Strong Solutions of the Neuronal Responses System

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Abstract: *We explore a connected reaction-diffusion system involving neural responses within limited convex open domains* $\boldsymbol{\Omega} \subset \mathbb{R}^{\boldsymbol{\eta}}(\boldsymbol{\eta} = \mathbf{1}, \mathbf{2}, \mathbf{3})$. *Utilizing the Classical Faedo-Galerkin technique and employing compactness arguments, we establish the existence, uniqueness, and continuous dependence on initial data for strong solutions.*

Keywords*: *Continuous dependence, Existence, Neumann boundary, Strong solution, uniquenes.

I. INTRODUCTION

Over the course of several years, extensive research has been dedicated to reaction-diffusion systems, nonlinear parabolic partial differential equations with broad applications in chemistry, ecology, physics, biology, and various other domains (see [2–7] for additional information). This investigation focuses specifically on the reaction-diffusion system describing the transmission of action potentials in cardiac muscle cells, analogous to the Hodgkin-Huxley model [8]. Fitzhugh and Nagumo, along with other researchers [9, 10], employed local dynamics to streamline the Hodgkin–Huxley system. Originally composed of four ordinary differential equations explaining potential changes across the membrane of a nerve cell in the giant axon of the squid, the Hodgkin-Huxley system is condensed to a system of two partial differential equations referred to as the FHN equations [8, 11]. Over the years, FHN equations have found applications in various fields [8, 12-17], including characterizing CO oxidation on Pt (110) [12], exploring Ca^(+2) waves on Xenopus oocytes [8] and Medaka eggs [11], and analyzing reentry in heart tissue [18]. For additional applications, readers are directed to [19, 21].

This research focuses on examining the neuronal responses system presented as four coupled reaction-diffusion equations with Neumann boundary conditions, taking the following form: (M) Find u_1, v_1, u_2 and v_2 such that.

$$\varepsilon \frac{\partial u_1}{\partial t} = d_1 \Delta \mathrm{u}_1 + a f(u_1) - v_1 + \alpha(u_2 - u_1), in\ Q_T, \tag{1.1}$$

$$\frac{\partial v_1}{\partial t} = d_2 \Delta \mathrm{v}_1 + u_1 - \delta v_1 + \beta(v_2 - v_1), in\ Q_T, \tag{1.2}$$

$$\varepsilon \frac{\partial u_2}{\partial t} = d_1 \Delta \mathrm{u}_2 + a f(u_2) - v_2 - \alpha(u_2 - u_1), in\ Q_T, \tag{1.3}$$

$$\frac{\partial v_2}{\partial t} = d_2 \Delta \mathrm{v}_2 + u_2 - \delta v_2 - \beta(v_2 - v_1), in\ \mathrm{Q_T}, \tag{1.4}$$

$$\frac{\partial u_1}{\partial v} = 0, \frac{\partial v_1}{\partial v} = 0, \frac{\partial u_2}{\partial v} = 0, \frac{\partial v_2}{\partial v} = 0, on\ S_T \tag{1.5}$$

$$u_1(.,0) = u_{1,0}, v_1(.,0) = v_{1,0}, u_2(.,0) = u_{2,0}, v_2(.,0) = v_{2,0}\ in\ \Omega, \tag{1.6}$$

where $Q_T = \Omega \times (0, T)$, Ω is an open bounded convex domain in $\mathbb{R}^{\eta}(\eta = 1,2,3)$, with smooth boundary $\partial\Omega, S_T = \partial\Omega \times (0, T), v$ denotes the exterior unit normal to $\partial\Omega, d_1$ and d_2 are known as the diffusion coefficients for u_1, v_1, u_2 and v_2, respectively, $f(u) = 3u - u^3$. ε and δ are small parameters; and α, β measure the coupling strength, which represents the interactions between neurons. [20] It is assumed that all parameters are real and finite. Although elliptic issues with Neumann boundary conditions were initially discussed in [23], the realization that this problem could be categorized as a variational problem in a Hilbert space was not widely recognized until later. [24] demonstrated that elliptic boundary value problems with Neumann boundary conditions can be interpreted as weak forms in Hilbert spaces.

Investigating the neuronal responses systems with Neumann boundary conditions is imperative for various reasons. Despite the considerable attention directed towards systems incorporating both Dirichlet and Neumann boundary conditions, there has been a relatively limited focus on Neumann boundary conditions specifically. Sherratt [25] introduces the notion of "oscillatory" reaction-diffusion equations, particularly in ecological applications, where the Dirichlet condition is often utilized as a fundamental approximation to a more realistic Neumann boundary condition. Al-Ofl [26] conducts a mathematical analysis of reaction-diffusion equations with Neumann boundary conditions, providing evidence for the existence, uniqueness, and continuous dependence on initial data for both weak and strong solutions. Given the significance of this problem, recent research has increasingly delved into the investigation of reaction-diffusion systems with Neumann boundary conditions, as demonstrated by studies such as [16-17].

The principal aim of this investigation is to ascertain the existence and uniqueness of a strong solution for the system (1.1)-(1.6) through the utilization of the Faedo-Galerkin method [1] and the Alaoglu compactness theorem [17]. The approach involves substituting the infinite-dimensional dynamical system with a finite-dimensional equivalent through the utilization of a truncated Eigenfunction expansion. The existence, uniqueness, and continuous dependence of strong solutions on initial data in $H^1\,(\Omega)$ have also been established, with these outcomes attributed to the low regularity of the initial data. The structure of this article is as follows: Section 2 provides essential notation, while Section 3 delves into the discussion of the existence and uniqueness of strong solutions, as well as their continuous dependence.

II. NOTATION AND PRELIMINARIES

Throughout this work, Ω represents a bounded domain in $\mathbb{R}^\eta$, $\eta = 1,2,3$ with a Lipschitz boundary $\partial\Omega$. We employ the typical Sobolev spaces $W^{l,\psi}\,(\Omega), l \in N, \psi \in [1,\infty]$, with the corresponding norms and semi-norms, denoted by $\|\cdot\|_{l,\psi}$ and $|\cdot|_{l,\psi}$, respectively. For $\psi = 2, W^{l,2}\,(\Omega)$ is indicated by $H^l\,(\Omega)$ with norm $\|\cdot\|_l$ and seminorm $|\cdot|_l$. If $l = 0, W^{0,2}\,(\Omega) = L^2\,(\Omega)$. The $(\cdot,\cdot)$ symbol represents the $L^2\,(\Omega)$ inner product over Ω with the norm $\|\cdot\|_0 = |\cdot|_0$. Additionally, $\langle\cdot,\cdot\rangle$ represents the duality pairing between $\left(H^1\,(\Omega)\right)'$ and $H^1\,(\Omega)$, where $\left(H^1\,(\Omega)\right)'$ is $H^1\,(\Omega)'s$ dual space. The norm $for\left(H^1\,(\Omega)\right)'$ is given by:

$$\|\varphi\|\left(H^1(\Omega)\right)' \coloneqq \sup_{\gamma\neq 0}\frac{|\langle\varphi,\gamma\rangle|}{\|\gamma\|_1} \equiv \sup_{\|\gamma\|_1=1}\|\langle\varphi,\gamma\rangle\|\,, \gamma \in H^1(\Omega). \tag{2.1}$$

We present the function spaces that depend on both time and space, denoted as $L^\psi\,(0,T;\,X)\,(1 \le \psi \le \infty)$, where X represents a Banach space. These spaces encompass all functions φ for which, almost everywhere in the interval $(0,T), \varphi$ belongs to X, and the associated norm is finite:

$$\|\varphi(t)\|_{L^\psi(0,T;X)} = \left(\int_0^T |\varphi(t)|_X^\psi dt\right)^{\frac{1}{\psi}},$$
$$\|\varphi(t)\|_{L^\infty(0,T;X)} = ess\sup_{t\in(o,T)}\|\varphi(t)\|x.$$

We also define $L^\psi(\Omega_T)$ as $L^\psi\left(0,T;L^\psi(\Omega)\right)$, with ψ ranging from 1 to ∞. In addition, we define $C([0,T];X)$ as the space of continuous functions from $[0,T]$ to X, consisting of $\varphi(t)\colon [0,T] \to X$ such that $\varphi(t) \to \varphi(t_0)$ in X as $t \to t_0$. We recall that $C([0,T];X)$ is a Banach space with a corresponding norm (see [27], page 43).

We also introduce the space $L^\psi(\Omega_T) = L^\psi\left(0,T;L^\psi(\Omega)\right)$, where $\psi \in [1,\infty]$. Additionally, we define $C([0,T];X)$, representing the space of continuous functions from $[0,T]$ into X, denoted as $\varphi(t)\colon [0,T] \to X$, such that $\varphi(t) \to \varphi(t_0)$ in X as $t \to t_0$. It is noteworthy that $C([0,T];X)$ forms a Banach space with the associated norm, as detailed in [27] on page 43.

We also bring to mind well-established Sobolev results:

$$H^1(\Omega)\overset{c}{\hookrightarrow}L^\psi(\Omega) \hookrightarrow \left(H^1(\Omega)\right)', \text{for } \psi \in \begin{cases}[1,\infty]\ if\ \eta = 1,\\ [1,\infty)\ if\ \eta = 2, (2.2)\\ [1,6]\ \ if\ \eta = 3,\end{cases}$$

Where $\hookrightarrow$ represents continuous embedding. In accordance with the Rellich-Kondrachov theorem, as detailed in [28] on page 114 and [29] on page 8, the embedding in equation (2.2) is compact. In the case of $\eta = 3$, the range for ψ is adjusted to $\psi \in [1,6]$.

The following Hӧlders inequality is also required: for $1 \le p_1, p_2 \le \infty$ such that $\frac{1}{p_1} + \frac{1}{p_2} = 1$ if $\varphi \in L^{P_1}(\Omega)$ and $\varrho \in L^{P_2}(\Omega)$ then $\varphi\varrho \in L^1(\Omega)$ and

$$\|\varphi\varrho\|_{0,1} = \int_\Omega |\varphi\varrho|dx \le \left(\int_\Omega |\varphi|^{P_1}dx\right)^{\frac{1}{P_1}}\left(\int_\Omega |\varrho|^{P_2}dx\right)^{\frac{1}{P_2}}$$
$$= \|\varphi\|_{0,P_1}\|\varrho\|_{0,P_2} \tag{2.3}$$

This previous inequality can be generalized by applying it twice to obtain

$$\|\varphi\varrho\vartheta\|_{0,1} = \int_\Omega |\varphi\varrho\vartheta|dx \le \left(\int_\Omega |\varphi|^{P_1}dx\right)^{\frac{1}{P_1}}\left(\int_\Omega |\varrho|^{P_2}dx\right)^{\frac{1}{P_2}}$$
$$\left(\int_\Omega |\vartheta|^{P_3}dx\right)^{\frac{1}{P_3}} = \|\varphi\|_{0,P_1}\|\varrho\|_{0,P_2}\|\vartheta\|_{0,P_3} \tag{2.4}$$

for $1 \le p_1, p_2, p_3 \le \infty$ such that $\frac{1}{P_1} + \frac{1}{P_2} + \frac{1}{P_3} = 1$.

The following Young's inequality is widely applied:

$$\ell_1\ell_2 \le \psi^{\frac{\psi_1}{\psi_2}}\frac{\ell_1^{\psi_1}}{\psi_1} + \psi^{-1}\frac{\ell_2^{\psi_2}}{\psi_2}\ , \qquad \frac{1}{\psi_1} + \frac{1}{\psi_2} = 1, \tag{2.5}$$

valid for any $\ell_1, \ell_2 \ge 0, \psi > 0$ and $\psi_1, \psi_2 > 1$. Another valuable consequence derived from Young's inequality is as follows:

$$\ell_1\ell_2 \ge -\psi\frac{\ell_1^2}{2} - \psi^{-1}\frac{\ell_2^2}{2}, \forall\ell_1, \ell_2 \in \mathbb{R}, \forall\psi > 0. \tag{2.6}$$

The necessity of the differential form of Gronwall's lemma is also emphasized: let $\Phi_1(t) \in W^{1,1}(0,T)$ and $\Phi_2(t), \Phi_3(t),\ \Phi_4(t) \in L^1(0,T)$, such that

$$\frac{d\Phi_1(t)}{dt} + \Phi_2(t) \le \Phi_3(t)\Phi_1(t) + \Phi_4(t), \qquad a.e.\, t \in [0,T],$$

then

$$\Phi_1(T) + \int_0^T \Phi_2(\theta)dt \le e^{\int_0^T \Phi_3 dt}\left[\Phi_1(0) + \int_0^T \Phi_4 dt\right]. \tag{2.7}$$

Theorem 2.1. Assuming $\Omega \subset \mathbb{R}^\eta$, where $\eta = 1,2,3$, is an open bounded convex domain. Let $u_{1,0}(.), v_{1,0}(.), u_{2,0}(.), v_{2,0}(.) \in L^2(\Omega)$. Then, System (M) has a unique weak solution u_1, v_1, u_2, v_2 satisfying:

$$u_1(x,t), u_2(x,t) \in L^4(\Omega_T) \cap L^2\big(0,T;H^1(\Omega)\big) \cap L^\infty\big(0,T;L^2(\Omega)\big) \cap C\big([0,T];L^2(\Omega)\big), \tag{2.8}$$

$$v_1(x,t), v_2(x,t) \in L^2(\Omega_T) \cap L^2\big(0,T;H^1(\Omega)\big) \cap L^\infty\big(0,T;L^2(\Omega)\big) \cap C\big([0,T];L^2(\Omega)\big), \tag{2.9}$$

and System (M) hold as equalities in $L^{\frac{4}{3}}(0,T;(H^1)')$, $L^2(0,T;(H^1)')$, $L^{\frac{4}{3}}(0,T;(H^1)')$, and $L^2(0,T;(H1)')$ respectively.

A. Local existence

We employ the Faedo-Galerkin method [1], to establish existence. Let $\{z_i\}_{i=1}^\infty$ denote an orthogonal basis for $H^1(\Omega)$ and an orthonormal basis for $L^2(\Omega)$, comprising eigen-functions for

$$-\Delta z_i + z_i = \mu_i z_i\,, in\ \Omega, \qquad \frac{\partial z_i}{\partial v} = 0\ on\ \partial\Omega, \tag{2.10}$$

where

$$1 \le \mu_1 \le \mu_2 \le \mu_3 \le \cdots \le \mu_k \le \cdots\ with \lim_{i\to\infty} \mu_i = \infty\,, \tag{2.11}$$

is an infinite set of associated eigenvalues. Note $\left(z_i, z_j\right)_{H^1(\Omega)} = \mu_i \delta_{ij}$ and $\left(z_i, z_j\right)_{L^2(\Omega)} = \delta_{ij}$. Now set $V^k := span\{z_i\}_{i=1}^k \subset H^1(\Omega)$, and seek a finite-dimensional weak form corresponding to (M):

(M^k) Find $u_1^k(.,t), v_1^k(.,t), u_2^k(.,t), v_2^k(.,t) \in V^k$ such that

$u_1^k(.,0) = u_{1,0,}^k\ v_1^k(.,0) = v_{1,0,}^k\ u_2^k(.,0) = u_{2,0,}^k\ v_2^k(.,0) = v_{2,0,}^k$

and for almost every $t \in (0,T)$,

$$\varepsilon\left(\frac{\partial u_1^k}{\partial t}, \eta^k\right) + d_1(\nabla u_1^k, \nabla\eta^k) = a(f(u_1^k),\eta^k) - (v_1^k,\eta^k) + \alpha(u_2^k - u_1^k,\eta^k) \tag{2.12}$$

$$\left(\frac{\partial v_1^k}{\partial t}, \eta^k\right) + d_2(\nabla v_1^k, \nabla\eta^k) = (u_1^k,\eta^k) - \delta(v_1^k,\eta^k) + \beta(v_2^k - v_1^k,\eta^k) \tag{2.13}$$

$$\varepsilon\left(\frac{\partial u_2^k}{\partial t}, \eta^k\right) + d_1(\nabla u_2^k, \nabla\eta^k) = a(f(u_2^k),\eta^k) - (v_2^k,\eta^k) - \alpha(u_2^k - u_1^k,\eta^k) \tag{2.14}$$

$$\left(\frac{\partial v_2^k}{\partial t}, \eta^k\right) + d_2(\nabla v_2^k, \nabla\eta^k) = (u_2^k,\eta^k) - \delta(v_2^k,\eta^k) - \beta(v_2^k - v_1^k,\eta^k) \tag{2.15}$$

Now, $u_1^k, v_1^k, u_2^k, v_2^k$ expressed as Galerkin approximations in the subsequent format

$$u_1^k(.,t) = \sum_{i=1}^k a_{ik}(t) z_i(.), \qquad v_1^k\,(.,t) = \sum_{i=1}^k b_{ik}(t) z_i(.), \tag{2.16}$$

$$u_2^k(.,t) = \sum_{i=1}^k c_{ik}(t) z_i(.), \qquad v_2^k(.,t) = \sum_{i=1}^k d_{ik}(t) z_i(.), \tag{2.17}$$

For $i = 1, \dots, k$, let $\eta^k = z_i$. The coefficients $a_{ik}(t), b_{ik}(t), c_{ik}(t)$, and $d_{ik}(t)$ are not yet known. The orthogonal projection from $L^2(\Omega)$ onto V^k is introduced as $P^k\colon L^2(\Omega) \mapsto V^k$. This projection ensures that $(P^k v, \eta^k) = (v, \eta^k)$ for all $\eta^k \in V^k$. For elements in $H^1(\Omega) \subset L^2(\Omega)$, this definition is valid.

Lemma 2.1. For any $\chi \in H^1(\Omega)$ we have

$$(\nabla(P^k\chi), \nabla\eta^k) = (\nabla\chi, \nabla\eta^k), \forall \eta^k \in V^k. \tag{2.18}$$

Upon direct computation, it becomes evident that this projection operator fulfills the subsequent properties:

$$\|\nabla P^k\chi\|_0 \le \|\nabla\chi\|_0, \forall\chi \in H^1(\Omega). \tag{2.19}$$

The initial values are selected in the following manner:

$$u_1^k(.,0) := P^k u_{1,0}^k, \quad v_1^k(.,0) := P^k v_{1,0}^k, \tag{2.20}$$

$$u_2^k(.,0) := P^k u_{2,0}^k, \quad v_2^k(.,0) := P^k v_{2,0}^k, \tag{2.21}$$

where the following property holds:

$$\left\{u_{1,0}^k, v_{1,0}^k, u_{2,0}^k, v_{2,0}^k\right\} \mapsto \left\{u_{1,0}, v_{1,0}, u_{2,0}, v_{2,0}\right\}\ in\ L^2(\Omega)\ as\ k \mapsto \infty. \tag{2.22}$$

The system of equations $(2.12) - (2.15)$ can be represented as a set of ordinary differential equations involving the variables $a_{ik}(t), b_{ik}(t), c_{ik}(t)$, and $d_{ik}(t)$. We express this system in a composite form that is equivalent to the original.

$$\epsilon \frac{du_1^k}{dt} = d_1 \Delta \mathrm{u}_1^k + aP^k f(u_1^k) - v_1^k + \alpha(u_2^k - u_1^k),$$
$$u_1^k(.,0) := P^k \tag{2.23}$$

$$\frac{dv_1^k}{dt} = d_2 \Delta \mathrm{v}_1^k + u_1^k - \delta v_1^k + \beta(v_2^k - v_1^k),$$
$$v_1^k(.,0) := P^k v_{1,0}^k, \tag{2.24}$$

$$\epsilon \frac{du_2^k}{dt} = d_1 \Delta \mathrm{u}_2^k + aP^k f(u_2^k) - v_2^k - \alpha(u_2^k - u_1^k),$$
$$u_2^k(.,0) := P^k u_{2,0}^k, \tag{2.25}$$

$$\frac{dv_2^k}{dt} = d_1 \Delta \mathrm{v}_2^k + u_2^k - \delta v_2^k - \beta(v_2^k - v_1^k),$$
$$v_2^k(.,0) := P^k v_{2,0}^k. \tag{2.26}$$

Our subsequent objective is to establish the local Lipschitz continuity of the nonlinearity within the system of ordinary differential equations.

Lemma 2.2. let $u \in C^\infty(\Omega)$ and Ω is an open bounded convex domain in $\mathbb{R}^\eta$. Then the nonlinear $F(u) = 3u - u^3$ in system (M) satisfies the inequality

$|F(u_1) - F(u_2)| \leq L|u_1 - u_2|$, where L is Lipschitz constant.

Proof: Based on the assumption, we obtain:

$\min\{u: u \in C^\infty(\Omega)\} \leq u \leq \max\{u: u \in C^\infty(\Omega)\}$, implying the existence of a positive integer C such that:

$$\max_{u \in C^\infty(\Omega)} |u| \leq C. \tag{2.27}$$

Now, we have

$$|f(u_1) - f(u_2)| = |((u_2)^3 - (u_1)^3) - 3(u_2 - u_1)|$$
$$= |(u_2 - u_1)((u_2)^2 + u_2 u_1 + (u_1)^2) - 3(u_2 - u_1)|$$
$$= |u_2 - u_1||(u_2)^2 + u_2 u_1 + (u_1)^2 - 3| \leq |u_2 - u_1|$$
$$(|u_2|^2 + |u_2||u_1| + |u_1|^2 + 3| \leq L|u_2 - u_1|. \tag{2.28}$$

Which completes the proof.

As a result, f is locally Lipschitz. According to local existence theorems, such as Picard's Theorem (see, for example, Hartman [30], p. 9), it can be concluded that the system of ordinary differential equations has a unique solution $u_1^k, v_1^K, u_2^k, v_2^k$ on a finite time interval $(0, t_k)$.

III. STRONG SOLUTIONS

We introduce a weak formulation of System (M), find $u_1(.,t), v_1(.,t), u_2(.,t), v_2(.,t) \in H^1(\Omega)$ such that $u_1(.,0) = u_{1,0}(.), v_1(.,0) = v_{1,0}(.), u_2(.,0) = u_{2,0}(.), v_2(.,0) = v_{2,0}(.)$, and for almost every $t \in (0, T)$,

$$\varepsilon\left(\frac{\partial u_1^k}{\partial t}, \eta^k\right) + d_1(\nabla u_1^k, \nabla \eta^k) =$$
$$a(f(u_1^k), \eta^k) - (v_1^k, \eta^k) + \alpha(u_2^k - u_1^k, \eta^k) \tag{3.1}$$

$$\left(\frac{\partial v_1^k}{\partial t}, \eta^k\right) + d_2(\nabla v_1^k, \nabla \eta^k) =$$
$$(u_1^k, \eta^k) - \delta(v_1^k, \eta^k) + \beta(v_2^k - v_1^k, \eta^k) \tag{3.2}$$

$$\varepsilon\left(\frac{\partial u_2^k}{\partial t}, \eta^k\right) + d_1(\nabla u_2^k, \nabla \eta^k) =$$
$$a(f(u_2^k), \eta^k) - (v_2^k, \eta^k) - \alpha(u_2^k - u_1^k, \eta^k) \tag{3.3}$$

$$\left(\frac{\partial v_2^k}{\partial t}, \eta^k\right) + d_2(\nabla v_2^k, \nabla \eta^k) =$$
$$(u_2^k, \eta^k) - \delta(v_2^k, \eta^k) - \beta(v_2^k - v_1^k, \eta^k) \tag{3.4}$$

Theorem 3.1. Assume $\Omega \subset \mathbb{R}^\eta (\eta = 1,2,3)$ is an open, bounded, convex domain with a boundary $\partial\Omega$ of class C^2. Assume that $u_{1,0}, v_{1,0}, u_{2,0}, v_{2,0} \in H^1(\Omega)$, then the system (1.1) - (1.6) possesses a unique, strong solution u_1, v_1, u_2 and v_2 satisfying

$$u_1(x,t), v_1(x,t), u_2(x,t), v_2(x,t) \epsilon$$
$$L_2\big(0, T; H^2(\Omega)\big) \cap C\big([0,T], H^1(\Omega)\big), \tag{3.5}$$

and System (M) hold as equalities in $L^2(\Omega_T)$. Furthermore, the

$$\big(u_{1,0}(x), v_{1,0}(x)\big), \longmapsto (u_1\big(x, t; u_{1,0}, u_{2,0}, v_{1,0}\big),$$
$$v_1\big(x, t; u_{1,0}, v_{1,0}, v_{2,0}\big)),$$

$$\big(u_{2,0}(x), v_{2,0}(x)\big), \longmapsto (u_2\big(x, t; u_{1,0}, u_{2,0}, v_{2,0}\big),$$
$$v_2\big(x, t; u_{2,0}, v_{1,0}, v_{2,0}\big)),$$

is continuous in $H^1(\Omega)$.

Proof: In order to establish the existence and uniqueness of strong solutions, additional regularity results are essential, and these can be acquired through the application of further a priori estimations.

A. Existence

The following estimates are essential to this section.

EstimateI: Selecting $\eta^k = -\Delta \mathrm{u}_1^k, \eta^k = -\Delta v_1^k, \eta^k = -\Delta u_2^k$, $\eta^k = -\Delta v_2^k$ in the weak forms (3.1) - (3.4), respectively, the results are combined and integrated by parts

$$\varepsilon \frac{1}{2}\frac{d}{dt}\int_\Omega \left|\nabla u_1^k\right|^2 dx + \frac{1}{2}\frac{d}{dt}\int_\Omega \left|\nabla v_1^k\right|^2 dx + \varepsilon \frac{1}{2}\frac{d}{dt}\int_\Omega \left|\nabla u_2^k\right|^2 dx +$$
$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left|\nabla v_2^k\right|^2 dx + d_1 \int_\Omega \left|\nabla u_1^k\right|^2 dx + d_1 \int_\Omega \left|\nabla u_2^k\right|^2 dx +$$
$$d_2 \int_\Omega \left|\nabla v_1^k\right|^2 dx + d_2 \int_\Omega \left|\nabla v_2^k\right|^2 dx + \alpha \int_\Omega \left|\nabla u_2^k - \nabla u_1^k\right|^2 dx +$$
$$\beta \int_\Omega \left|\nabla v_2^k - \nabla v_1^k\right|^2 dx + 3a \int_\Omega \left|u_1^k \nabla u_1^k\right|^2 dx +$$
$$3a \int_\Omega \left|u_2^k \nabla u_2^k\right|^2 dx + \delta \int_\Omega \left|\nabla v_1^k\right|^2 dx + \delta \int_\Omega \left|\nabla v_2^k\right|^2 dx =$$
$$3a \int_\Omega \left|\nabla u_1^k\right|^2 dx + 3a \int_\Omega \left|\nabla u_2^k\right|^2 dx. \tag{3.6}$$

Add $\left(\int_\Omega \left|\nabla v_1^k\right|^2 dx + \int_\Omega \left|\nabla v_2^k\right|^2 dx\right)$, on the right-hand side, multiplying by 2,

$$\varepsilon \frac{d}{dt} \int_\Omega \left|\nabla u_1^k\right|^2 dx + \frac{d}{dt} \int_\Omega \left|\nabla v_1^k\right|^2 dx + \varepsilon \frac{d}{dt} \int_\Omega \left|\nabla u_2^k\right|^2 dx + \frac{d}{dt} \int_\Omega \left|\nabla v_2^k\right|^2 dx + 2d_1 \int_\Omega \left|\Delta u_1^k\right|^2 dx + 2d_1 \int_\Omega \left|\Delta u_2^k\right|^2 dx + 2d_2 \int_\Omega \left|\Delta v_1^k\right|^2 dx + 2d_2 \int_\Omega \left|\Delta v_2^k\right|^2 dx + 2\alpha \int_\Omega \left|\nabla u_2^k - \nabla u_1^k\right|^2 dx + 2\beta \int_\Omega \left|\nabla v_2^k - \nabla v_1^k\right|^2 dx + 6a \int_\Omega \left|u_1^k \nabla u_1^k\right|^2 dx + 6a \int_\Omega \left|u_2^k \nabla u_2^k\right|^2 dx + 2\delta \int_\Omega \left|\nabla v_1^k\right|^2 dx + 2\delta \int_\Omega \left|\nabla v_2^k\right|^2 dx \leq 6a \left(\varepsilon \int_\Omega \left|\nabla u_1^k\right|^2 dx + \varepsilon \int_\Omega \left|\nabla u_2^k\right|^2 dx + \int_\Omega \left|\nabla v_1^k\right|^2 dx + \int_\Omega \left|\nabla v_2^k\right|^2 dx\right). \tag{3.7}$$

Application of Grönwall lemma (2.7) gives

$$\varepsilon \left|u_1^k(T)\right|_1^2 + \left|v_1^k(T)\right|_1^2 + \varepsilon \left|u_2^k(T)\right|_1^2 + \left|v_2^k(T)\right|_1^2 + 2d_1 \left\|u_1^k\right\|_{L^2(0,T,H^2)}^2 + 2d_1 \left\|u_2^k\right\|_{L^2(0,T,H^2)}^2 + 2d_2 \left\|v_1^k\right\|_{L^2(0,T,H^2)}^2 + 2d_2 \left\|v_2^k\right\|_{L^2(0,T,H^2)}^2 + 2\alpha \left\|u_2^k - u_1^k\right\|_{L^2(0,T,H^1)}^2 + 2\beta \left\|v_2^k - v_1^k\right\|_{L^2(0,T,H^1)}^2 + 6a \left\|u_1^k \nabla u_1^k\right\|_{L^2(\Omega_T)+}^2 6a \left\|u_2^k \nabla u_2^k\right\|_{L^2(\Omega_T)}^2 + 2\delta \left\|v_1^k\right\|_{L^2(0,T,H^1)}^2 + 2\delta \left\|v_2^k\right\|_{L^2(0,T,H^1)}^2 \leq e^{6aT} \left(\varepsilon \left|u_1^k(0)\right|_1^2 + \left|v_1^k(0)\right|_1^2 + \varepsilon \left|u_2^k(0)\right|_1^2 + \left|v_2^k(0)\right|_1^2\right) \tag{3.8}$$

Then, we deduce that $u_1^k, v_1^k, u_2^k, v_2^k$, are uniformly bounded in $L^\infty\left(0, T; H^1(\Omega)\right)$, see Theorem 2.1. We now recall that $L^1(0, T; H^1(\Omega)')$, the pre-dual of $L^\infty\left(0, T; H^1(\Omega)\right)$, is a separable Banach space but lacks reflexivity. Consequently, we deduce from the initial and subsequent bounds in equation (3.8) that

$$\{u_1^k, v_1^k, u_2^k, v_2^k\} \rightharpoonup^* \{u_1, v_1, u_2, v_2\}\ in\ L^\infty\left(0, T; H^1(\Omega)\right), \tag{3.9}$$

Then, we have u_1, v_1, u_2 and $v_2 \in L^\infty\left(0, T; H^1(\Omega)\right)$. We employ established elliptic regularity results on domains that are bounded, convex, and open. From the eigenvalue equations (2.10) and (2.11) (see [17]), we have for finite k that $z_i \in H^2(\Omega) (i = 1, \dots, k)$, and hence, $u_1^k(.,t), v_1^k(.,t), u_2^k(.,t), v_2^k(.,t) \in L^2(\Omega)$ for a.e. $t \in (0, T)$. Thus, by [35] Theorem 3.1.3.3, we have $\left\|u_1^k\right\|_2 \leq C \left\|\Delta u_1^k\right\|_0, \left\|u_2^k\right\|_2 \leq C \left\|\Delta u_2^k\right\|_0$, for some positive constant C and a.e. $t \in (0, T)$. Therefore, from the fifth to eighth bounds in (3.8), we conclude that $u_1^k, v_1^k, u_2^k, v_2^k$ are uniformly bounded in $L^2\left(0, T; H^2(\Omega)\right)$. Since $L^2\left(0, T; H^2(\Omega)\right)$ is a reflexive Banach space (see [33] page 40), then, by compactness arguments (see [31] page 289), we deduce the existence of subsequences u_1^k, v_1^k, u_2^k and $v_2^k \in L^2\left(0, T; H^2(\Omega)\right)$ such that

$$\{u_1^k, v_1^k, u_2^k, v_2^k\} \rightharpoonup \{u_1, v_1, u_2, v_2\}\ in\ L^2\left(0, T; H^2(\Omega)\right), \tag{3.10}$$

Thus, we arrive at u_1, v_1, u_2 and $v_2 \in L^2\left(0, T; H^2(\Omega)\right)$. Furthermore since $\frac{\partial u_1^k}{\partial v} = 0, \frac{\partial v_1^k}{\partial v} = 0, \frac{\partial u_2^k}{\partial v} = 0$, and $\frac{\partial v_2^k}{\partial v} = 0$ on $\partial\Omega$, it follows by the weak convergence of $u_1^k \to u_1, v_1^k \to v_1, u_2^k \to u_2$, and $v_2^k \to v_2$ in $H^2(\Omega)$, that $\frac{\partial u_1}{\partial v} = 0, \frac{\partial v_1}{\partial v} = 0, \frac{\partial u_2}{\partial v} = 0$ and $\frac{\partial v_2}{\partial v} = 0$ on $L^2(\partial\Omega)$.

Estimate II: Set $\eta^k = \frac{\partial u_1^k}{\partial t}, \eta^k = \frac{\partial v_1^k}{\partial t}, \eta^k = \frac{\partial u_2^k}{\partial t}, \eta^k = \frac{\partial v_2^k}{\partial t}$ in the weak form (3.1) - (3.4) respectively, we get the result when we combine the results

$$\varepsilon \int_\Omega \left|\frac{\partial u_1^k}{\partial t}\right|^2 dx + \int_\Omega \left|\frac{\partial v_1^k}{\partial t}\right|^2 dx + \varepsilon \int_\Omega \left|\frac{\partial u_2^k}{\partial t}\right|^2 dx + \int_\Omega \left|\frac{\partial v_2^k}{\partial t}\right|^2 dx + \frac{d_1}{2} \frac{d}{dt} \int_\Omega \left|\nabla u_1^k\right|^2 dx + \frac{d_1}{2} \frac{d}{dt} \int_\Omega \left|\nabla u_2^k\right|^2 dx + \frac{d_2}{2} \frac{d}{dt} \int_\Omega \left|\nabla v_1^k\right|^2 dx + \frac{d_2}{2} \frac{d}{dt} \int_\Omega \left|\nabla v_2^k\right|^2 dx + \frac{a}{4} \frac{d}{dt} \int_\Omega \left|u_1^k\right|^4 dx + \frac{a}{4} \frac{d}{dt} \int_\Omega \left|u_2^k\right|^4 dx + \frac{\alpha}{2} \frac{d}{dt} \int_\Omega \left|u_2^k - u_1^k\right|^2 dx + \frac{\beta}{2} \frac{d}{dt} \int_\Omega \left|v_2^k - v_1^k\right|^2 dx + \delta \frac{1}{2} \frac{d}{dt} \int_\Omega \left|v_1^k\right|^2 dx + \delta \frac{1}{2} \frac{d}{dt} \int_\Omega \left|v_2^k\right|^2 dx = \frac{3a}{2} \frac{d}{dt} \int_\Omega \left|u_1^k\right|^2 dx + \frac{3a}{2} \frac{d}{dt} \int_\Omega \left|u_2^k\right|^2 dx + \int_\Omega u_1^k \frac{\partial v_1^k}{\partial t} dx + \int_\Omega u_2^k \frac{\partial v_2^k}{\partial t} dx - \int_\Omega v_1^k \frac{\partial u_1^k}{\partial t} dx - \int_\Omega v_2^k \frac{\partial u_2^k}{\partial t} dx. \tag{3.11}$$

Using Inequality (2.5) on the last four terms on the right, we get

$$\int_\Omega u_1^k \frac{\partial v_1^k}{\partial t} dx \leq \frac{1}{2} \int_\Omega \left|u_1^k\right|^2 dx + \frac{1}{2} \int_\Omega \left|\frac{\partial v_1^k}{\partial t}\right|^2 dx \tag{3.12}$$

$$\int_\Omega u_2^k \frac{\partial v_2^k}{\partial t} dx \leq \frac{1}{2} \int_\Omega \left|u_2^k\right|^2 dx + \frac{1}{2} \int_\Omega \left|\frac{\partial v_2^k}{\partial t}\right|^2 dx \tag{3.13}$$

$$-\int_\Omega v_1^k \frac{\partial u_1^k}{\partial t} dx \leq \frac{1}{2\varepsilon} \int_\Omega \left|v_1^k\right|^2 dx + \frac{\varepsilon}{2} \int_\Omega \left|\frac{\partial u_1^k}{\partial t}\right|^2 dx \tag{3.14}$$

$$-\int_\Omega v_2^k \frac{\partial u_2^k}{\partial t} dx \leq \frac{1}{2\varepsilon} \int_\Omega \left|v_2^k\right|^2 dx + \frac{\varepsilon}{2} \int_\Omega \left|\frac{\partial u_2^k}{\partial t}\right|^2 dx \tag{3.15}$$

Combining (3.12) - (3.15) in (3.11) and multiplying through by 2, gives

$$\varepsilon \int_\Omega \left|\frac{\partial u_1^k}{\partial t}\right|^2 dx + \int_\Omega \left|\frac{\partial v_1^k}{\partial t}\right|^2 dx + \varepsilon \int_\Omega \left|\frac{\partial u_2^k}{\partial t}\right|^2 dx + \int_\Omega \left|\frac{\partial v_2^k}{\partial t}\right|^2 dx + d_1 \frac{d}{dt} \int_\Omega \left|\nabla u_1^k\right|^2 dx + d_1 \frac{d}{dt} \int_\Omega \left|\nabla u_2^k\right|^2 dx + d_2 \frac{d}{dt} \int_\Omega \left|\nabla v_1^k\right|^2 dx + d_2 \frac{d}{dt} \int_\Omega \left|\nabla v_2^k\right|^2 dx + \frac{a}{2} \frac{d}{dt} \int_\Omega \left|u_1^k\right|^4 dx + \frac{a}{2} \frac{d}{dt} \int_\Omega \left|u_2^k\right|^4 dx + \alpha \frac{d}{dt} \int_\Omega \left|u_2^k - u_1^k\right|^2 dx + \beta \frac{d}{dt} \int_\Omega \left|v_2^k - \right.$$

$$v_1^k\big|^2dx + \delta\tfrac{d}{dt}\int_\Omega\big|v_1^k\big|^2dx + \delta\tfrac{d}{dt}\int_\Omega\big|v_2^k\big|^2dx \le 3a\tfrac{d}{dt}\int_\Omega\big|u_1^k\big|^2dx + 3a\tfrac{d}{dt}\int_\Omega\big|u_2^k\big|^2dx + \int_\Omega\big|u_1^k\big|^2dx + \int_\Omega\big|u_2^k\big|^2dx + \tfrac{1}{\varepsilon}\int_\Omega\big|v_1^k\big|^2dx + \tfrac{1}{\varepsilon}\int_\Omega\big|v_2^k\big|^2dx. \tag{3.16}$$

Integrating over time $(0,t)$, leads to

$$\varepsilon\left\|\frac{\partial u_1^k}{\partial t}\right\|_{L^2(\Omega_T)}^2 + \left\|\frac{\partial v_1^k}{\partial t}\right\|_{L^2(\Omega_T)}^2 + \varepsilon\left\|\frac{\partial u_2^k}{\partial t}\right\|_{L^2(\Omega_T)}^2 + \left\|\frac{\partial v_2^k}{\partial t}\right\|_{L^2(\Omega_T)}^2 + d_1\big|u_1^k(T)\big|_1^2 + d_1\big|u_2^k(T)\big|_1^2 + d_2\big|v_1^k(T)\big|_1^2 + d_2\big|v_2^k(T)\big|_1^2 + \tfrac{a}{2}\big|u_1^k(T)\big|_{0,4}^2 + \tfrac{a}{2}\big|u_2^k(T)\big|_{0,4}^2 + \alpha\big\|u_2^k(T) - u_1^k(T)\big\|_0^2 + \beta\big\|v_2^k(T) - v_1^k(T)\big\|_0^2 + \delta\big\|v_1^k(T)\big\|_0^2 + \delta\big\|v_2^k(T)\big\|_0^2 + 3a\big\|u_1^k(0)\big\|_0^2 + 3a\big\|u_2^k(0)\big\|_0^2 \le 3a\big\|u_1^k(T)\big\|_0^2 + 3a\big\|u_2^k(T)\big\|_0^2 + \tfrac{1}{\varepsilon}\big\|v_1^k\big\|_{L^2(\Omega_T)}^2 + \tfrac{1}{\varepsilon}\big\|v_2^k\big\|_{L^2(\Omega_T)}^2 + \big\|u_1^k\big\|_{L^2(\Omega_T)}^2 + \big\|u_2^k\big\|_{L^2(\Omega_T)}^2 + d_1\big|u_1^k(0)\big|_1^2 + d_1\big|u_2^k(0)\big|_1^2 + d_2\big|v_1^k(0)\big|_1^2 + d_2\big|v_2^k(0)\big|_1^2 + \tfrac{a}{2}\ \big\|u_1^k(0)\big\|_{0,4}^2 + \tfrac{a}{2}\ \big\|u_2^k(0)\big\|_{0,4}^2 + \alpha\big\|u_2^k(0) - u_1^k(0)\big\|_0^2 + \beta\big\|v_2^k(0) - v_1^k(0)\big\|_0^2 + \delta\big\|v_1^k(0)\big\|_0^2 + \delta\big\|v_2^k(0)\big\|_0^2. \tag{3.17}$$

Now, on noting bounds in Estimates $I, L^4(\Omega_T), \hookrightarrow L^2(\Omega_T), H^1(\Omega_T) \hookrightarrow L^4(\Omega_T), H^1(\Omega_T) \hookrightarrow L^2(\Omega_T)$, so the initial condition $u_{1,0}, v_{1,0}, u_{2,0}$ and $v_{2,0} \in H^1$, From this deduction, it follows that the right-hand side of equation (3.17) is constrained by a positive constant. It can be expressed as: $\frac{\partial u_1^k}{\partial t}, \frac{\partial v_1^k}{\partial t}, \frac{\partial u_2^k}{\partial t}$ and $\frac{\partial v_2^k}{\partial t}$ are uniformly bounded in $L^2(\Omega_T)$. Since $L^2(\Omega_T)$ is a reflexive Banach space, we can employ compactness arguments to infer the existence of subsequences u_1^k, v_1^k, u_2^k and $v_2^k \in L^2(\Omega_T)$ such that

$$\left\{\frac{\partial u_1^k}{\partial t}, \frac{\partial v_1^k}{\partial t}, \frac{\partial u_2^k}{\partial t}, \frac{\partial v_2^k}{\partial t}\right\} \rightharpoonup \left\{\frac{\partial u_1}{\partial t}, \frac{\partial v_1}{\partial t}, \frac{\partial u_2}{\partial t}, \frac{\partial v_2}{\partial t}\right\} \quad in\ L^2(\Omega_T) \tag{3.18}$$

Thus, we have that $\frac{\partial u_1}{\partial t}, \frac{\partial v_1}{\partial t}, \frac{\partial u_2}{\partial t}$, and $\frac{\partial v_2}{\partial t} \in L^2(\Omega_T), u_1, v_1, u_2$ and $v_2 \in L^\infty\big(0,T; H^1(\Omega)\big), u_1$ and $u_2 \in L^\infty\big(0,T; L^4(\Omega)\big), v_1$ and $v_2 \in L^\infty\big(0,T; L^2(\Omega)\big)$ and $(u_2 - u_1), (v_2 - v_1) \in L^\infty\big(0,T; L^2(\Omega)\big)$.

Estimate III: Set $\eta^k = (u_1^k)^3, \eta^k = (v_1^k)^3, \eta^k = (u_2^k)^3$, $\eta^k = (v_2^k)^3$ in the weak form (3.1) - (3.4) respectively, combine the results, yields

$$\tfrac{\varepsilon d}{4dt}\big\|u_1^k\big\|_{0,4}^4 + \tfrac{d}{4dt}\big\|v_1^k\big\|_{0,4}^4 + \tfrac{\varepsilon d}{4dt}\big\|u_2^k\big\|_{0,4}^4 + \tfrac{d}{4dt}\big\|v_2^k\big\|_{0,4}^4 + 3d_1\big\|u_1^k\nabla u_1^k\big\|_0^2 + 3d_2\big\|v_1^k\nabla v_1^k\big\|_0^2 + 3d_1\big\|u_2^k\nabla u_2^k\big\|_0^2 + 3d_2\big\|v_2^k\nabla v_2^k\big\|_0^2 + a\big\|u_1^k\big\|_{0,6}^6 + a\big\|u_2^k\big\|_{0,6}^6 + \delta\big\|v_1^k\big\|_{0,4}^4 + \delta\big\|v_2^k\big\|_{0,4}^4 = 3a\big\|u_1^k\big\|_{0,4}^4 + 3a\big\|u_2^k\big\|_{0,4}^4 + (u_1^k, (v_1^k)^3) + (u_2^k, (v_2^k)^3) - (v_1^k, (u_1^k)^3) - (v_2^k, (u_2^k)^3) + \alpha(u_2^k - u_1^k, (u_1^k)^3) - \alpha(u_2^k - u_1^k,\ (u_2^k)^3) + \beta\alpha(v_2^k - v_1^k,\ (v_1^k)^3) + \beta(v_2^k - v_1^k,\ (v_2^k)^3) \tag{3.19}$$

Using applying Inequality (2.5) on the final eight terms in the right hand side, we have that

$$\int_\Omega u_2^k(v_2^k)^3dx \le \frac{\alpha^{\frac{1}{3}}}{4}\big\|u_2^k\big\|_{0,4}^4 + \frac{3}{4\alpha}\ \big\|v_2^k\big\|_{0,4}^4, \tag{3.21}$$

$$-\int_\Omega v_1^k(u_1^k)^3dx \le \frac{\alpha^{\frac{1}{3}}}{4}\big\|v_1^k\big\|_{0,4}^4 + \frac{3}{4\alpha}\ \big\|u_1^k\big\|_{0,4}^4, \tag{3.22}$$

$$-\int_\Omega v_2^k(u_2^k)^3dx \le \frac{\alpha^{\frac{1}{3}}}{4}\big\|v_2^k\big\|_{0,4}^4 + \frac{3}{4\alpha}\ \big\|u_2^k\big\|_{0,4}^4, \tag{3.23}$$

$$\alpha(u_2^k - u_1^k, (u_1^k)^3) - \alpha(u_2^k - u_1^k, (u_2^k)^3) = \alpha(u_2^k, (u_1^k)^3) - \alpha(u_1^k, (u_1^k)^3) - \alpha(u_2^k, (u_2^k)^3) + \alpha(u_1^k, (u_2^k)^3) \le -\alpha\big\|u_1^k\big\|_{0,4}^4 - \alpha\big\|u_2^k\big\|_{0,4}^4 + \alpha\tfrac{\alpha^{\frac{1}{3}}}{4}\big\|u_2^k\big\|_{0,4}^4 + \alpha\ \tfrac{3}{4\alpha}\ \big\|u_1^k\big\|_{0,4}^4 + \alpha\tfrac{\alpha^{\frac{1}{3}}}{4}\big\|u_1^k\big\|_{0,4}^4 + \alpha\ \tfrac{3}{4\alpha}\ \big\|u_2^k\big\|_{0,4}^4, \tag{3.24}$$

$$\beta(v_2^k - v_1^k, (v_1^k)^3) - \beta(v_2^k - v_1^k, (v_2^k)^3) = \beta(v_2^k, (v_1^k)^3) - \beta(v_1^k, (v_1^k)^3) - \beta(v_2^k, (v_2^k)^3) + \beta(v_1^k, (v_2^k)^3) \le -\beta\big\|v_1^k\big\|_{0,4}^4 - \beta\big\|v_2^k\big\|_{0,4}^4 + \beta\tfrac{\alpha^{\frac{1}{3}}}{4}\big\|v_2^k\big\|_{0,4}^4 + \beta\ \tfrac{3}{4\alpha}\ \big\|v_1^k\big\|_{0,4}^4 + \beta\tfrac{\alpha^{\frac{1}{3}}}{4}\big\|v_1^k\big\|_{0,4}^4 + \beta\ \tfrac{3}{4\alpha}\ \big\|v_2^k\big\|_{0,4}^4\ . \tag{3.25}$$

Where $\alpha > 0$, combining (3.19) - (3.25) , gives

$$\tfrac{\varepsilon d}{4dt}\big\|u_1^k\big\|_{0,4}^4 + \tfrac{d}{4dt}\big\|v_1^k\big\|_{0,4}^4 + \tfrac{\varepsilon d}{4dt}\big\|u_2^k\big\|_{0,4}^4 + \tfrac{d}{4dt}\big\|v_2^k\big\|_{0,4}^4 + 3d_1\big\|u_1^k\nabla u_1^k\big\|_0^2 + 3d_2\big\|v_1^k\nabla v_1^k\big\|_0^2 + 3d_1\big\|u_2^k\nabla u_2^k\big\|_0^2 + 3d_2\big\|v_2^k\nabla v_2^k\big\|_0^2 + a\big\|u_1^k\big\|_{0,6}^6 + a\big\|u_2^k\big\|_{0,6}^6 + \alpha\big\|u_1^k\big\|_{0,4}^4 + \alpha\big\|u_2^k\big\|_{0,4}^4 + (\delta+\beta)\big\|v_1^k\big\|_{0,4}^4 + (\delta+\beta)\big\|v_2^k\big\|_{0,4}^4 \le C[\big\|u_1^k\big\|_{0,4}^4 + \big\|v_1^k\big\|_{0,4}^4 + \big\|u_2^k\big\|_{0,4}^4 + \big\|v_2^k\big\|_{0,4}^4]. \tag{3.26}$$

Multiplying through by 4 and application (2.7), gives

$$\varepsilon\big\|u_1^k(T)\big\|_{0,4}^4 + \big\|v_1^k(T)\big\|_{0,4}^4 + \varepsilon\big\|u_2^k(T)\big\|_{0,4}^4 + \big\|v_2^k(T)\big\|_{0,4}^4 + 12d_1\big\|u_1^k\nabla u_1^k\big\|_{L^2(\Omega_T)}^2 + 12d_2\big\|v_1^k\nabla v_1^k\big\|_{L^2(\Omega_T)}^2 + 12d_1\big\|u_2^k\nabla u_2^k\big\|_{L^2(\Omega_T)}^2 + 12d_2\big\|v_2^k\nabla v_2^k\big\|_{L^2(\Omega_T)}^2 + 4a\big\|u_1^k\big\|_{L^6(\Omega_T)}^6 + 4a\big\|u_2^k\big\|_{L^6(\Omega_T)}^6 + 4\alpha\big\|u_1^k\big\|_{L^4(\Omega_T)}^4 + 4\alpha\big\|u_2^k\big\|_{L^4(\Omega_T)}^4 + 4(\delta+\beta)\big\|v_1^k\big\|_{L^4(\Omega_T)}^4 + 4(\delta+\beta)\big\|v_2^k\big\|_{L^4(\Omega_T)}^4 \le \exp(CT)\Big(\varepsilon\big\|u_1^k(0)\big\|_{0,4}^4 + \big\|v_1^k(0)\big\|_{0,4}^4 + \varepsilon\big\|u_2^k(0)\big\|_{0,4}^4 + \big\|v_2^k(0)\big\|_{0,4}^4\Big). \tag{3.27}$$

Now, on noting the initial condition $u_{1,0}, v_{1,0}, u_{2,0}, v_{2,0} \in H^1$, and, $H^1(\Omega_T) \hookrightarrow L^2(\Omega_T)$, we deduce that the right-hand side of (3.27) is bounded by a positive constant. We have that

$u_1, v_1, u_2, v_2 \in L^{\infty}(0,T;L^2(\Omega))$, $u_1, u_2 \in L^6(\Omega_T)$ and $v_1, v_2 \in L^4(\Omega_T)$.

Lemma 3.1. For some $\eta \geq 0$, suppose that

$$\varsigma \in L^2\big(0,T;H^{\eta+1}(\Omega)\big), \frac{\partial \varsigma}{\partial t} \in L^2\big(0,T;H^{\eta-1}(\Omega)\big).$$

It follows that $\varsigma \in C\big([0,T];H^1(\Omega)\big)$

Proof: (See [32], pages 191-194).

Here, in our case, $\eta = 1, H^{\eta+1}(\Omega) = H^2(\Omega), H^{\eta}(\Omega) = H^1(\Omega), H^{\eta-1}(\Omega) = L^2(\Omega)$. Thus, from Lemma (3.1) we have that $u_1^k, v_1^k, u_2^k, v_2^k \in C\big([0,T];H^1(\Omega)\big)$

B. Continuous dependence and uniqueness.

Assume $u_1^1, u_1^2, v_1^1, v_1^2, u_2^1, u_2^2$ and v_2^1, v_2^2 satisfy the weak form (3.1) - (3.4), with initial conditions $u_1^1(.,0) = u_{1,0}^1(.)$, $u_1^2(.,0) = u_{1,0}^2(.)$, $v_1^1(.,0), v_{1,0}^1(.)$, $v_1^2(.,0) = v_{1,0}^2(.)$, $u_2^1(.,0) = u_{2,0}^1(.)$, $u_2^2(.,0) = u_{2,0}^2(.)$ and $v_2^1(.,0) = v_{2,0}^1(.)$, $v_2^2(.,0) = v_{2,0}^2(.)$, respectively, such that $u_{1,0}^1(.) \neq u_{1,0}^2(.), v_{1,0}^1(.) \neq v_{2,0}^1(.)$, $u_{2,0}^1(.) \neq u_{2,0}^2(.)$ and $v_{2,0}^1(.) \neq v_{2,0}^2(.)$. Setting $\omega_1 = u_1^1 - u_1^2, \omega_2 = v_1^1 - v_1^2,, \omega_3 = u_2^1 - u_2^2$ and $\omega_4 = v_2^1 - v_2^2$ and setting $\eta = -\Delta\omega_1 + \omega_1$ and $\eta = -\Delta\omega_2 + \omega_2, \eta = -\Delta\omega_3 + \omega_3, \eta = -\Delta\omega_4 + \omega_4$ in (3.1) - (3.4), Subtracting weak forms results in, after integration by parts,

$$\begin{aligned}&\varepsilon\frac{1}{2}\frac{d}{dt}\int_{\Omega}(|\omega_1|^2 + |\nabla\omega_1|^2)dx + \frac{1}{2}\frac{d}{dt}\int_{\Omega}(|\omega_2|^2 + |\nabla\omega_2|^2)dx + \\ &\varepsilon\frac{1}{2}\frac{d}{dt}\int_{\Omega}(|\omega_3|^2 + |\nabla\omega_3|^2)dx + \frac{1}{2}\frac{d}{dt}\int_{\Omega}(|\omega_4|^2 + |\nabla\omega_4|^2)dx + \\ &d_1\int_{\Omega}(|\nabla\omega_1|^2 + |\Delta\omega_1|^2)dx + d_2\int_{\Omega}(|\nabla\omega_2|^2 + |\Delta\omega_2|^2)dx + \\ &d_1\int_{\Omega}(|\nabla\omega_3|^2 + |\Delta\omega_3|^2)dx + d_2\int_{\Omega}(|\nabla\omega_4|^2 + |\Delta\omega_4|^2)\,dx + \\ &a\int_{\Omega}((u_1^1)^3 - (u_1^2)^3)(-\Delta\omega_1 + \omega_1)dx + a\int_{\Omega}((u_2^1)^3 - \\ &(u_2^2)^3)(-\Delta\omega_3 + \omega_3)dx \quad + \alpha\int_{\Omega}(|\omega_3 - \omega_1|^2 + \\ &|\nabla\omega_3 - \nabla\omega_1|^2)dx + \beta\int_{\Omega}(|\omega_4 - \omega_2|^2 + |\nabla\omega_4 - \nabla\omega_2|^2)dx + \\ &\delta\int_{\Omega}|\omega_2|^2dx + \delta\int_{\Omega}|\nabla\omega_2|^2dx + \delta\int_{\Omega}|\omega_4|^2dx + \\ &\delta\int_{\Omega}|\nabla\omega_4|^2dx = 3a\int_{\Omega}(|\omega_1|^2 + |\nabla\omega_1|^2)dx + 3a\int_{\Omega}(|\omega_3|^2 + \\ &|\nabla\omega_3|^2)dx \qquad (3.28)\end{aligned}$$

Applying Young's inequalities (2.6), yields that

$$\begin{aligned}&a\int_{\Omega}((u_1^1)^3 - (u_1^2)^3)(-\Delta\omega_1 + \omega_1)dx = a\int_{\Omega}(u_1^1 - \\ &u_1^2)((u_1^1)^2 + u_1^1u_1^2 + (u_1^2)^2)(-\Delta\omega_1 + \omega_1)dx \geq a\int_{\Omega}\left(\frac{(u_1^1)^2}{2} + \right. \\ &\left.\frac{(u_1^2)^2}{2}\right)(|\nabla\omega_1|^2 + |\omega_1|^2)dx. \qquad (3.29)\end{aligned}$$

In the same way,

$$\begin{aligned}&a\int_{\Omega}((u_2^1)^3 - (u_2^2)^3)(-\Delta\omega_3 + \omega_3)dx = a\int_{\Omega}(u_2^1 - \\ &u_2^2)((u_2^1)^2 + u_2^1u_2^2 + (u_2^2)^2)(-\Delta\omega_3 + \omega_3)dx \geq a\int_{\Omega}\left(\frac{(u_2^1)^2}{2} + \right. \\ &\left.\frac{(u_2^2)^2}{2}\right)(|\nabla\omega_3|^2 + |\omega_3|^2)dx. \qquad (3.30)\end{aligned}$$

Substitute (3.30) and (3.29) in to (3.28), add $(\|\omega_2\|_1^2 + \omega_4\|_1^2)$, on the right-hand side, multiplying by 2, and neglecting final terms, leads to

$$\begin{aligned}&\varepsilon\frac{d}{dt}\|\omega_1\|_1^2 + \frac{d}{dt}\|\omega_2\|_1^2 + \varepsilon\frac{d}{dt}\|\omega_3\|_1^2 + \frac{d}{dt}\|\omega_4\|_1^2 + \leq 6a(\|\omega_1\|_1^2 + \\ &\|\omega_3\|_1^2 + \|\omega_2\|_1^2 + \|\omega_4\|_1^2), \qquad (3.31)\end{aligned}$$

application of Grӧnwall lemma (2.7) we get

$$\begin{aligned}&\varepsilon\|\omega_1(T)\|_1^2 + \|\omega_2(T)\|_1^2 + \varepsilon\|\omega_3(T)\|_1^2 + \|\omega_4(T)\|_1^2 \leq \\ &\exp(6aT)\,(\varepsilon\|\omega_1(0)\|_1^2 + \|\omega_2(0)\|_1^2 + \varepsilon\|\omega_3(0)\|_1^2 + \\ &\|\omega_4(0)\|_1^2). \qquad (3.32)\end{aligned}$$

Thus, if $\big(u_1^1(0), v_1^1(0), u_2^1(0), v_2^1(0)\big) = (u_1^2(0), \; v_1^2(0), \; u_2^2(0), \; v_2^2(0))$ then $\big(\omega_1(0), \omega_2(0), \omega_3(0), \omega_4(0)\big) = (0, 0,0,0)$ and hence it follows from (3.23) that $(\omega_1(T) = 0, \omega_2(T) = 0, \omega_3(T) = 0, \; \omega_4(T) = 0)$ and hence $u_1^1(T) = u_1^2(T), v_1^1(T) = v_1^2(T), u_2^1(T) = u_2^2(T)$ and $v_2^1(T) = v_2^2(T)$ for all t, we deduce uniqueness of solution. However, if $(u_1^1(0), v_1^1(0), u_2^1(0), v_2^1(0)) \neq (u_1^2(0), \; v_1^2(0), \; u_2^2(0), v_2^2(0))$, then we have continuous dependence in $H^1(\Omega)$. This completes proof.

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