Legendre Operational Differential Matrix for Solving Fuzzy Differential Equations with Trapezoidal Fuzzy Function Coefficients

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Abstract; *In this work, we used the Legendre operational differential matrix method based on Tau method to obtain the fuzzy approximate-analytical solutions of the fuzzy differential equations in which the coefficients are trapezoidal fuzzy function. This method allows for the fuzzy solution of the fuzzy initial (or boundary) value problems to be computed in the form of an infinite fuzzy series .Also, this method enables to approximate the fuzzy exact-analytical solutions with high efficiency, as these solutions can be resorted to if it is not possible to find the exact solutions of these fuzzy problems. We introduced a comparison between the approximate solutions that we computed and the exact solutions of the chosen problem, as we found the absolute error. According to the numerical results, the series solutions that we found are accurate solutions and very close to the exact solutions.*

Keywords; Fuzzy differential equation, triangular fuzzy function, fuzzy approximate-analytical solution

I. INTRODUCTION

In fuzzy differential equations (FDEs), the coefficients may be non-fuzzy variable coefficients or fuzzy variable coefficients. The first type means that the coefficients are real-valued functions while the second type means that the coefficients are fuzzy-valued functions. There are many different types of the fuzzy variable coefficients, the most important of which are the triangular fuzzy function coefficients and the trapezoidal fuzzy function coefficients.

In 2013, the researcher KadakIntroduceda new concept in the fuzzy functions called the trapezoidal fuzzy functions (TrFFs).The aforementioned researcher used the TrFF in the sequences of fuzzy-valued functions and presented characterizations of uniform convergence in sequences with TrFF coefficients [21]. Also, in 2014, Kadak and the researcher Basar used the TrFF in the Fourier series of the periodic fuzzy-valued functions and studied the convergence of the Fourier series in which the coefficients are TrFF [22]. The aforementioned researchers proved many properties of the fuzzy derivative of fuzzy-valued functions, including the TrFF. Regarding FDE, so far no researchers have touched on solving the FDE with TrFF coefficients, but many researchers have touched on solving the FDE in which the coefficients are trapezoidal fuzzy numbers (TrFNs). We mention among them:

In 2015, the researchers Mondal and Roy obtained the exact-analytical solutions and the numerical solutions of the second order linear FDE with TrFN coefficients in which the conditions are fuzzy boundary conditions [27].

In 2016, the researchers Mondal and Roy obtained the exact-analytical solutions and the numerical solutions of the second order linear FDE with TrFN coefficients in which the conditions are fuzzy initial conditions [7].

In 2017, the researchers Patel and Desai used the fuzzy Laplace transform method to find the exact-analytical solutions of second order linear FDE with TrFN coefficients in which the conditions are fuzzy initial (and boundary) conditions [27].

In 2022, the researchers Malak and Attili used the Picard method and the general linear method to find the numerical solution of the first order FDE in which the coefficients are TrFN [26].

In 2023, the researchers Shams and Kausar used the generalized modified Adomian decomposition method to find the approximate-analytical solution of the linear system of first order (and second order) FDE with TrFN coefficients in which the fuzzy conditions are initial [25].

In 2023, the researchers Shams and Kausar used the generalized modified Adomian decomposition method to find the approximate-analytical solution of higher order linear FDE with TrFN coefficients in which the fuzzy conditions are initial [16].

The researchers mentioned above, in addition to other researchers, used numerical, approximate-analytical, and exact-analytical methods to solve the FDE with TrFN coefficients, but none of them dealt with the FDE with TrFF coefficients. Therefore, in this section we will deal with the FDE with TrFF coefficients, as we will obtain the approximate-analytical solutions to these equations using the fuzzification of Legendre operational differential matrix method that we have used in chapter three for solving the FDE with TFF coefficients.

II. FUNDAMENTAL CONCEPT IN FUZZY SET THEORY

The fundamental definitions in the fuzzy set theory, which are: fuzzy set, $\alpha-$ level set, fuzzy number,...etc. can be found in details in [7,11,16,18]. In this section, we will touch on definitions that are directly related to our work**.**

Definition (2.1), Trapezoidal Fuzzy Number:

Let u_1, u_2, u_3 and u_4 are real numbers with $u_1 \leq u_2 \leq u_3 \leq u_4$. Then the trapezoidalfuzzy number (TrFN) can be written as $\tilde{u} = (u_1, u_2, u_3, u_4)$ and it is a fuzzy number with membership function [23,24]:

$$\mu_{\tilde{u}}(x) = \begin{cases} (x-u_1)/(u_2-u_1) & \text{if } u_1 \leq x \leq u_2 \\ 1 & \text{if } u_2 \leq x \leq u_3 \\ (u_4-x)/(u_4-u_3) & \text{if } u_3 \leq x \leq u_4 \\ 0 & \text{otherwise} \end{cases}$$

(1)

Remark (2.2):

The parametric form of the TrFN $\tilde{u} = (u_1, u_2, u_3, u_4)$ can be defined as [b12]:

$$[\tilde{u}]_\alpha = [[u]_\alpha^L, [u]_\alpha^U] = [(u_2-u_1)\alpha + u_1, (u_3-u_4)\alpha + u_4]$$

$\forall\ \alpha \in [0,1]$. (2)

Example (2.3):

The parametric form of the TrFN $\tilde{u} = (3, 6, 8, 12)$ is:

$$[\tilde{u}]_\alpha = [[u]_\alpha^L, [u]_\alpha^U] = [3\alpha + 3, -4\alpha + 12]$$

$$[u]_\alpha^L = 3\alpha + 3$$

$$[u]_\alpha^U = -4\alpha + 12$$

The functions $[u]_\alpha^L$ and $[u]_\alpha^U$ represents the lower bound and the upper bound of parametric form of $\tilde{u}$, respectively.

Definition (2.4), Trapezoidal Fuzzy Function:

Let F_a, F_b, F_c and $F_d : I \to R$ for some interval $I \subseteq R$ arecontinuous real-valued functions such that:

$$F_a(t) \leq F_b(t) \leq F_c(t) \leq F_d(t), \quad \forall\ t \in I.$$

We call the fuzzy set $\tilde{F}(t)$, determined by the membership function [21,22]:

$$\mu_{\tilde{u}}(x) = \begin{cases} (x-F_a(t))/(F_b(t)-F_a(t)) & \text{if } F_a(t) \leq x \leq F_b(t) \\ 1 & \text{if } F_b(t) \leq x \leq F_c(t) \\ (F_d(t)-x)/(F_d(t)-F_c(t)) & \text{if } F_c(t) \leq x \leq F_d(t) \\ 0 & \text{otherwise} \end{cases}$$

(3)

as trapezoidal fuzzy function (TrFF) for all $x, t \in I$,and it is denoted by:

$$\tilde{F}(t) = (F_a(t), F_b(t), F_c(t), F_d(t))$$

According to this definition, the TrFFis a fuzzy set of real-valued functions and it is a special kind of the fuzzy functions.

From the above definition, we can conclude that any fuzzy function produces a TrFN for any real input can be described as TrFF.

Remark (2.5):

The parametric form of the TrFF$\tilde{F}(t) = (F_a(t), F_b(t), F_c(t), F_d(t))$ can be defined as [22]:

$[\tilde{F}(t)]_\alpha$=$[[F(t)]^L_\alpha, [F(t)]^U_\alpha]$

$=[(F_b(t) - F_a(t))\alpha + F_a(t), (F_c(t) - F_d(t))\alpha + F_d(t)]$ (4)

$\forall\ \alpha \in [0,1]$ and $\forall\ t \in I$

Example (2.6):

The parametric form of the TrFF $\tilde{F}(t)$ can be written as:

$[\tilde{F}(t)]_\alpha$=$[[F(t)]^L_\alpha, [F(t)]^U_\alpha] =$

$[(5.2e^t - 1.199t^3 + 1.1)\alpha - 7e^t + 2.353t^3 + 2.1, (-8.7e^t + 2.172t^3 - 1.9)\alpha + 10.5e^t - 2.941t^3 + 6.6]$

Then, we get:

$[F(t)]^L_\alpha = (5.2e^t - 1.199t^3 + 1.1)\alpha - 7e^t + 2.353t^3 + 2.1$

$[F(t)]^U_\alpha = (-8.7e^t + 2.172t^3 - 1.9)\alpha + 10.5e^t - 2.941t^3 + 6.6$

The functions $[F(t)]^L_\alpha$ and $[F(t)]^U_\alpha$ represents the lower bound and the upper bound of parametric form of $\tilde{F}(t)$, respectively.

III. SHIFTED LEGENDRE POLYNOMIALS

The Legendre polynomials of order r are defined on the interval $[-1, 1]$ and are denoted by$L_r(z)$. These polynomials can be described as [5]:

$$L_0(z) = 1 \quad (5)$$

$$L_1(z) = z \quad (6)$$

$$L_2(z) = \frac{3}{2}z^2 - \frac{1}{2} \quad (7)$$

$$L_3(z) = \frac{5}{2}z^3 - \frac{3}{2}z \quad (8)$$

$$L_4(z) = \frac{35}{8}z^4 - \frac{15}{4}z^2 + \frac{3}{8} \quad (9)$$

$\vdots$

$$L_{r+1}(z) = \frac{2r+1}{r+1} zL_r(z) - \frac{r}{r+1} L_{r-1}(z) \quad ;\ r = 1, 2, 3, \dots \quad (10)$$

In order to use the Legendre polynomials on the interval $[0, 1]$, the so-called shifted Legendre polynomials (SLPs) are defined by introducing $z = 2t - 1$.

Let the SLPs $L_r(2t - 1)$ be denoted by $p_r(t)$, then $p_r(t)$ can be obtained as follows:

$$p_0(t) = 1 \quad (11)$$

$$p_1(t) = 2t - 1 \quad (12)$$

$$p_2(t) = 6t^2 - 6t + 1 \quad (13)$$

$$p_3(t) = 20t^3 - 30t^2 + 12t - 1 \quad (14)$$

$$p_4(t) = 70t^4 - 140t^3 + 90t^2 - 20t + 1 \quad (15)$$

$\vdots$

$$p_{r+1}(t) = \frac{2r+1}{r+1}(2t - 1)p_r(t) - \frac{r}{r+1} p_{r-1}(t) \ ;\ r = 1, 2, 3, \dots$$

(16)

In this work, we will use the SLPs as a prime factor to get the FAAS of the FDE. And this will be done based on Tau method, since the basis of Tau method is a definite integral with in the period$[0,1]$, which is the same period for which the SLPs are defined.

Finding the FAAS of the FDE is based on an infinite fuzzy series. This series is called the solution series, which is a convergent series from which the first terms are taken to approximate the FEAS of FDE. The mathematical formula for this series consists of SLPs and shifted Legendre coefficients (SLCs), which means that the SLP constitute the basic element in finding the desired approximate solution, as we will notice in the next section.

IV. Description Of legendre Operational Differential Matrix Method

To describe LODMM in a simple way, we will consider the following general form of the second order NFDD:

$$x''(t) = f\big(t,\ x(t),\ x'(t)\big)\ ,t \geq 0 \qquad (17)$$

With:

$$x(0) = a\ \ ,\ \ x'(0) = b \qquad (18)$$

The solution x(t) of problem (17) can be approximated as [5,9]:

$$x(t) = \sum_{r=0}^{\infty} c_r\, p_r(t) \qquad (19)$$

Where:

$p_r(t)$ are SLPs,

c_r are SLCs.

The coefficients c_r are given by:

$$c_r = (2r+1)\int_0^1 x(t)\, p_r(t)\, dt \quad ; r = 0,1,2,\ldots \qquad (20)$$

Finding the approximate solution x(t) depends mainly on finding the constants c_r as we will notice later.

By considering only the first (m+1) terms of the series solution in equation (19), we have:

$$(t) \approx \sum_{r=0}^{m} c_r p_r(t) \qquad (21)$$

This means that:

$$x(t) \approx c_0\, p_0(t) + c_1\, p_1(t) + c_2\, p_2(t) + \cdots + c_m\, p_m(t) \qquad (22)$$

In matrix form, we get:

$$x(t) \approx C^T W(t) \qquad (23)$$

Where:

$C^T = [c_0\,, c_1\,, c_2\,, \ldots, c_m]$is SLCs vector,

$W(t) = [p_0(t),\ p_1(t),\ p_2(t)\,, \ldots,\ p_m(t)]^T$is SLPs vector.

The derivative of W(t) is:

$$\frac{d(W(t))}{dt} = D^{(1)}W(t) \qquad (24)$$

Where:

$D^{(1)}$ is $(m+1)\times(m+1)$operational differential matrix, which is given by:

$$D^{(1)} = \big(d_{ij}\big) = \begin{cases} 4j-2\,, & \text{if } j = i-k \\ 0 & \text{, otherwise} \end{cases} \qquad (25)$$

Where:

$$k = \begin{cases} 1,3,5,\ldots m & ,\ \text{if m is odd} \\ 1,3,5,\ldots m-1, & \text{if m is even} \end{cases} \qquad (26)$$

In this work, based on many applied problems that we solved by using different values of m, we will consider $m = 4$, since this value of m is suitable for the approximation.

Therefore, form $= 4$, we get:

$$D^{(1)} = (d_{ij}) = \begin{cases} 4j-2\,, & \text{if } j = i-1 \text{ or } j = i-3 \\ 0\,, & \text{otherwise} \end{cases} \quad (27)$$

Thus, the operational differential matrix will be:

$$D^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 2 & 0 & 10 & 0 & 0 \\ 0 & 6 & 0 & 14 & 0 \end{bmatrix} \quad (28)$$

For the nth order derivative, we obtain:

$$\frac{d^n(W(t))}{dt^n} = \left(D^{(1)}\right)^n W(t) = D^{(n)} W(t) \quad ; \quad n = 1\,, 2\,, 3, \ldots \quad (29)$$

Where $\left(D^{(1)}\right)^n$ denotes the matrix powers.

Thus, we find:

$$D^{(2)} = D^{(1)} \times D^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 12 & 0 & 0 & 0 & 0 \\ 0 & 60 & 0 & 0 & 0 \\ 40 & 0 & 140 & 0 & 0 \end{bmatrix} \quad (30)$$

Therefore, we get:

$$x(t) = C^T W(t) \quad (31)$$

This means that:

$$x'(t) = \frac{d(x(t))}{dt} = \frac{d\left(C^T W(t)\right)}{dt} = C^T \frac{d(W(t))}{dt} \quad (32)$$

This gives:

$$x'(t) = C^T D^{(1)} W(t) \quad (33)$$

Also, we have:

$$x''(t) = \frac{d\left(x'(t)\right)}{dt} = \frac{d\left(C^T D^{(1)} W(t)\right)}{dt} = C^T \frac{d\left(D^{(1)} W(t)\right)}{dt} \quad (34)$$

This gives:

$$x''(t) = C^T D^{(2)} W(t) \quad (35)$$

Where:

$$C^T = [\, c_0\,, c_1\,, c_2\,, c_3, c_4] \quad (36)$$

$$W(t) = [\, p_0(t)\,,\ p_1(t)\,,\ p_2(t)\,,\ p_3(t),\ p_4(t)]^T \quad (37)$$

Therefore, from the above description, we will conclude the following:

- From equation(31), we find:

$$x(t) = C^T W(t) =$$
$$[\, c_0\,, c_1\,, c_2\,, c_3, c_4][\, p_0(t)\,,\ p_1(t)\,,\ p_2(t)\,,\ p_3(t),\ p_4(t)]^T \quad (38)$$

This gives:

$$x(t) = c_0\, p_0(t) + c_1\, p_1(t) + c_2\, p_2(t) + c_3\, p_3(t) + c_4\, p_4(t) \quad (39)$$

- From equation(33), we find:

$$x'(t) = C^T D^{(1)} W(t) =$$
$$[c_0, c_1, c_2, c_3, c_4] \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 2 & 0 & 10 & 0 & 0 \\ 0 & 6 & 0 & 14 & 0 \end{bmatrix} [p_0(t), p_1(t), p_2(t), p_3(t), p_4(t)]^T \quad (40)$$

This gives:

$$x'(t) = 2c_1p_0(t) + 2c_3p_0(t) + 6c_2p_1(t) + 6c_4p_1(t) +$$
$$10c_3p_2(t) + 14c_4p_3(t) \quad (41)$$

- From equation(35), we find:

$$x''(t) = C^T D^{(2)} W(t) =$$
$$[c_0, c_1, c_2, c_3, c_4] \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 12 & 0 & 0 & 0 & 0 \\ 0 & 60 & 0 & 0 & 0 \\ 40 & 0 & 140 & 0 & 0 \end{bmatrix} [p_0(t), p_1(t), p_2(t), p_3(t), p_4(t)]^T \quad (42)$$

This gives:

$$x''(t) = 12c_2p_0(t) + 40c_4p_0(t) + 60c_3p_1(t) + 140c_4p_2(t) \quad (43)$$

Now, by using the equations (39), (41) and (43) we can find the residual function R(t) of problem (17) as follows:

From equation (17), we have:

$$R(t) = x''(t) - f\big(t\,, x(t), x'(t)\big) \tag{44}$$

This gives:

$$R(t) = 12c_2p_0(t) + 40c_4p_0(t) + 60c_3p_1(t) + 140c_4p_2(t) - f\big(t, c_0\,p_0(t) + c_1\,p_1(t) + c_2\,p_2(t) + c_3\,p_3(t) + c_4\,p_4(t), 2c_1p_0(t) + 2c_3p_0(t) + 6c_2p_1(t) + 6c_4p_1(t) + 10c_3p_2(t) + 14c_4p_3(t)\big) \tag{45}$$

Then, we can apply Tau method, which can be defined as:

$$\int_0^1 R(t)\,p_r(t)\,dt = 0 \quad ; r = 0,1,2,\dots, m-2 \tag{46}$$

For m=4, we get:

$$\int_0^1 R(t)\,p_0(t)\,dt = 0 \tag{47}$$

$$\int_0^1 R(t)\,p_1(t)\,dt = 0 \tag{48}$$

$$\int_0^1 R(t)\,p_2(t)\,dt = 0 \tag{49}$$

From the equations (47), (48) and (49), we get three linear (or non-linear) equations. In addition, two linear equations can be obtained by applying the initial conditions of equation (18). Therefore, we will get a system of five linear (or non-linear) equations, and then by solving this system, we will obtain the constants c_0, c_1, c_2, c_3 and c_4

Through these constants, the approximate-analytical solution of the problem (17) can be obtained, which is:

$$x(t) \approx c_0\,p_0(t) + c_1\,p_1(t) + c_2\,p_2(t) + c_3\,p_3(t) + c_4\,p_4(t) \tag{50}$$

It is necessary to note that the above mathematical description can be repeated if the NFDD of problem (17) is boundary value problem or if it is higher order initial (or boundary) value problems.

From the above, the solution steps for LODMM based on Tau method can be summarized as follows:

1. We adjust the order of the DE so that it is always an equation of the second order.
2. We use the equations (39), (41) and (43) to obtain the residual function R(t) of the DE in step (1).
3. We use the equations (47), (48) and (49), to get three linear (or non-linear) algebraic equations.
4. We apply the initial conditions (or the boundary conditions) of the DE in step (1) to get two linear algebraic equations.
5. From the steps (3) and (4), we have a system of five
6. linear (or non-linear) algebraic equations.
7. We solve the system in step (5) to find the constants c_0, c_1, c_2, c_3 and c_4.
8. We substitute the constants of step (6) in the equation (50) to get the approximate-analytical solution of the DE in step (1).

The advantage of the LODMM based on Tau method over the other methods is therefore:

- It is computational less cost,
- It needs less computational time and effort and
- It has better accuracy.

The propositions that ensure the convergence of the solution and the stability of the errors analysis can be found in [5,9].

V. FUZZY DIFFERENTIAL EQUATIONS WITH TRFF COEFFICIENTS

The general form of the nth-order linear FDE with TrFF coefficients is:

$$\tilde{x}^{(n)}(t) + \tilde{b}_{n-1}(t)\tilde{x}^{(n-1)}(t) + \tilde{b}_{n-2}(t)\tilde{x}^{(n-2)}(t) + \cdots + \tilde{b}_1(t)\tilde{x}'(t) + \tilde{b}_0(t)\tilde{x}(t) = \tilde{k}(t) \quad , \quad t \geq 0 \tag{51}$$

With fuzzy initial conditions:

$$\tilde{x}(0) = x(0\,;\alpha) = w_0(\alpha) \tag{52}$$

$$\tilde{x}'(0) = x'(0\,;\alpha) = w_1(\alpha) \tag{53}$$

$\tilde{x}''(0) = x''(0\,;\alpha) = w_2(\alpha)$ (54)

$\vdots$

$\tilde{x}^{(n-1)}(0) = x^{(n-1)}(0\,;\alpha) = w_{n-1}(\alpha)$ (55)

Where:

$\tilde{b}_{n-1}(t), \tilde{b}_{n-2}(t), \dots, \tilde{b}_1(t), \tilde{b}_0(t)$ and $\tilde{k}(t)$ are TrFFs

$w_0(\alpha)$, $w_1(\alpha)$, $w_2(\alpha)$, … , $w_{n-1}(\alpha)$ are TrFNs.

Since $\tilde{b}_{n-1}(t), \tilde{b}_{n-2}(t), \dots, \tilde{b}_1(t), \tilde{b}_0(t)$ and $\tilde{k}(t)$ are TrFFs, then we must have:

$\tilde{b}_{n-1}(t) = b_{n-1}(t\,;\alpha) = (g_1(t),\ g_2(t),\ g_3(t),\ g_4(t))$ (56)

$\tilde{b}_{n-2}(t) = b_{n-2}(t\,;\alpha) = (g_5(t),\ g_6(t),\ g_7(t),\ g_8(t))$ (57)

$\vdots$

$\tilde{b}_1(t) = b_1(t\,;\alpha) = (g_9(t),\ g_{10}(t),\ g_{11}(t),\ g_{12}(t))$ (58)

$\tilde{b}_0(t) = b_0(t\,;\alpha) = (g_{13}(t),\ g_{14}(t),\ g_{15}(t),\ g_{16}(t))$ (59)

$\tilde{k}(t) = k(t\,;\alpha) = (g_{17}(t),\ g_{18}(t),\ g_{19}(t),\ g_{20}(t))$ (60)

Where $g_1(t),\ g_2(t), \dots, g_{20}(t)$ are continuous real-valued functions.

Since $w_0(\alpha)$, $w_1(\alpha)$, $w_2(\alpha)$, … , $w_{n-1}(\alpha)$ are TrFNs, then we must have:

$w_0(\alpha) = (r_1\,, r_2\,, r_3,\ r_4)$ (61)

$w_1(\alpha) = (r_5\,, r_6\,, r_7,\ r_8)$ (62)

$w_2(\alpha) = (r_9\,, r_{10}\,, r_{11},\ r_{12})$ (63)

$\vdots$

$w_{n-1}(\alpha) = (r_{13}\,, r_{14}\,, r_{15},\ r_{16})$ (64)

Where $r_1,\ r_2, \dots, r_{16}$ are real numbers.

By using the concepts that we introduced in section two, we write the parametric form of the equations (56-64) as follows:

$b_{n-1}(t\,;\alpha) = [\,b_{n-1}(t)\,]_\alpha = [[b_{n-1}(t)]^L_\alpha\,,\ [b_{n-1}(t)]^U_\alpha]$

$= [(g_2(t) - g_1(t))\alpha + g_1(t), (g_3(t) - g_4(t))\alpha + g_4(t)]$ (65)

$b_{n-2}(t\,;\alpha) = [\,b_{n-2}(t)\,]_\alpha = [[b_{n-2}(t)]^L_\alpha\,,\ [b_{n-2}(t)]^U_\alpha]$

$= [(g_6(t) - g_5(t))\alpha + g_5(t)\,, (g_7(t) - g_8(t))\alpha + g_8(t)]$ (66)

$\vdots$

$b_1(t\,;\alpha) = [\,b_1(t)\,]_\alpha = [[b_1(t)]^L_\alpha\,,\ [b_1(t)]^U_\alpha]$

$= [(g_{10}(t) - g_9(t))\alpha + g_9(t)\,, (g_{11}(t) - g_{12}(t))\alpha + g_{12}(t)]$ (67)

$b_0(t\,;\alpha) = [\,b_0(t)\,]_\alpha = [[b_0(t)]^L_\alpha\,,\ [b_0(t)]^U_\alpha]$

$= [(g_{14}(t) - g_{13}(t))\alpha + g_{13}(t)\,, (g_{15}(t) - g_{16}(t))\alpha + g_{16}(t)]$ (68)

$k(t\,;\alpha) = [\,k(t)\,]_\alpha = [[k(t)]^L_\alpha\,,\ [k(t)]^U_\alpha]$

$= [(g_{18}(t) - g_{17}(t))\alpha + g_{17}(t)\,, (g_{19}(t) - g_{20}(t))\alpha + g_{20}(t)]$ (69)

$w_0(\alpha) = [\,w_0\,]_\alpha = [[w_0]^L_\alpha\,,\ [w_0]^U_\alpha]$

$= [(r_2 - r_1)\alpha + r_1\,, (r_3 - r_4)\alpha + r_4]$ (70)

$w_1(\alpha) = [\,w_1\,]_\alpha = [[w_1]^L_\alpha\,,\ [w_1]^U_\alpha]$

$= [(r_6 - r_5)\alpha + r_5\,, (r_7 - r_8)\alpha + r_8]$ (71)

$w_2(\alpha) = [\,w_2\,]_\alpha = [[w_2]^L_\alpha\,,\ [w_2]^U_\alpha]$

$= [(r_{10} - r_9)\alpha + r_9\,, (r_{11} - r_{12})\alpha + r_{12}]$ (72)

$\vdots$

$w_{n-1}(\alpha) = [\,w_{n-1}\,]_\alpha = [[w_{n-1}]^L_\alpha\,,\ [w_{n-1}]^U_\alpha]$

$= [(r_{14} - r_{13})\alpha + r_{13}\,, (r_{15} - r_{16})\alpha + r_{16}]$ (73)

There are many main approaches in solving the FDE. The most popular approach is so called the defuzzification. The principal idea in this approach is converting the FDE into a system of NFDEs by using the properties of the α-level sets. (For more details, see [12,14,17,19]).

Therefore, for solving problem (51), we convert it into a system of nth-order linear NFDEs as follows:

$$[\, x^{(n)}(t) \;+ b_{n-1}(t)\, x^{(n-1)}(t\,) \;+ b_{n-2}(t)\, x^{(n-2)}(t\,) \;+ \cdots + b_1(t)\, x'(t) + b_0(t)\, x(t)]_\alpha = [\, k(t)\,]_\alpha \quad (74)$$

With fuzzy initial conditions:

$$[\, x(0)]_\alpha = [\, w_0\,]_\alpha \quad (75)$$

$$[\, x'(0)]_\alpha = [\, w_1\,]_\alpha \quad (76)$$

$$[\, x''(0)]_\alpha = [\, w_2\,]_\alpha (77)$$

⋮

$$[\, x^{(n-1)}(0)]_\alpha = [\, w_{n-1}\,]_\alpha (78)$$

Then, we get:

$$[\, x^{(n)}(t)\,]_\alpha + [\, b_{n-1}(t)\, x^{(n-1)}(t\,)\,]_\alpha + [\, b_{n-2}(t)\, x^{(n-2)}(t\,)\,]_\alpha + \cdots + [\, b_1(t)\, x'(t)]_\alpha + [\, b_0(t)\, x(t)]_\alpha = [\, k(t)\,]_\alpha \quad (79)$$

Therefore, we have:

$$[x^{(n)}(t)]_\alpha + [b_{n-1}(t)]_\alpha [x^{(n-1)}(t\,)]_\alpha + [b_{n-2}(t)]_\alpha\, [x^{(n-2)}(t\,)]_\alpha + \cdots + [b_1(t)]_\alpha [\, x'(t)]_\alpha + [b_0(t)]_\alpha [x(t)]_\alpha = [\, k(t)\,]_\alpha \quad (80)$$

Then, we write the lower bound and the upper bound of equation (80) as follows:

●) The lower bound of the parametric form:

$$\left[\, x^{(n)}(t)\right]_\alpha^L + [\, b_{n-1}(t)]_\alpha^L \left[x^{(n-1)}(t\,)\right]_\alpha^L + [\, b_{n-2}(t)]_\alpha^L \left[x^{(n-2)}(t\,)\right]_\alpha^L + \cdots + [\, b_1(t)]_\alpha^L [x'(t\,)]_\alpha^L + [\, b_0(t)]_\alpha^L [x(t)]_\alpha^L = [k(t)]_\alpha^L \quad (81)$$

With initial conditions:

$$[x(0)]_\alpha^L = [w_0]_\alpha^L \quad (82)$$

$$[x'(0)]_\alpha^L = [w_1]_\alpha^L \quad (83)$$

$$[x''(0)]_\alpha^L = [w_2]_\alpha^L \quad (84)$$

⋮

$$[x^{(n-1)}(0)]_\alpha^L = [w_{n-1}]_\alpha^L \quad (85)$$

Where:

$$[b_{n-1}(t)]_\alpha^L = (g_2(t) - g_1(t))\alpha + g_1(t) \quad (86)$$

$$[b_{n-2}(t)]_\alpha^L = (g_6(t) - g_5(t))\alpha + g_5(t) \quad (87)$$

⋮

$$[b_1(t)]_\alpha^L = (g_{10}(t) - g_9(t))\alpha + g_9(t) \quad (88)$$

$$[b_0(t)]_\alpha^L = (g_{14}(t) - g_{13}(t))\alpha + g_{13}(t) \quad (89)$$

$$[k(t)]_\alpha^L = (g_{18}(t) - g_{17}(t))\alpha + g_{17}(t) \quad (90)$$

$$[w_0]_\alpha^L = (\, r_2 - \, r_1)\alpha + \, r_1 \quad (91)$$

$$[w_1]_\alpha^L = (\, r_6 - \, r_5)\alpha + \, r_5 \quad (92)$$

$$[w_2]_\alpha^L = (\, r_{10} - \, r_9)\alpha + \, r_9 \quad (93)$$

⋮

$$[w_{n-1}]_\alpha^L = (\, r_{14} - \, r_{13})\alpha + \, r_{13} \quad (94)$$

Now, by using LODMM that we described in section four, we will solve equation (81) subject to the initial conditions in the equations (82-85), we can obtain the lower bound of the fuzzy solution of problem (51) which is $[x(t)]_\alpha^L$.

●) The upper bound of the parametric form:

$$\left[\, x^{(n)}(t)\right]_\alpha^U + [\, b_{n-1}(t)]_\alpha^U \left[x^{(n-1)}(t\,)\right]_\alpha^U + [\, b_{n-2}(t)]_\alpha^U \left[x^{(n-2)}(t\,)\right]_\alpha^U + \cdots + [\, b_1(t)]_\alpha^U [x'(t\,)]_\alpha^U + [\, b_0(t)]_\alpha^U [x(t)]_\alpha^U = [k(t)]_\alpha^U \quad (95)$$

With initial conditions:

$$[x(0)]_\alpha^U = [w_0]_\alpha^U \quad (96)$$

$$[x'(0)]_\alpha^U = [w_1]_\alpha^U \quad (97)$$

$$[x''(0)]_\alpha^U = [w_2]_\alpha^U \quad (98)$$

⋮

$[x^{(n-1)}(0)]^U_\alpha = [w_{n-1}]^U_\alpha$ (99)

Where:

$[b_{n-1}(t)]^U_\alpha = (g_3(t) - g_4(t))\alpha + g_4(t)$ (100)

$[b_{n-2}(t)]^U_\alpha = (g_7(t) - g_8(t))\alpha + g_8(t)$ (101)

⋮

$[b_1(t)]^U_\alpha = (g_{11}(t) - g_{12}(t))\alpha + g_{12}(t)$ (102)

$[b_0(t)]^U_\alpha = (g_{15}(t) - g_{16}(t))\alpha + g_{16}(t)$ (103)

$[k(t)]^U_\alpha = (g_{19}(t) - g_{20}(t))\alpha + g_{20}(t)$ (104)

$[w_0]^U_\alpha = (r_3 - r_4)\alpha + r_4$ (105)

$[w_1]^U_\alpha = (r_7 - r_8)\alpha + r_8$ (106)

$[w_2]^U_\alpha = (r_{11} - r_{12})\alpha + r_{12}$ (107)

⋮

$[w_{n-1}]^U_\alpha = (r_{15} - r_{16})\alpha + r_{16}$ (108)

Now, by using LODMM that we described in section four, we will solve equation (95) subject to the initial conditions in the equations (96-99), we can obtain the upper bound of the fuzzy solution of problem (51) which is $[x(t)]^U_\alpha$.

Finally, we obtain the fuzzy solution of the problem (51), which is:

$\tilde{x}(t) = [x(t)]_\alpha = [[x(t)]^L_\alpha , [x(t)]^U_\alpha$
(109)

The following proposition ensures the existence and uniqueness of the fuzzy solution of the problem (51).

Proposition (5.1):

Let $\tilde{F}:[0,\infty) \times \tilde{E} \times \tilde{E} \times \dots \times \tilde{E} \to \tilde{E}$ be continuous fuzzy function and assume that there exist a real numbers $q_1, q_2, \dots, q_n > 0$ such that [1]:

$$D\left(\tilde{F}(t, \tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n), \tilde{F}(t, \tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_n)\right) \le \sum_{i=1}^{n} q_i D(\tilde{x}_i, \tilde{y}_i),$$

for all $t \in [a,b], \tilde{x}_i, \tilde{y}_i \in \tilde{E}, i = 1,2,\dots,n$.

Then the nth-order FDE described by the problem (51) has a unique solution on $[0,\infty)$.

VI. EXAMPLES

In this section, we will solve three fuzzy problems. For each problem, we obtain the absolute errors:

$[error]^L_\alpha = \left| [x_{exact}(t)]^L_\alpha - [x_{app}(t)]^L_\alpha \right|$ (110)

$[error]^U_\alpha = \left| [x_{exact}(t)]^U_\alpha - [x_{app}(t)]^U_\alpha \right|$ (111)

Where:

$[x_{exact}(t)]^L_\alpha$ and $[x_{exact}(t)]^U_\alpha$ are the lower bound and upper bound of the FEAS, respectively.

$[x_{app}(t)]^L_\alpha$ and $[x_{app}(t)]^U_\alpha$ are the lower bound and upper bound of the FAAS , respectively.

Example (1): Consider the first order non-linear FDE:

$(0.6, 0.9, 1.3, 1.7)\tilde{x}'(t) + (\tilde{x}'(t))^2 = (0, t^2, 2t^2, 8)\ ;\ \ t \in [0,2]$
(112)

With the fuzzy initial condition:

$\tilde{x}(0) = (0,0,1,1).$ (113)

Solution:

The parametric form is:

$[0.3\alpha + 0.6, -0.4\alpha + 1.7][x'(t)]_\alpha + [(x'(t))^2]_\alpha = [\alpha t^2, (2t^2 - 8)\alpha + 8]$ (114)

With:

$[x(0)]_\alpha = [0, 1]$ (115)

Then, we have the following system:

$(0.3\alpha + 0.6)[x'(t)]^L_\alpha + [(x'(t))^2]^L_\alpha = \alpha t^2$ (116)

$(-0.4\alpha + 1.7)[x'(t)]^U_\alpha + [(x'(t))^2]^U_\alpha = (2t^2 - 8)\alpha + 8$ (117)

With:

$[x(0)]^L_\alpha = 0$ (118)

$[x(0)]^U_\alpha = 1$ (119)

For $\alpha = 0.4$, we get:

$0.72[x'(t)]^L_{0.4} + [(x'(t))^2]^L_{0.4} = 0.4t^2$ (120)

$1.54[x'(t)]^U_{0.4} + [(x'(t))^2]^U_{0.4} = 0.8t^2 + 4.8$ (121)

With:

$[x(0)]^L_{0.4} = 0$ (122)

$[x(0)]^U_{0.4} = 1$ (123)

Then, the fuzzy approximate-solution of the problem (112) will be:

$\tilde{x}(t) = [\ [x(t)]^L_{0.4}\ , [x(t)]^U_{0.4}\ ]$ (124)

Where:

$[x(t)]^L_{0.4} = (0.004747651525656)t^2 + (0.183037497865060)t^3 - (0.047717895175020)t^4$ (125)

$[x(t)]^U_{0.4} = 1 + (1.552261828476712)t - (0.268449988712856)t^2 + (0.250695421048220)t^3 - (0.012223579327190)t^4$ (126)

The fuzzy exact-analytical solutions:

$[x(t)]_{0.4} = [[x(t)]^L_{0.4}\ , [x(t)]^U_{0.4}]$ (127)

Where:

$[x(t)]^L_{0.4} = -0.36t + \sqrt{0.0324t^2 + 0.1t^4} + \frac{0.1296}{\sqrt{1.6}} \ln\left(\frac{\sqrt{0.4t^2} + \sqrt{0.4t^2 + 0.1296}}{\sqrt{0.1296}} \right)$ (128)

$[x(t)]^U_{0.4} = -0.77t + \sqrt{1.348225t^2 + 0.2t^4} + \frac{5.3929}{\sqrt{3.2}} \ln\left(\frac{\sqrt{0.8t^2} + \sqrt{0.8t^2 + 5.3929}}{\sqrt{5.3929}} \right) + 1$ (129)

Numerical solutions for this problem can be found in the following tables:

Table (1): Numerical result for example (1), $\alpha = 0.4$

t	$[x_{app}(t)]^L_\alpha$	$[error]^L_\alpha$	$[x_{app}(t)]^U_\alpha$	$[error]^U_\alpha$
0	0	0	1	0
0.0000741	0.002614292332250	2.61e-11	1.000115021127584	1.47e-9
0.0001482	0.010486956517651	1.04 e-10	1.000230039307765	5.90 e-9
0.0002223	0.023662670749611	2.35 e-10	1.000345054541153	1.33 e-8
0.0002964	0.042186109768789	4.17 e-10	1.000460066828362	2.36 e-8
0.0003705	0.066101944863091	6.52 e-10	1.000575076170003	3.68 e-8
0.0004446	0.095454843867673	9.38 e-10	1.000690082566689	5.30 e-8
0.0005187	0.130289471164938	1.28 e-9	1.000805086019030	7.22 e-8
0.0005928	0.170650487684536	1.67 e-9	1.000920086527639	9.43 e-8
0.0006669	0.216582550903367	2.11 e-9	1.001035084093128	1.19 e-7

0.000741	0.268130314845580	2.61 e-9	1.001150078716109	1.47 e-7

Table (2): Numerical result for example (1), $\alpha = 0.4$

t	$[x_{app}(t)]^L_\alpha$	$[error]^L_\alpha$	$[x_{app}(t)]^U_\alpha$	$[error]^U_\alpha$
0	0	0	1	0
0.00000458	0.009960622220416	9.96e-14	1.000007109353543	5.63e-12
0.00000916	0.039849522761424	3.98 e-13	1.000014218695825	2.25 e-11
0.00001374	0.089677252404870	8.96 e-13	1.000021328026844	5.07 e-11
0.00001832	0.159454361882208	1.59 e-12	1.000028437346601	9.10 e-11
0.0000229	0.249191401874501	2.49 e-12	1.000035546655097	1.41 e-10
0.00002748	0.358898923012423	3.59 e-12	1.000042655952332	2.03 e-10
0.00003206	0.488587475876254	4.88 e-12	1.000049765238305	2.76 e-10
0.00003664	0.638267610995883	6.37 e-12	1.000056874513017	3.60 e-10
0.00004122	0.807949878850811	8.07 e-12	1.000063983776467	4.56 e-10
0.0000458	0.997644829870144	9.96 e-12	1.000071093028657	5.63 e-10

Example (2): Consider the second order non-linear FDE:

$$(\tilde{x}''(t))^2 - (3.2, 3.7, 4.1, 4.8) = \frac{(-3t,-t,2t,5t)}{(-2,2,4,8)}\ ; t \in [0,2] \quad (130)$$

With the fuzzy boundary conditions:

$$\tilde{x}(0) = (0.2, 0.4, 0.7, 0.8) \quad (131)$$

$$\tilde{x}(2) = (2.3, 3.5, 3.8, 4.9) \quad (132)$$

Solution:

The parametric form is:

$$[(x''(t))^2]_\alpha - [0.5\alpha{+}3.2\ , -0.7\alpha{+}4.8] = [2\alpha t - \tfrac{5}{2}t\ , -\tfrac{1}{2}\alpha t + \tfrac{3}{2}t] \quad (133)$$

With:

$$[x(0)]_\alpha = [0.2\alpha{+}0.2\ , -0.1\alpha{+}0.8] \quad (134)$$

$$[x(2)]_\alpha = [1.2\alpha{+}2.3\ , -1.1\alpha{+}4.9] \quad (135)$$

Then, we have the following system:

$$[(x''(t))^2]^L_\alpha - (0.5\alpha{+}3.2) = 2\alpha t - \tfrac{5}{2}t \quad (136)$$

$$[(x''(t))^2]^U_\alpha - (-0.7\alpha{+}4.8) = -\tfrac{1}{2}\alpha t + \tfrac{3}{2}t \quad (137)$$

With:

$$[x(0)]^L_\alpha = 0.2\alpha{+}0.2 \quad (138)$$

$$[x(0)]^U_\alpha = -0.1\alpha{+}0.8 \quad (139)$$

$$[x(2)]^L_\alpha = 1.2\alpha{+}2.3 \quad (140)$$

$$[x(2)]^U_\alpha = -1.1\alpha{+}4.9 \quad (141)$$

For α=0.8, we get:

$$[(x''(t))^2]^L_{0.8} - 3.6 = -0.9t \quad (142)$$

$$[(x''(t))^2]^U_{0.8} - 4.24 = 1.1t \quad (143)$$

With:

$$[x(0)]^L_{0.8} = 0.36 \quad (144)$$

$$[x(0)]^U_{0.8} = 0.72 \quad (145)$$

$$[x(2)]^L_{0.8} = 3.26 \quad (146)$$

$$[x(2)]^U_{0.8} = 4.02 \quad (147)$$

Then, the fuzzy approximate-solution of the problem (130) will be:

$$\tilde{x}(t) = [\ [x(t)]^{L}_{0.8}\ ,\ [x(t)]^{U}_{0.8}\ ] \quad (148)$$

Where:

$$[x(t)]^{L}_{0.8} = 0.36 - (1.695667290353982)t + (1.728488400528276)t^2 - (3.844199876030980)t^3 + (1.883186249177660)t^4 \quad (149)$$

$$[x(t)]^{U}_{0.8} = 0.72 - (2.409382950485358)t + (1.875898128786384)t^2 - (4.573787381436181)t^3 + (2.325342027332170)t^4 \quad (150)$$

The fuzzy exact-analytical solution is:

$$[x(t)]_{0.8} = [[x(t)]^{L}_{0.8}\ , [x(t)]^{U}_{0.8}] \quad (151)$$

Where:

$$[x(t)]^{L}_{0.8} = \frac{80}{243}(3.60 - 0.9t)^{\frac{5}{2}} + (4.782173652215594)t - (7.735430810031052) \quad (152)$$

$$[x(t)]^{U}_{0.8} = \frac{80}{363}(4.24 + 1.1t)^{\frac{5}{2}} - (5.868463229949957)t - (7.438268668765088) \quad (153)$$

Numerical solutions for this problem can be found in the following tables:

Table (3): Numerical result for example (2), $\alpha = 0.8$

t	$[x_{app}(t)]^{L}_{\alpha}$	$[error]^{L}_{\alpha}$	$[x_{app}(t)]^{U}_{\alpha}$	$[error]^{U}_{\alpha}$
0	0.36	0	0.72	0
0.00000584	0.359990097361975	8.28e-6	0.719985929267547	1.07 e-5
0.00001168	0.359980194841847	1.66 e-5	0.719971858663045	2.14 e-5
0.00001752	0.359970292439613	2.48 e-5	0.719957788186491	3.21 e-5
0.00002336	0.359960390155266	3.31 e-5	0.719943717837876	4.28 e-5
0.0000292	0.359950487988804	4.14 e-5	0.719929647617198	5.35 e-5
0.00003504	0.359940585940221	4.97 e-5	0.719915577524449	6.42 e-5
0.00004088	0.359930684009513	5.80 e-5	0.719901507559625	7.50 e-5
0.00004672	0.359920782196675	6.63 e-5	0.719887437722719	8.56 e-5
0.00005256	0.359910880501703	7.45 e-5	0.719873368013727	9.63 e-5
0.0000584	0.359900978924591	8.28 e-5	0.719859298432644	1.07 e-4

Table (4): Numerical result for example (2), $\alpha = 0.8$

t	$[x_{app}(t)]^{L}_{\alpha}$	$[error]^{L}_{\alpha}$	$[x_{app}(t)]^{U}_{\alpha}$	$[error]^{U}_{\alpha}$
0	0.36	0	0.72	0
-0.00000327	0.359994455186443	4.64e-6	0.719992121337811	5.99 e-6
-0.00000654	0.359988910409850	9.27 e-6	0.719984242715738	1.20 e-5
-0.0000098	0.359983365670221	1.39 e-5	0.719976364133781	1.80 e-5

1				
-0.00001308	0.359977820967554	1.85 e-5	0.719968485591938	2.40 e-5
-0.00001635	0.359972276301850	2.32 e-5	0.719960607090209	3.00 e-5
-0.00001962	0.359966731673106	2.78 e-5	0.719952728628593	3.59 e-5
-0.00002289	0.359961187081323	3.25 e-5	0.719944850207089	4.19 e-5
-0.00002616	0.359955642526499	3.71 e-5	0.719936971825696	4.79 e-5
-0.00002943	0.359950098008634	4.17 e-5	0.719929093484413	5.39 e-5
-0.0000327	0.359944553527726	4.64 e-5	0.719921215183238	5.99 e-5

Example (3): Consider the third order non-linear FDE:

$$\tilde{x}''(t) + \left(t, \sqrt{t}, \sqrt[3]{t}, \sqrt[4]{t}\right)(\tilde{x}'''(t))^2 = (1, 4, 6, 8)\ ;\ t \in [0,1] \quad (154)$$

With the fuzzy initial conditions:

$$\tilde{x}(0) = (2, 3, 7, 9) \quad (155)$$

$$\tilde{x}'(0) = (2, 3, 5, 6) \quad (156)$$

$$\tilde{x}''(0) = (3, 4, 6, 8) \quad (157)$$

Solution:

The parametric form is:

$$[x''(t)]_\alpha + \left[\left(\sqrt{t} - t\right)\alpha + t\ , \left(\sqrt[3]{t} - \sqrt[4]{t}\right)\alpha + \sqrt[4]{t}\ \right][(x'''(t))^2]_\alpha = [3\alpha + 1, -2\alpha + 8] \quad (158)$$

With:

$$[x(0)]_\alpha = (\alpha+2, -2\alpha+9) \quad (159)$$

$$[x'(0)]_\alpha = (\alpha+2, -\alpha+6) \quad (160)$$

$$[x''(0)]_\alpha = (\alpha+3, -2\alpha+8) \quad (161)$$

Then, we have the following system:

$$[x''(t)]_\alpha^L + ((\sqrt{t} - t)\alpha+t)\left[\left(x'''(t)\right)^2\right]_\alpha^L = 3\alpha + 1 \quad (162)$$

$$[x''(t)]_\alpha^U + ((\sqrt[3]{t} - \sqrt[4]{t})\alpha+\sqrt[4]{t})\left[\left(x'''(t)\right)^2\right]_\alpha^U = -2\alpha + 8 \quad (163)$$

With:

$$[x(0)]_\alpha^L = \alpha + 2 \quad (164)$$

$$[x(0)]_\alpha^U = -2\alpha + 9 \quad (165)$$

$$[x'(0)]_\alpha^L = \alpha + 2 \quad (166)$$

$$[x'(0)]_\alpha^U = -\alpha + 6 \quad (167)$$

$$[x''(0)]_\alpha^L = \alpha + 3 \quad (168)$$

$$[x''(0)]_\alpha^U = -2\alpha + 8 \quad (169)$$

For $\alpha = 1$, we get:

$$[x''(t)]_1^L + \sqrt{t}\left[\left(x'''(t)\right)^2\right]_1^L = 4 \quad (170)$$

$$[x''(t)]_1^U + \sqrt[3]{t}\left[\left(x'''(t)\right)^2\right]_1^U = 6 \quad (171)$$

With:

$[x(0)]_1^L = 3$ (172)

$[x(0)]_1^U = 7$ (173)

$[x'(0)]_1^L = 3$ (174)

$[x'(0)]_1^U = 5$ (175)

$[x''(0)]_1^L = 4$ (176)

$[x''(0)]_1^U = 6$ (177)

By reduction of order, we get:

$$[y'(t)]_1^L + \sqrt{t}\left[\left(y''(t)\right)^2\right]_1^L = 4 \quad (178)$$

$$[y'(t)]_1^U + \sqrt[3]{t}\left[\left(y''(t)\right)^2\right]_1^U = 6 \quad (179)$$

With:

$[y(0)]_1^L = 3$ (180)

$[y(0)]_1^U = 5$ (181)

$[y'(0)]_1^L = 4$ (182)

$[y'(0)]_1^U = 6$ (183)

Then, the fuzzy approximate-solution of the system (4.168-4.173) will be:

$[y(t)]_1^L = 3 + 4t - (0.780370873116012)t^2 + (1.103883564354000)t^3 - (0.442486144789010)t^4$ (184)

$[y(t)]_1^U = 5 + 6t - (0.487977118633692)t^2 + (0.715705173854900)t^3 - (0.291825462876080)t^4$ (185)

Therefore, the fuzzy approximate-solution of the problem (154) is:

$$[x(t)]_1 = [[x(t)]_1^L, [x(t)]_1^U] \quad (186)$$

Where:

$[x(t)]_1^L = 3 + 3t + 2t^2 - (0.260123624372004)t^3 + (0.275970891088500)t^4 - (0.088497228957802)t^5$ (187)

$[x(t)]_1^U = 7 + 5t + 3t^2 - (0.162569039544564)t^3 + (0.178926293463725)t^4 - (0.058365092575216)t^5$ (111)

The fuzzy exact-analytical solution is:

$$[x(t)]_1 = [[x(t)]_1^L, [x(t)]_1^U] \quad (188)$$

Where:

$$[x(t)]_1^L = 3 + 3t + 2t^2 - \frac{16}{315}t^{\frac{7}{2}} \quad (189)$$

$$[x(t)]_1^U = 7 + 5t + 3t^2 - \frac{27}{440}t^{\frac{11}{3}} \quad (190)$$

Numerical solutions for this problem can be found in the following tables:

Table (5): Numerical result for example (3), $\alpha = 1$

t	$[x_{app}(t)]_\alpha^L$	$[error]_\alpha^L$	$[x_{app}(t)]_\alpha^U$	$[error]_\alpha^U$
0	3	0	7	0
0.0000252	3.000075601270076	4.00e-15	7.000126001905117	2.66 e-15
0.0000504	3.000151205080287	3.33e-14	7.000252007620460	2.04 e-14
0.0000756	3.000226811430608	1.12e-13	7.000378017146010	7.02 e-14
0.0001008	3.000302420321014	2.66e-13	7.000504030481754	1.66 e-13
0.000126	3.000378031751480	5.19e-13	7.000630047627675	3.25 e-13
0.0001512	3.000453645721981	8.97e-13	7.000756068583757	5.61 e-13
0.0001764	3.000529262232492	1.42e-12	7.000882093349987	8.92 e-13
0.0002016	3.000604881282989	2.12e-12	7.001008121926348	1.33 e-12
0.0002268	3.000680502873446	3.02e-12	7.001134154312825	1.89 e-12

0.000252	3.000756127003839	4.15e-12	7.001260190509400	2.60 e-12

Table (6): Numerical result for example (3), $\alpha = 1$

t	$[x_{app}(t)]^L_\alpha$	$[error]^L_\alpha$	$[x_{app}(t)]^U_\alpha$	$[error]^U_\alpha$
0	3	0	7	0
0.00000619	3.000018570076632	0	7.000030950114948	0
0.00001238	3.000037140306528	4.44 e-16	7.000061900459794	0
0.00001857	3.000055710689688	1.78 e-15	7.000092851034534	8.88e-16
0.00002476	3.000074281226111	4.00 e-15	7.000123801839170	2.66 e-15
0.00003095	3.000092851915797	7.55 e-15	7.000154752873703	4.44 e-15
0.00003714	3.000111422758746	1.33 e-14	7.000185704138131	8.00 e-15
0.00004333	3.000129993754956	2.13 e-14	7.000216655632453	1.33 e-14
0.00004952	3.000148564904429	3.15 e-14	7.000247607356672	1.95 e-14
0.00005571	3.000167136207163	4.49 e-14	7.000278559310784	2.84 e-14
0.0000619	3.000185707663158	6.17 e-14	7.000309511494792	3.82 e-14

VII. DISCUSSION

Through the applied examples that solved in this work, it can be seen that LODMM based on Tau method has a high efficiency in approximating the exact-analytical solution, as the comparison that we conducted with other approximation methods showed the accuracy of the results that can be obtained when using this method. These results can improve further when increasing the number of terms of the solution series. This means, using a larger value for m, such as m = 5, m = 6, and so on.

From the solved examples in this work, we can conclude that several factors affect the accuracy of the results, namely:

- The number of terms of the solution series. The more terms in the solution series, the more accurate results will be obtained.
- The value of the variable t. If the value of t is close to the initial value, the results will be more accurate.
- The value of the constant α. In fact, the best value of α cannot be determined, as it changes from one problem to another.
- The mathematical nature of the problem, whether it is linear or non-linear.
- The order of the FDE, whether it is first order or higher order.

It is necessary to note that the lower absolute error is not related to the upper absolute error. In the same problem, with the same value of α and the same value of t, the lower absolute error may be higher than the upper absolute error and vice versa. As for the case of equality between the two errors, it is rare.

VIII. CONCLUSION

In this work, we have used the fuzzy function of LODMM based on Tau method to obtain the FAAS of the FDEs in which the coefficients are TrFFs .The approximate solutions that we obtained are accurate solutions and very close to the FEAS. In comparison with the other methods, we can determine many advantages for the LODMM, namely: it is computational less cost, it needs less computational time and effort and it has better accuracy.

For the future works, one can extend and use this method for solving other types of the FDE such as fuzzy fractional differential equations, fuzzy partial differential equations, fuzzy delay differential equations, etc. Also, one can use this method for solving FDEs with other types of the fuzzy function coefficients such as exponential fuzzy function coefficients, etc.

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