On Some Operators in Anti-Topological Spaces

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Abstract—*This article introduces some anti-topological operators and provides various examples and results regarding the new operators. Specifically, We introduce the anti-frontier operator and examine essential properties of such concept. Furthermore, we introduce anti-accumulation points and anti-exterior of sets. We also introduce minimal and maximal anti-open sets and their dual. We show that every anti-open and anti-closed set are both minimal and maximal.*

Keywords— Anti-topology; anti-open; anti-closed; anti-frontier; anti-accumulation.

I. INTRODUCTION

Dropping some axioms of topological spaces can lead to new types of topological spaces. Such a modification proved to be interesting for many researchers during the period of almost the past three decades, and yet this research line seems to last longer and further investigations and studies to be conducted.

If we dispense with the closeness of an arbitrary union of open sets, then new types of topologies are introduced. In the literature, such topologies are called infra-topologies. Studies like [1-4] dealt with such topological spaces, and many interesting results and properties were presented. In [5], a new investigation carried out on infra-topological spaces, and further results and findings were exhibited.

Other types of topologies were initiated by removing the condition of closeness of the intersection of finite open sets. Such topologies are called supra-topologies and were first introduced in [6]. Other investigations followed introducing supra-topologies such as in [7]. The procedure for getting rid of some conditions showed potential research findings need to be explored.

Removing the closure of intersections and unions of open sets leaded to the definition of another kind of interested topologies, which are named minimal structures, [8]. Then another further scarification of removing some conditions of topological spaces occurred when all but one condition survived. The condition is just keeping the empty set in a collection of sets. Such a structure is called a weak structure, which is introduced in [9]. The reader might think about the possibility of removing all conditions that topological spaces possess. This kind of modification occurred when anti-topologies were defined in [10]. In such structures, all the conditions of topological spaces are removed. Yet, interesting results concerning anti-topological spaces were already captured in [11].

The main motivation of this article is to continue investigating the properties of anti-topologies and define new anti-topological operators, such as the anti-frontier operator. We also define and investigate minimal and maximal anti-open sets and their dual.

II. REVIEWING ANTI-TOPOLOGICAL SPACES

We start with recalling the concept of anti-topological spaces and other related terms.

Definition 2.1: [11] An *anti-topological space* is the pair $(X^a,\ \mathcal{A})$, where $\mathcal{A}$ is a collection of subsets of a non-empty set X^a such that the following conditions are held.

1. $\phi, X^a \notin \mathcal{A}$.

2. The intersection of any finite collection of is not contained in $\mathcal{A}$ provided that the intersection is not trivial.

3. The union of any collection of members of is not contained in $\mathcal{A}$ provided that the union is not trivial.

Elements of $\mathcal{A}$ are said to be *anti-open* sets. The complements of anti-open sets are called *anti-closed* sets. We denote the collection of all anti-closed sets by $F\left(X^a\right)$.

Example 2.2: Let $X^a = \{1,2\}$ and $= \{\{1\},\{2\}\}$. It is clear that $\mathcal{A}$ is anti-topology.

Example 2.3: Let $X^a = \{1,2,3,4\}$ and

$\mathcal{A} = \{\{1,2\},\{2,3\},\{3,4\}\}$. Clearly, $\mathcal{A}$ is an anti-topology.

Example 2.4: Let $X^a = \{a,b,c\}$ and $\mathcal{A} = \{\{a\},\{b\},\{c\}\}$. It is clear that is an anti-topology.

Definition 2.5: Assume that $\mathcal{A}$ defines an anti-topology on X^a and $x \in X^a$. An *anti-neibourhood* of x is a subset H of X^a such that there exists an anti-open set P of X^a such that $x \in P \subset H$. By $N_a(x)$ we denote all anti-neighbourhoods of x.

Example 2.6: Consider the anti-topological space $(X^a, \mathcal{A})$ where $X^a = \{1,2,3,4,5\}$ and

$\mathcal{A} = \{\{1,2,3\},\{1,4\},\{3,4,5\},\{2,5\},\{1,5\}\}$. Then $\{1,4,5\} \in N_a(4)$ and $\{2,3,4,5\} \in N_a(5)$.

For $x \in X^a$, one can easily notice that $N_a(x)$ can be void if there is no $U \in \mathcal{A}$ with $x \in U$. The reason behind this is because $X^a \notin \mathcal{A}$.

The *anti-interior* of a set A in an anti-topological space $(X^a, \mathcal{A})$ is the set $Int^a(A) = \bigcup\{U \in \mathcal{A} : U \subseteq A\}$. The *anti-closure* of A is the set $Cl^a(A) = \bigcap\{F \in F_a(X) : A \subseteq F\}$. See [11] for more results on such two concepts.

Example 2.7: Consider the anti-topological space $(X^a, \mathcal{A})$ where $X^a = \{a,b,c,d,e\}$ and $\mathcal{A} = \{\{a,b\},\{a.c\},\{d\},\{e\}\}$ $, F\left(X^a\right) = \{\{c,d,e\},\{b,d,e\},\{a,b,c,e\},\{a,b,c,d\}\}$.

Consider the set $\{d,e\}$. So, $Int^a(\{d,e\}) = \{d,e\}$ and $Cl^a(\{d,e\}) = \{c,d,e\} \cap \{b,d,e\} = \{d,e\}$.

Unlike interior and closure operators in topological spaces, anti-interior needs not be anti-open and anti-closure needs not be anti-closed, see Example 2.7.

Proposition 2.8: Suppose that $\mathcal{A}$ defines an anti-topology on X^a. Let P and S be two subsets of X^a. Then, $Cl^a(P \cap S) \subset Cl^a(P) \cap Cl^a(S)$.

Proof. Since $P \cap S \subset P$ and $P \cap S \subset S$, so $Cl^a(P \cap S) \subset Cl^a(P)$ and $Cl^a(P \cap S) \subset Cl^a(S)$. Consequently, $Cl^a(P \cap S) \subset Cl^a(P) \cap Cl^a(S)$.

The equality of the inclusion in Proposition 2.8 cannot be occurred in general as we clarify that in the following example.

Example 2.9: Consider Example 2.6. Let $P = \{1,4\}$ and $S = \{2,3,5\}$. Then,

$$Cl^a(P) \cap Cl^a(S) = Cl^a(\{1,4\}) \cap Cl^a(\{2,3,5\}) = \{3\} .$$

$$Cl^a(P \cap S) = \phi$$

Minimal and maximal open sets were defined in [12] and [13]. In the following, we introduce minimal and maximal sets in terms of anti-open and anti-closed sets.

Definition 2.10: A proper anti-open subset M in an anti-topology $\mathcal{A}$ over the set X^a is called *minimal* (resp. *maximal*) if M is the only anti-open set contained in (resp. containing) M.

Definition 2.11: A proper anti-closed subset M in an anti-topology $\mathcal{A}$ over the set X^a is called *minimal* (resp. *maximal*) if M is the only anti-closed set contained in (resp. containing) M.

Proposition 2.12: Let $\mathcal{A}$ be an anti-topology defined over the set X^a. Then any anti-open subset of X^a is both minimal and maximal.

Proof: On contrary, assume that M is an anti-open set in X^a. Now, let U be an anti-open set properly contained in M. Thus, $M \cup U = M \in \mathcal{A}$ which contradicts the third axiom of the anti-topology definition.

Now, assume that M is an anti-open set in X^a and U is an anti-open set contains M. Hence, $M \cap U = M \in \mathcal{A}$ which also contradicts the second axiom of the anti-topology definition.

Proposition 2.13: Let $\mathcal{A}$ be an anti-topology defined over a set X^a. Then, every anti-closed subset of X^a is both minimal and maximal.

We remake that an anti-interior of a set cannot be anti-open unless the set itself is an anti-open and in this case, such a set is the only anti-open set contained in it. Similarly, anti-closure of a set cannot be anti-closed unless the set itself is an anti-closed.

Lemma 2.14: Assume that $\mathcal{A}$ defines an anti-topology on X^a and P be a subset of X^a. If P is anti-open in X^a then $Int^a(P) = P$ and P is the only anti-open set contained in $Int^a(P)$.

Proof**:** Suppose that $P \in \mathcal{A}$. By the definition of $Int^a(P)$, it is clear that P is the largest one. So, $Int^a(P) = P$. Now, suppose that Q is an anti-open set contained in P. Hence, $P \bigcup Q = P \in \mathcal{A}$ which leads to a contradiction.

Lemma 2.15: Assume that $\mathcal{A}$ defines an anti-topology on X^a and $P \subset X^a$. If P is an anti-closed in X^a then $Cl^a(P) = P$.

In Lemma 2.14 and Lemma 2.15, we can easily notice that the other directions are not true in general. In Example 2.7, the set $\{d,e\}$ is neither anti-open nor anti-closed.

Proposition 2.16: Assume that $\mathcal{A}$ defines an anti-topology on X^a and $P \subset X^a$. If $x \in Cl^a(P)$ then, for every $U \in N_a(\mathrm{x})$, we have $U \bigcap P \neq \phi$.

Proof: Let $x \in Cl^a(P)$ and assume that $U \bigcap P = \phi$, for some $U \in N_a(\mathrm{x})$. So there exists $O \in \mathcal{A}$ such that $x \in O \subset U$. It is clear that $O \bigcap P = \phi$. Thus, $P \subset X^a - O$. But $Cl^a(P)$ is the smallest anti-closed set containing P. Hence, $Cl^a(P) \subset X^a - O$. But, $x \notin X^a - O$, so $x \notin Cl^a(P)$ which leads to a contradiction.

Proposition 2.17: Suppose that $\mathcal{A}$ defines an anti-topology on X^a and $P \subset X^a$. Then

1. $Cl^a(X^a - P) = X^a - Int^a(P)$
2. $Int^a(X^a - P) = X^a - Cl^a(P)$

Proof:

(1) $X^a\text{-}Int^a(P) = X^a\text{-}\bigcup\{U \in \mathcal{A} : U \subset P\}$

$= \bigcap\{X^a - U \in F(X^a) : X^a - P \subset X^a - U\}$

$= Cl^a(X^a - P)$

(2) $X^a\text{-}Cl^a(P) = X^a\text{-}\bigcap\{F \in F(X^a) : P \subset F\}$

$= \bigcup\{X^a - F \in \mathcal{A} : X^a - F \subset X^a - P\}$

$= Int^a(X^a - P)$

Definition 2.18 [11]: A function f from an anti-topological space (X^a,) into an anti-topological space (Y^a, $\mathfrak{B}$) is called anti-continuous if $f^{-1}(B) \in \mathcal{A}$ for any $B \in \mathfrak{B}$.

Example 2.19: Consider the two anti-topological spaces $(X^a, \mathcal{A})$ and $(Y^a, \mathfrak{B})$ where $X^a = \{1,2,3\}$, $= \{\{1\},\{2,3\}\}$, $Y^a = \{a,b,c\}$ and $\mathfrak{B} = \{\{a\},\{b\}\}$. Let f be a function from the anti-topological space (X^a, $\mathcal{A}$) into the anti-topological space (Y^a, $\mathfrak{B}$) defined by $f(1) = a, f(2) = f(3) = b$. Then f is an anti-continuous.

Proposition 2.20: Let f be a function from an anti-topological space (X^a,) into an anti-topological space (Y^a, $\mathfrak{B}$). Then f is anti-continuous if and only if $f^{-1}(B)$ is anti-closed in X^a for any anti-closed set B in Y^a.

Proof. Suppose that f is an anti-continuous and let B be an anti-closed set in Y^a. So, $Y^a - B$ is anti-open set in Y^a. But f is anti-continuous, so $f^{-1}(Y^a - B) \in \mathcal{A}$. But $f^{-1}(Y^a - B) = f^{-1}(Y^a) - f^{-1}(B) = X^a - f^{-1}(B)$. Thus, $f^{-1}(B)$ is an anti-closed in X^a.

Now, let $B \in \mathfrak{B}$, we want to show that $f^{-1}(B) \in \mathcal{A}$. It is clear that $Y^a - B$ is anti-closed in Y^a. Thus, by hypothesis, $f^{-1}(Y^a - B)$ is anti-closed in X^a.

But, $f^{-1}(Y^a - B) = f^{-1}(Y^a - B) = X^a - f^{-1}(B)$

is anti-closed in X^a. Therefore, $f^{-1}(B) \in \mathcal{A}$.

III. SOME ANTI-TOPOLOGICAL OPERATORS

Definition 3.1: An anti-frontier (anti-boundary) of a set P in an anti-topology $\mathcal{A}$ over the set X^a denoted by $Fr^a(P)$, is defined by the set $Fr^a(P) = Cl^a(P) - Int^a(P)$.

Example 3.2: Consider Example 2.6. Then, $F(X^a) = \{\{4,5\},\{2,3,5\},\{1,2\},\{1,3,4\},\{2,3,4\}\}$. Now, let $P = \{1,3\} \subset X^a$. Then, $Cl^a(P) = \{1,3,4\}$ and $Int^a(P) = \phi$. Hence, $Fr^a(P) = \{1,3,4\} - \phi = \{1,3,4\}$. Let us consider the set $Q = \{1,4\}$, $Cl^a(Q) = \{1,3,4\}$ and $Int^a(Q) = \{1,4\}$. Consequently, $Fr^a(Q) = \{1,3,4\} - \{1,4\} = \{3\}$.

Proposition 3.3: Assume that $\mathcal{A}$ defines an anti-topology on X^a and $P \subset X^a$. Then,

1. If P is an anti-open , then $Fr^a(P) = Cl^a(P) - P$

2. If Q is an anti-closed, then if $Fr^a(Q) = Q - Int^a(Q)$.

Proof: 1. Assume that P is an anti-open. So, $Fr^a(P) = Cl^a(P) - Int^a(P)$

$= Cl^a(P) - A$ since P is anti-open.

2. Suppose that Q is an anti-closed. Then $Fr^a(P) = Cl^a(P) - Int^a(P)$

$= P - \operatorname{int}^a(P)$ since P is anti-closed.

Proposition 3.4: Given a subset P of an anti-topological space (X^a, A). Then $Fr^a(X^a - P) = Fr^a(P)$

Proof: $Fr^a(X^a - P) = Cl^a(X^a - P) - Int^a(X^a - P)$

$$= (X^a - Int^a(P)) - (X^a - Cl^a(P))$$

$$= Cl^a(P) - Int^a(P)$$

$$= Fr^a(P)$$

Definition 3.5: An anti-clopen set in an anti-topology $\mathcal{A}$ over a set X^a is a set that is both anti-open and anti-closed.

Example 3.6: Let $X^a = \{r, s, t, u, v, w\}$ and $\mathcal{A} = \{\{r, s, t\}, \{u, v, w\}\}$. It is clear that both the two sets $\{r, s, t\}$ and $\{u, v, w\}$ are anti-clopen sets.

Proposition 3.7: Assume that $\mathcal{A}$ defines an anti-topology on X^a and let P be a subset X^a. Then, if P is an anti-clopen set then $Fr^a(P) = \phi.$

Proof. Assume that P is an anti-clopen. Hence P is an anti-open and so by Proposition 3.3(1), we have $Fr^a(P) = Cl^a(P) - P$. However, P is also anti-closed. So by Proposition 3.3(2), we get $Fr^a(P) = Cl^a(P) - P = P - P = \phi$.

Definition 3.8: An anti-exterior of a set P in an anti-topology $\mathcal{A}$ over a set X^a is the set $Ext^a(P) = Int^a(X^a - P)$.

Example 3.9: Let $X^a = \{a, b, c, d, e\}$ and $\mathcal{A} = \{\{a, b\}, \{a.c\}, \{d\}\}$. Let $P = \{a, c, e\}$, so $Ext^a(P) = Int^a(\{b, d\}) = \{d\}$.

Proposition 3.10: Suppose that $\mathcal{A}$ defines an anti-topology on X^a and $P \subset X^a$. Then $Cl^a(P) = X^a - Ext^a(P)$.

Proof: Assume that $x \notin Cl^a(P)$. So there exists $U_x \in N_a(x)$ such that $U_x \cap P = \phi$. So, $U_x \subset X^a - A$. Thus, $x \in Int^a(X^a - P)$ which implies that $x \in Ext^a(P)$. Consequently, $x \notin X^a - Ext^a(P)$. The other implication follows similarly.

Definition 3.11: Let $\mathcal{A}$ be an anti-topology on X^a. A point $x \in X^a$ is called an anti-accumulation point of a subset A of X^a if every anti-neibourhood of x contains points of A other than x. The set of all anti-accumulation points of A is called the anti-derived set of A and it is denoted by $d^a(A)$.

Example 3.12: Consider the anti-topological space

$(X^a, \mathcal{A})$ where $X^a = \{\mathrm{a,b,c,d}\}$ and

$\mathcal{A} = \{\{\mathrm{a}\}, \{b, c\}, \{c, d\}\}$. Let $A = \{\mathrm{a,b,c}\}$, it is clear that $b \in d^a(A)$ since for each $U \in N_a(\mathrm{b}) = \{\{b, c\}\}$ we have $(U - \{b\}) \cap A \neq \phi$. However, $a \notin d^a(A)$ since $\{a\} \in N_a(\mathrm{a})$, but $(\{a\} - \{a\}) \cap A = \phi$. Similarly, $c \notin d^a(A)$. Therefore, $d^a(A) = \{b.d\}$

Proposition 3.13: Let be an anti-topology on X^a and A be a subset of X^a. Then

1. If $A \subset B$, then $d^a(\mathrm{A}) \subset d^a(B)$.
2. $d^a(A) \cup d^a(B) \subset d^a(A \cup B)$.

Proof:

1. Let $x \in d^a(A)$, hence $(U - \{x\}) \cap A \neq \phi$ for any $U \in N_a(\mathrm{x})$. But $A \subset B$, so $(U - \{x\}) \cap B \neq \phi$ for every $U \in N_a(\mathrm{x})$. Hence, $x \in d^a(B)$.

2. Since $A \subset A \cup B$ and $B \subset A \cup B$, then by (1), we have $d^a(A) \subset d^a(A \cup B)$ and $d^a(B) \subset d^a(A \cup B)$. Hence $d^a(A) \cup d^a(B) \subset d^a(A \cup B)$.

Proposition 3.14: Let $\mathcal{A}$ be an anti-topology on X^a and $A \subset X^a$. Then, A is an anti-closed if and only if $d^a(A) \subset A$.

Proof. Suppose that A is an anti-closed and let $x \notin A$. Thus, $X^a - A$ is anti-open and $x \in X^a \quad A$. But $A \cap (X^a - A) = \phi$ and $x \in X^a - A$. So, $x \notin d^a(A)$. This means that $d^a(A) \subset A$.

Now, suppose that $d^a(A) \subset A$. Let $x \in X^a - A$, it is clear that $x \notin d^a(A)$. So, there is an anti-neibourhood O with $x \in O$ and $A \cap (O - \{x\}) = \phi$. But $x \notin A$ so $A \cap O = \phi$. Therefore, $O \subset X^a - A$, i.e. $x \in Int^a(X^a - A)$. Therefore, A is an anti-closed.

Proposition 3.15: Let $\mathcal{A}$ be an anti-topology on X^a and A be an anti-closed subset of X^a. If $B \subset A$, then $d^a(B) \subset A$.

Proof. Since $B \subset A$, so $d^a(B) \subset d^a(A)$, by Proposition 3.13(1). Now, since A is anti-closed, so by Proposition 3.14 we have $d^a(A) \subset A$ Hence, $d^a(B) \subset d^a(A) \subset A$.

IV. CONCLUSION

In this article, we examined collections of sets that don't satisfy the axioms of topological spaces. Such collections are called anti-topological spaces. Specifically, we introduced new anti-topological operators such as anti-frontier and anti-exterior, as well as anti-accumulation points and anti-accumulation sets. We gave examples and some properties of each new concept introduced in this article. In addition, we introduced the concepts of minimal and maximal anti-open sets and consequently their dual. The main result regarding such sets states that every anti-open and anti-closed set is both minimal and maximal.

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