i=\bar{0}$$

$$For\ p=1 \quad then\ \ c_1=\sum_{i=0}^{k}(i\alpha_i-\beta_i)=\bar{0}$$

$$For\ p=2 \quad then\ \ c_2=\sum_{i=0}^{k}(\frac{i^2}{2!}\alpha_i-i\beta_i-\gamma_i)=\bar{0}$$

$$For\ p=3 \quad then\ \ c_3=\sum_{i=0}^{k}(\frac{i^3}{3!}\alpha_i-\frac{i^2}{2!}\beta_i-i\gamma_i-\delta_i)=\bar{0}$$

$$\vdots$$

$$For\ p=15 \quad then\ \ c_{15}=\sum_{i=0}^{k}(\frac{i^{15}}{15!}\alpha_i-\frac{i^{14}}{14!}\beta_i-\frac{i^{13}}{13!}\gamma_i-\frac{i^{12}}{12!}\delta_i)=\bar{0}$$

$$For\ p=16 \quad then\ \ c_{15}=\sum_{i=0}^{k}(\frac{i^{16}}{16!}\alpha_i-\frac{i^{15}}{15!}\beta_i-\frac{i^{14}}{14!}\gamma_i-\frac{i^{13}}{13!}\delta_i)\neq\bar{0}$$

$$c_{16}=\begin{bmatrix}\frac{42479}{876550474800}\\ \frac{53644141}{1795175372390400}\\ \frac{13339}{4382752374000}\\ \frac{2188393}{1282268123136000}\\ \frac{3901}{4382752374000}\\ \frac{293}{634023936000}\\ \frac{3901}{4382752374000}\\ \frac{3841247}{8975876861952000}\\ \frac{13339}{4382752374000}\\ \frac{11999521}{8975876861952000}\\ \frac{42479}{876550474800}\\ \frac{1588231}{85484541542400}\end{bmatrix}.$$

Where $\bar{0}$is the zero vector, represented by the expression $\bar{0}$ =$[0,0,0,0,0,0,0,0,0,0,0,0]^T$. we can get $c_1=c_2=c_3=c_4=c_5=c_6=c_7=c_8=c_9=c_{10}=c_{11}=c_{12}=c_{13}=c_{14}=c_{15}=\bar{0}, c_{16}\neq\bar{0}$ is found at$p=16$. as a result $p=14$ is the order of a six points implicit block method with a constant error:

$$c_{16}=[\frac{42479}{876550474800},\frac{53644141}{1795175372390400},\frac{13339}{4382752374000},\frac{2188393}{1282268123136000},\frac{3901}{4382752374000},\frac{293}{634023936000},\frac{3901}{4382752374000},\frac{3841247}{8975876861952000},\frac{13339}{4382752374000},\frac{11999521}{8975876861952000},\frac{42479}{876550474800},\frac{1588231}{85484541542400}]^T.$$

IV. ZERO-STABILITY OF NEW SIX-POINT BLOCK METHOD

Eq (55) results from changing Eq (11) into Eq (19) using the six-point implicit block method.

$$y'_{n+2}=y'_n+\frac{h}{2432430000}(783720817\,f_n+706775424\,f_{n+1}-457058625f_{n+2}+1387808000\,f_{n+3}+1957353375\,f_{n+4}+466919808\,f_{n+5}+19341201\,f_{n+6})+\frac{h^2}{243243000}(7057013\,g_n-117681984\,g_{n+1}-337970925g_{n+2}-336793600\,g_{n+3}-134615475g_{n+4}-16742592\,g_{n+5}-380629\,g_{n+6}) \quad (55)$$

Eq (56) results from changing Eq's (11) and (12) into Eq (20) using the six-point implicit block method.

$$y_{n+2}=y_n+2hy'_n+\frac{h^2}{3648645000}(2090453732f_n+94702128f_{n+1}-3226160625f_{n+2}+2970192000f_{n+3}+$$

$$4442896500f_{n+4} + 1070094768f_{n+5} + 44515753f_{n+6}) + \frac{h^3}{364864500}(18010763g_n - 290292912g_{n+1} - 711062100g_{n+2} - 758920000g_{n+3} - 307215675g_{n+4} - 38428272g_{n+5} - 876502g_{n+6}) \quad (56)$$

Eq (57) results from changing Eq (55) into Eq (27) using the six-point implicit block method.

$$y'_{n+3} = y'_n + \frac{h}{2562560000}(826473395f_n + 775497456f_{n+1} + 688759875f_{n+2} + 2699264000f_{n+3} + 2168488125f_{n+4} + 508254480f_{n+5} + 20942669f_{n+6}) + \frac{3h^2}{256256000}(2483365g_n - 41010192g_{n+1} - 111229875g_{n+2} - 130187200g_{n+3} - 49194675g_{n+4} - 6063120g_{n+5} - 137307g_{n+6}) \quad (57)$$

Eq (58) results from changing Eq's (55) and (56) into Eq (28) using the six-point implicit block method.

$$y_{n+3} = y_n + 3hy'_n + \frac{h^2}{622702080000}(+557510320665f_n + 168880769856f_{n+1} + 639159878375f_{n+2} + 949064688000f_{n+3} + 1271957286375f_{n+4} + 304163760960f_{n+5} + 12618815769f_{n+6}) + \frac{9h^3}{256256000}(2232485g_n - 36371808g_{n+1} - 93524625g_{n+2} - 100352000g_{n+3} - 40049775g_{n+4} - 4989600g_{n+5} - 113573g_{n+6}) \quad (58)$$

Eq (59) results from changing Eq (57) into Eq (35) using the six-point implicit block method.

$$y'_{n+4} = y'_n + \frac{2h}{152026875}(24532563f_n + 23488800f_{n+1} + 23587500f_{n+2} + 116768000f_{n+3} + 99037875f_{n+4} + 15993312f_{n+5} + 645700f_{n+6}) + \frac{4h^2}{30405375}(221317g_n - 3633120g_{n+1} - 9727200g_{n+2} - 10524800g_{n+3} - 5041125g_{n+4} - 567648g_{n+5} - 12680g_{n+6}) \quad (59)$$

Eq (60) results from changing Eq's (57) and (58) into Eq (36) using the six-point implicit block method.

$$y_{n+4} = y_n + 4hy'_n + \frac{h^2}{1824322500}(2221907120f_n + 1052631552f_{n+1} + 2401664625f_{n+2} + 5328384000f_{n+3} + 5501094000f_{n+4} + 1262906880f_{n+5} + 52152448f_{n+6}) + \frac{16h^3}{91216125}(612865g_n - 10013616g_{n+1} - 26085150g_{n+2} - 28328000g_{n+3} - 11494350g_{n+4} - 1413360g_{n+5} - 32066g_{n+6}) \quad (60)$$

Eq (61) results from changing Eq (59) into Eq (43) using the six-point implicit block method.

$$y'_{n+5} = y'_n + \frac{5h}{996323328}(64425317f_n + 64519824f_{n+1} + 75895125f_{n+2} + 323584000f_{n+3} + 343912875f_{n+4} + 121889136f_{n+5} + 2097051f_{n+6}) + \frac{25h^2}{498161664}(582409g_n - 9420624g_{n+1} - 24598575g_{n+2} - 24939200g_{n+3} - 8412975g_{n+4} - 2563920g_{n+5} - 40439g_{n+6}) \quad (61)$$

Eq (62) results from changing Eq's (59) and (60) into Eq (44) using the six-point implicit block method.

$$y_{n+5} = y_n + 5hy'_n + \frac{h^2}{2988969984}(4605800785f_n + 2668106400f_{n+1} + 4958488443f_{n+2} + 13448400000f_{n+3} + 13852959375f_{n+4} + 3000764640f_{n+5} + 113346725f_{n+6}) + \frac{25h^3}{1494484992}(8169863g_n - 133433280g_{n+1} - 348798375g_{n+2} - 376144000g_{n+3} - 152650125g_{n+4} - 20385792g_{n+5} - 444955g_{n+6}) \quad (62)$$

Eq (63) results from changing Eq (61) into Eq (51) using the six-point implicit block method.

$$y'_{n+6} = y'_n + \frac{h}{10010000}(3310219f_n + 5014656f_{n+1} + 11161125f_{n+2} + 21088000f_{n+3} + 11161125f_{n+4} + 5014656f_{n+5} + 3310219f_{n+6}) + \frac{3h^2}{1001000}(10237g_n - 136512g_{n+1} - 242325g_{n+2} + 242325g_{n+4} + 136512g_{n+5} - 10237g_{n+6}) \quad (63)$$

Eq (64) results from changing Eq's (61) and (62) into Eq (52) using the six-point implicit block method.

$$y_{n+6} = y_n + 6hy'_n + \frac{h^2}{1216215000}(2270816046f_n + 1563617520f_{n+1} + 2832297625f_{n+2} + 7686576000f_{n+3} + 7454936250f_{n+4} + 2092066704f_{n+5} + 142333605f_{n+6}) + \frac{9h^3}{500500}(9257g_n - 149040g_{n+1} - 382050g_{n+2} - 392000g_{n+3} - 139725g_{n+4} - 12528g_{n+5} - 980g_{n+6}) \quad (64)$$

$$\begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix} \begin{bmatrix} y'_{n+1}\\ y_{n+1}\\ y'_{n+2}\\ y_{n+2}\\ y'_{n+3}\\ y_{n+3}\\ y'_{n+4}\\ y_{n+4}\\ y'_{n+5}\\ y_{n+5}\\ y'_{n+6}\\ y_{n+6} \end{bmatrix} + \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix} \begin{bmatrix} y'_{n-5}\\ y_{n-5}\\ y'_{n-4}\\ y_{n-4}\\ y'_{n-3}\\ y_{n-3}\\ y'_{n-2}\\ y_{n-2}\\ y'_{n-1}\\ y_{n-1}\\ y'_n\\ y_n \end{bmatrix}$$

$$+h\begin{bmatrix}0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&2&0\\0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&3&0\\0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&4&0\\0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&5&0\\0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0&6&0\end{bmatrix}\begin{bmatrix}y'_{n-5}\\y_{n-5}\\y'_{n-4}\\y_{n-4}\\y'_{n-3}\\y_{n-3}\\y'_{n-2}\\y_{n-2}\\y'_{n-1}\\y_{n-1}\\y'_{n}\\y_{n}\end{bmatrix}$$

$$+h\begin{bmatrix}0&0&0&0&0&\frac{199368819177}{622702080000}&\frac{-68951829552}{622702080000}&\frac{-380416470375}{622702080000}&\frac{300642304000}{622702080000}&\frac{457138998375}{622702080000}&\frac{110327270448}{622702080000}&\frac{4592987927}{622702080000}\\0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&\frac{783720817}{2432430000}&\frac{706775424}{2432430000}&\frac{-457058625}{2432430000}&\frac{1387808000}{2432430000}&\frac{1957353375}{2432430000}&\frac{466919808}{2432430000}&\frac{19341201}{2432430000}\\0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&\frac{826473395}{2562560000}&\frac{775497456}{2562560000}&\frac{688759875}{2562560000}&\frac{2699264000}{2562560000}&\frac{2168488125}{2562560000}&\frac{508254480}{2562560000}&\frac{20942669}{2562560000}\\0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&\frac{49065126}{152026875}&\frac{46977600}{152026875}&\frac{47175000}{152026875}&\frac{233536000}{152026875}&\frac{198075750}{152026875}&\frac{31986624}{152026875}&\frac{1291400}{152026875}\\0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&\frac{322126585}{996323328}&\frac{322599120}{996323328}&\frac{379475625}{996323328}&\frac{1617920000}{996323328}&\frac{1719564375}{996323328}&\frac{609445680}{996323328}&\frac{10485255}{996323328}\\0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&\frac{3310219}{10010000}&\frac{5014656}{10010000}&\frac{11161125}{10010000}&\frac{21088000}{10010000}&\frac{11161125}{10010000}&\frac{5014656}{10010000}&\frac{3310219}{10010000}\\0&0&0&0&0&0&0&0&0&0&0&0\end{bmatrix}\begin{bmatrix}f_{n-5}\\f_{n-4}\\f_{n-3}\\f_{n-2}\\f_{n-1}\\f_{n}\\f_{n+1}\\f_{n+2}\\f_{n+3}\\f_{n+4}\\f_{n+5}\\f_{n+6}\end{bmatrix}$$

$$+h^2\begin{bmatrix}0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&\frac{469983596501}{1868106240000}&\frac{-402081396000}{1868106240000}&\frac{-709748678625}{1868106240000}&\frac{534196176000}{1868106240000}&\frac{831579916875}{1868106240000}&\frac{201705150624}{1868106240000}&\frac{8418354625}{1868106240000}\\0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&\frac{2090453732}{3648645000}&\frac{94702128}{3648645000}&\frac{-3226160625}{3648645000}&\frac{2970192000}{3648645000}&\frac{4442896500}{3648645000}&\frac{1070094768}{3648645000}&\frac{44515753}{3648645000}\\0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&\frac{557510320665}{622702080000}&\frac{168880769856}{622702080000}&\frac{639159878375}{622702080000}&\frac{949064688000}{622702080000}&\frac{1271957286375}{622702080000}&\frac{304163760960}{622702080000}&\frac{12618815769}{622702080000}\\0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&\frac{2221907120}{1824322500}&\frac{1052631552}{1824322500}&\frac{2401664625}{1824322500}&\frac{5328384000}{1824322500}&\frac{5501094000}{1824322500}&\frac{1262906880}{1824322500}&\frac{52152448}{1824322500}\\0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&\frac{4605800785}{2988969984}&\frac{2668106400}{2988969984}&\frac{4958488443}{2988969984}&\frac{13448400000}{2988969984}&\frac{13852959375}{2988969984}&\frac{3000764640}{2988969984}&\frac{113346725}{2988969984}\\0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&\frac{2270816046}{1216215000}&\frac{1563617520}{1216215000}&\frac{2832297625}{1216215000}&\frac{7686576000}{1216215000}&\frac{7454936250}{1216215000}&\frac{2092066704}{1216215000}&\frac{142333605}{1216215000}\end{bmatrix}\begin{bmatrix}f_{n-5}\\f_{n-4}\\f_{n-3}\\f_{n-2}\\f_{n-1}\\f_{n}\\f_{n+1}\\f_{n+2}\\f_{n+3}\\f_{n+4}\\f_{n+5}\\f_{n+6}\end{bmatrix}$$

$$+h^2\begin{bmatrix}0&0&0&0&0&\frac{1784098013}{62270208000}&\frac{-33488665488}{62270208000}&\frac{-71514207675}{62270208000}&\frac{-77935000000}{62270208000}&\frac{-31646886075}{62270208000}&\frac{-3963034512}{62270208000}&\frac{-90441763}{62270208000}\\0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&\frac{7057013}{243243000}&\frac{-117681984}{243243000}&\frac{-337970925}{243243000}&\frac{-336793600}{243243000}&\frac{-134615475}{243243000}&\frac{-16742592}{243243000}&\frac{-380629}{243243000}\\0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&\frac{7450095}{256256000}&\frac{-123030576}{256256000}&\frac{-333689625}{256256000}&\frac{-390561600}{256256000}&\frac{-147584025}{256256000}&\frac{-18189360}{256256000}&\frac{-411921}{256256000}\\0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&\frac{885268}{30405375}&\frac{-14532480}{30405375}&\frac{-38908800}{30405375}&\frac{-42099200}{30405375}&\frac{-20164500}{30405375}&\frac{-2270592}{30405375}&\frac{-50720}{30405375}\\0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&\frac{14560225}{498161664}&\frac{-235515600}{498161664}&\frac{-614964375}{498161664}&\frac{-623480000}{498161664}&\frac{-210324375}{498161664}&\frac{-64098000}{498161664}&\frac{-1010975}{498161664}\\0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&\frac{30711}{1001000}&\frac{-409536}{1001000}&\frac{-726975}{1001000}&\frac{0}{1001000}&\frac{726975}{1001000}&\frac{409536}{1001000}&\frac{-30711}{1001000}\\0&0&0&0&0&0&0&0&0&0&0&0\end{bmatrix}\begin{bmatrix}g_{n-5}\\g_{n-4}\\g_{n-3}\\g_{n-2}\\g_{n-1}\\g_{n}\\g_{n+1}\\g_{n+2}\\g_{n+3}\\g_{n+4}\\g_{n+5}\\g_{n+6}\end{bmatrix}$$

$$+h^3\begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&\frac{3829846679}{186810624000}&\frac{-54808104960}{186810624000}&\frac{-128008032975}{186810624000}&\frac{-141368944000}{186810624000}&\frac{-57728681325}{186810624000}&\frac{-7251690816}{186810624000}&\frac{-165820915}{186810624000}\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&\frac{18010763}{364864500}&\frac{-290292912}{364864500}&\frac{-711062100}{364864500}&\frac{-758920000}{364864500}&\frac{-307215675}{364864500}&\frac{-38428272}{364864500}&\frac{-876502}{364864500}\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&\frac{20092365}{256256000}&\frac{-327346272}{256256000}&\frac{-841721625}{256256000}&\frac{-903168000}{256256000}&\frac{-360447975}{256256000}&\frac{-44906400}{256256000}&\frac{-1022157}{256256000}\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&\frac{9805840}{91216125}&\frac{-160217856}{91216125}&\frac{-417362400}{91216125}&\frac{-453248000}{91216125}&\frac{-183909600}{91216125}&\frac{-22613760}{91216125}&\frac{-513056}{91216125}\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&\frac{204246575}{1494484992}&\frac{-3335832000}{1494484992}&\frac{-8719959375}{1494484992}&\frac{-9403600000}{1494484992}&\frac{-3816253125}{1494484992}&\frac{-509644800}{1494484992}&\frac{-11123875}{1494484992}\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&\frac{83313}{500500}&\frac{-1341360}{500500}&\frac{-3438450}{500500}&\frac{-3528000}{500500}&\frac{-1257525}{500500}&\frac{-112752}{500500}&\frac{-8820}{500500} \end{bmatrix}\begin{bmatrix} g_{n-5}\\ g_{n-4}\\ g_{n-3}\\ g_{n-2}\\ g_{n-1}\\ g_n\\ g_{n+1}\\ g_{n+2}\\ g_{n+3}\\ g_{n+4}\\ g_{n+5}\\ g_{n+6} \end{bmatrix}$$

The six-point implicit block method's first characteristic polynomial is stated as

$$\wp(\mathcal{Z}) = \det\left[\mathcal{Z}\mathcal{B}^{(0)} - \mathcal{B}^{(1)}\right] = 0$$

$$\mathcal{B}^{(0)} = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix}$$

$$\mathcal{B}^{(1)} = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix}$$

$$\wp(\mathcal{E}) = det\begin{bmatrix} \mathcal{Z}&0&0&0&0&0&0&0&0&0&-1&0\\ 0&\mathcal{Z}&0&0&0&0&0&0&0&0&0&-1\\ 0&0&\mathcal{Z}&0&0&0&0&0&0&0&-1&0\\ 0&0&0&\mathcal{Z}&0&0&0&0&0&0&0&-1\\ 0&0&0&0&\mathcal{Z}&0&0&0&0&0&-1&0\\ 0&0&0&0&0&\mathcal{Z}&0&0&0&0&0&-1\\ 0&0&0&0&0&0&\mathcal{Z}&0&0&0&-1&0\\ 0&0&0&0&0&0&0&\mathcal{Z}&0&0&0&-1\\ 0&0&0&0&0&0&0&0&\mathcal{Z}&0&-1&0\\ 0&0&0&0&0&0&0&0&0&\mathcal{Z}&0&-1\\ 0&0&0&0&0&0&0&0&0&0&\mathcal{Z}-1&0\\ 0&0&0&0&0&0&0&0&0&0&0&\mathcal{Z}-1 \end{bmatrix} = 0$$

$$\mathcal{Z}^{10}(\mathcal{Z}-1)^2 = 0, \mathcal{Z} = 0,0,0,0,0,0,0,0,0,0,1,1\ , |\mathcal{Z}| \le 1$$

According to [3] the six-point implicit block technique is zero-stable since the initial characteristic polynomial $\wp(\mathcal{Z}) = 0$ satisfies; $|\mathcal{Z}_j| \le 1; j = 0, \dots, n$. The six-point implicit block technique is also consistent since their order is less than 1. [2] suggests that since the six-point block method is consistent and zero-stable, then it represents convergence.

V. TEST OF PROBLEMS

Problem 1 : [11] . Suppose that a 64 lb weight stretches a spring 6 inches in equilibrium and a dash pot provides a damping force of 32 lb for each $\mathrm{f\ t/sce}$ of velocity, find the displacement y for > 0 if the motion is critically damped and the initial conditions are $y(0) = 0$ and $y'(0) = 20$. It follows from the equation of motion $Ay'' + By' + Cy = D$ Substituting $A = 2, B = 32Ib\ , C = 128\frac{Ib}{f}t\ , D = 0$, $y''(t) + 64y'(t) + 64y(t) = 0$ the exact solution is: $y(t) = e^{-8t}(1 + 28t)$.

Problem 2: [11]. An object stretches a spring 6 inches in equilibrium find the displacement of the object for , if it’s initially displaced 18 inches above equilibrium and given a downward velocity of 3 $\mathrm{f/st}$. From Newton’s second law of motion, we have $y'' + 64y = 0$

The initial upward displacement of 18 inches is positive and must be expressed in feet. The initial downward velocity is negative thus $y(0) = 1.5\ , y'(0) = -3$ the exact solution is: $y(t) = -0.375 \sin 8t + 1.5\ \cos 8t$.

Problem 3 :[7].

$y'' = x(y')^2\ \ , y(0) = 1\ , y'(0) = 0.5\ \ , [0,1]$, exact solution: $y(x) = 1 + 0.5 \log\frac{x+2}{2-x}$.

Problem 4 :[4].

$y'' = -y' + 2y\ \ , y(0) = 0\ , y' = 1\ , [0,10]$, exact solution: $y(x) = \frac{e^x - e^{-2x}}{3}$.

VI. NUMERICAL EXPERIMENTS

Notations used are:

- h: step size.

- Error: Maximum error $|y(x_i) - y_i|$ where y_i the computed solution and $y(x_i)$ is the exact solution.
- SPI : 6-point implicit extra derivative block method in this paper .
- OSH: One-eight step hybrid block method for solving second order initial value problems of directly proposed by Emmanuelet al.[11].
- 4PIA:4-point block method proposed by Omar and Adeyeye [10].
- 4PIM: 4-point implicit block method proposed by Mukhtar et al. [6].
- 3PIM : 3-point implicit block method with addition second derivative proposed by Turki et al. [17].

TABLE 1 : Performance of Problem 1.

t	Method	ERR	Time
3.0	SPI	1.302857e-13	0.2273
	OSH	7.627261e-11	0.9991
3.1	NSP	2.456558e-13	0.3144
	OSH	3.843052e-11	1.0304
3.2	SPI	5.356783e-13	0.2682
	OSH	1.927790e-11	1.0553
3.3	SPI	9.032457e-13	0.3325
	OSH	9.630866e-12	1.0833
3.4	SPI	7.935732e-14	0.3967
	OSH	4.793171e-12	1.1105
3.5	SPI	1.957483e-14	0.2899
	OSH	2.377102e-12	1.1383
3.6	SPI	9.139554e-14	0.3213
	OSH	1.175015e-12	1.1648
3.7	SPI	6.937136e-14	0.2732
	OSH	5.790320e-13	1.2014
3.8	SPI	4.193425e-15	0.2255
	OSH	2.845176e-13	1.2608
3.9	SPI	1.536280e-15	0.1861
	OSH	1.394245e-13	1.2863

TABLE 2 : Performance of Problem 2.

t	Method	ERR	Time
0.1	SPI	2.137583e-09	0.0294
	OSH	3.349623e-07	0.0442
0.2	SPI	3.543729e-09	0.0475
	OSH	1.637121e-06	0.0736
0.3	SPI	1.458392e-09	0.0533
	OSH	3.271566e-06	0.0978
0.4	SPI	2.984731e-09	0.0315
	OSH	3.597893e-06	0.1227
0.5	SPI	4.278190e-09	0.0387
	OSH	1.358909e-06	0.1502
0.6	SPI	1.342572e-09	0.0130
	OSH	2.914248e-06	0.1756
0.7	SPI	3.569042e-09	0.0432
	OSH	6.722640e-06	0.2002
0.8	SPI	1.895235e-08	0.0298
	OSH	7.058882e-06	0.2254
0.9	SPI	8.103405e-08	0.0502
	OSH	2.654331e-06	0.2536
1.0	SPI	5.153423e-08	0.0635
	OSH	4.605548e-06	0.2844

TABLE 3 : Performance of Problem 3.

h	Method	ERR	Time
0.05	SPI	6.234762 e-14	0.003
	3PIM	5.564653 e-11	0.005
	4PIA	4.447243 e-09	0.003
	4PIM	4.447340 e-09	0.002
0.025	SPI	3.923654 e-15	0.008
	3PIM	1.368330 e-12	0.015
	4PIA	7.217795 e-11	0.013
	4PIM	7.217813 e-11	0.012
0.0125	SPI	7.139034 e-16	0.009
	3PIM	2.121577 e-14	0.031
	4PIA	1.138397 e-13	0.029
	4PIM	1.138136 e-12	0.028
0.00625	SPI	1.542780 e-17	0.023
	3PIM	4.116625 e-16	0.062
	4PIA	1.733290 e-15	0.058
	4PIM	1.698450 e-14	0.057

TABLE 4 : Performance of Problem 4.

h	Method	ERR	Time
0.05	SPI	2.824357 e-13	0.023
	3PIM	1.100932 e-09	0.072
	4PIA	3.747866e-09	0.069
	4PIM	3.747862 e-09	0.068
0.025	SPI	5.234536 e-14	0.040
	3PIM	1.718955 e-12	0.119
	4PIA	5.905707 e-11	0.116
	4PIM	5.905831e-11	0.115
0.0125	SPI	2.146583 e-16	0.072
	3PIM	6.197166 e-14	0.148
	4PIA	9.799913 e-13	0.145
	4PIM	9.826201e-13	0.144
0.00625	SPI	4.729823e-17	0.096
	3PIM	2.981210 e-16	0.225
	4PIA	1.799913 e-15	0.223
	4PIM	2.052406 e-15	0.222

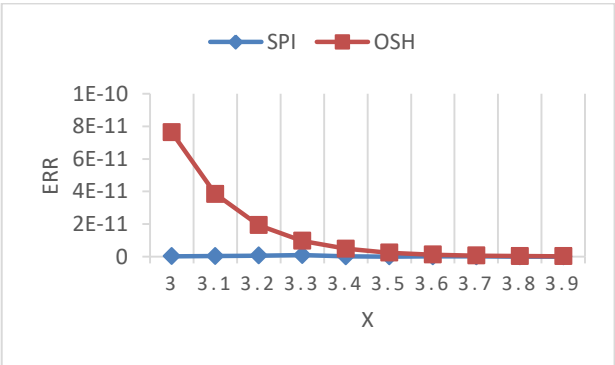

FIGURE 1: The efficiency curves for Problem1.

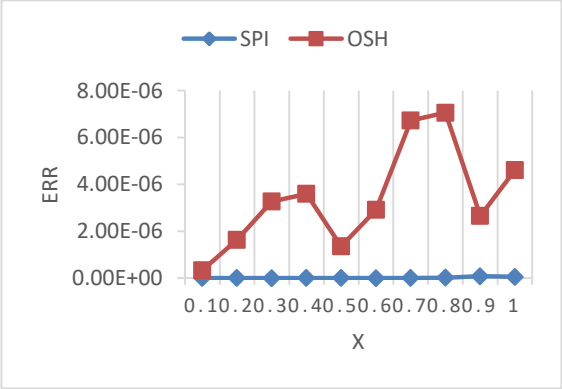

FIGURE 2: The efficiency curves for Problem 2.

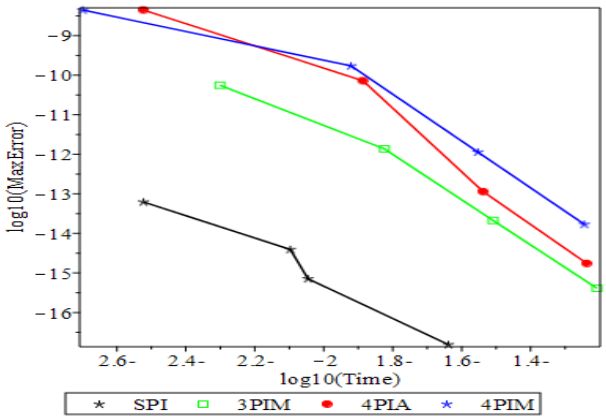

FIGURE 3: The efficiency curves for Problem 3.

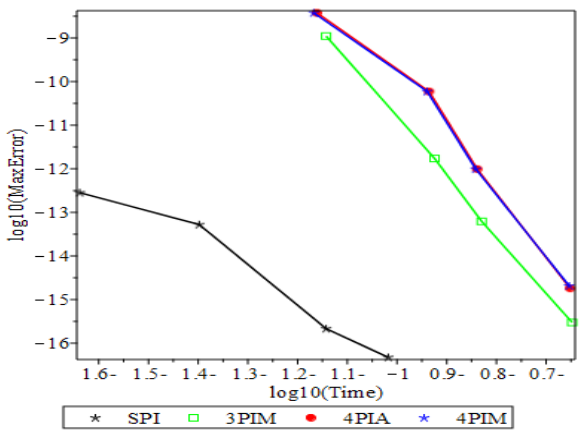

FIGURE 4: The efficiency curves for Problem 4.

VII. CONCLUSION

We observe that the approach developed in this study yields numerical results more quickly and accurately than earlier approaches, demonstrating the method's efficiency.

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