Al-Zughair Transform for Solving Ordinary Differential Equations with Constant Coefficients

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Abstract—*Ordinary differential equations (ODEs) are important in the field of mathematics due to their spread in various applications. Ordinary differential equations are of two types, one of which is equations with variable coefficients and the other is equations with constant coefficients, both types are important in mathematics.*

The purpose of any integral transforms is to convert functions and equations from their original form to another simpler form, or at least another form known to us where in it, we transform the independent variable of the original function into another variable, thus changing the scope and extent of the original function.

Many integral transformations have appeared that solve equations with variable coefficients, including the Al-Zughair transform, the Expansion of Al-Zughair, the Extension of Al-Zughair, Al-Tememe transform, and many other transforms that solve equations with variable coefficients, as for the integral transforms that solve equations with constant coefficients, it was limited.

The formula of equations with variable coefficients that solved it Al-Zughair transform is given by:

$$a_0(lnx)^n \frac{d^n y}{dx^n} + a_1(lnx)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(lnx) \frac{dy}{dx} + a_n y = f(x)$$

In this paper our aim is to solve ordinary differential equations with constant coefficients by using Al-Zughair transform. Although Al-Zughair transform is used only to solve equations with variable coefficients, we were able to use it to solve ordinary differential equations with constant coefficients because of its important in solving more complex equations. We will use Al-Zughair transform to solve equations with constant coefficients by put $y = y(\ln(lnx))$ *as the general form of the equations it will solve it is given by:*

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = f(x)$$

Keywords— Al-Zughair transform, ordinary differential equations, derivatives

I. INTRODUCTION

For many years, many methods have been used for solving differential equations[1-3], after that, integral transformations appeared that were used to solve these equations for many researchers, including the French scientist Pepper Simon Laplace who find Laplace transform [4], Fourier, Hankel transforms[5,6], also the researcher Ali Hassan Mohammad who discovered Al-Tememe transform [7,8], and other researchers who discovered other transforms, such as Al-Zughair transform [9,10], the Expansion of Al-Zughair [11], and the Extension of Al-Zughair transform [12], in addition to the transforms of Batoor Al-Tememe, Batoor Al-Zughair, Kuffi Al-Tememe, and Kuffi Al-Zughair [13]. All these conversions are used to solve different types of ordinary and partial differential equations, as well as integral equations.

Let f is a function defined on an interval (u, v) then, the integral transform for f whose symbol $\mathcal{F}(\omega)$ is defined as:

$$\mathcal{F}(\omega) = \int_{u}^{v} R(\omega, x) f(x) dx$$

where R is called the kernel of the transform and it is a function of two variables and u, v are real numbers or $\mp\infty$, such that the above integral is convergent, this integral transform was presented by Gabriel Nagy [14].

Mohammed [9] has found Al-Zughair transform for the ordinary equations with variable coefficients. In this research, we found it for the ordinary equations with constant coefficients.

II. AL-ZUGHAIR TRANSFORM

2.1 Definition [9]

Let $\varpi(g)$ be a function, then Al-Zughair transform for this function where $g \in [1, e]$ is defined by the integral:

$$\mathcal{Z}[\varpi(g)] = \int_{1}^{e} \frac{(lng)^{\ell}}{g} \varpi(g) dg = \mathcal{W}(g)$$

Such that this integral is convergent, ℓ is constant $(\ell > -1)$.

2.2 Property (Linear property) [9]

Let $\mathfrak{f}(g)$ and $T(g)$ are two functions defined when $g \in [1, e]$ and let c, d are constants then, Al-Zughair transform has the linear property which it is:

$\mathcal{Z}[c\,\mathfrak{f}(g) \pm d\,T(g)] = c\mathcal{Z}[\mathfrak{f}(g)] \pm d\mathcal{Z}[T(g)]$

Proof:

$$\mathcal{Z}[c\,\mathfrak{f}(g) \pm d\,T(g)] = \int_{1}^{e} \frac{(lng)^{\ell}}{g} [c\,\mathfrak{f}(g) \pm d\,T(g)] dg$$

$$= \int_{1}^{e} \frac{(lng)^{\ell}}{g} c\,\mathfrak{f}(g)\, dg + \int_{1}^{e} \frac{(lng)^{\ell}}{g} d\,T(g)\, dg$$

$$= c\int_{1}^{e} \frac{(lng)^{\ell}}{g} \mathfrak{f}(g)\, dg + d\int_{1}^{e} \frac{(lng)^{\ell}}{g} T(g)\, dg$$

$$= c\mathcal{Z}[\mathfrak{f}(g)] \pm d\mathcal{Z}[T(g)]$$

III. AL-ZUGHAIR TRANSFORM FOR THE DERIVATIVES

In this section, we will find Al-Zughair transform for the derivatives.

Let us consider that $\varpi = \varpi(\ln(lng))$ we define

$$\mathcal{Z}[\varpi(\ln(lng)] = \int_{1}^{e} \frac{(lng)^{\ell}}{g} \varpi(\ln(lng)) dg$$, then:

1) $\boldsymbol{\mathcal{Z}[\varpi'(ln(lng))] = \varpi(0) - (\ell + 1)\, \mathcal{Z}[\varpi(ln(lng))]}$

Proof:

$$\mathcal{Z}[\varpi'(\ln(lng))] = \int_{1}^{e} \frac{(lng)^{\ell}}{g} \varpi'(\ln(lng))\, dx$$

$$= \int_{1}^{e} \frac{(lng)^{\ell+1}}{g(lng)} \varpi'(\ln(lng))\, dg$$

Integrate by part:

Let , $u = (lng)^{\ell+1} \Rightarrow du = (\ell + 1)\frac{(lng)^{\ell}}{g} dg$ and $dv = \frac{\varpi'(\ln(lng))}{g(lng)} \Rightarrow v = \varpi(\ln(lng))$

$$so, \int_{1}^{e} \frac{(lng)^{\ell+1}}{g(lng)} \varpi'(\ln(lng))\, dg$$

$$= (lng)^{\ell+1}. \varpi(\ln(lng))|_{1}^{e} - (\ell + 1)\int_{1}^{e} \frac{(lng)^{\ell}}{g} \varpi(\ln(lng))\, dg$$

$$= \varpi(0) - (\ell + 1)\int_{1}^{e} \frac{(lng)^{\ell}}{g} \varpi(\ln(lng))\, dx$$

$$= \varpi(0) - (\ell + 1)\, \mathcal{Z}[\varpi(\ln(lng))]$$

2) $\boldsymbol{\mathcal{Z}[\varpi''(ln(lng))] = \varpi'(0) - (\ell + 1)\varpi(0) + (\ell + 1)^2\mathcal{Z}[\varpi(ln(lng))]}$

Proof:

$$\mathcal{Z}[\varpi''(\ln(lng))] = \int_{1}^{e} \frac{(lng)^{\ell}}{g} \varpi''(\ln(lng))\, dg$$

$$= \int_{1}^{e} \frac{(lng)^{\ell+1}}{g(lng)} \varpi''(\ln(lng))\, dg$$

Integrate by part:

Let , $u = (lng)^{\ell+1} \Rightarrow du = (\ell + 1)\frac{(lng)^{\ell}}{g} dg$

and $dv = \frac{\varpi''(\ln(lng))}{g(lng)} \Rightarrow v = \varpi'(\ln(lng))$

$$so, \int_{1}^{e} \frac{(lng)^{\ell+1}}{g(lng)} \varpi''(\ln(lng))\, dg$$

$$= (lng)^{\ell+1}. \varpi'(\ln(lng))|_{1}^{e} - (\ell + 1)\int_{1}^{e} \frac{(lng)^{\ell}}{g} \varpi'(\ln(lng))\, dg$$

$$= \varpi'(0) - (\ell + 1)\int_{1}^{e} \frac{(lng)^{\ell}}{g} \varpi'(\ln(lng))\, dg$$

$$= \varpi'(0) - (\ell + 1)\left[\varpi(0) - (\ell + 1)\, \mathcal{Z}[\varpi(\ln(lng))]\right]$$

$$= \varpi'(0) - (\ell + 1)\varpi(0) + (\ell + 1)^2\mathcal{Z}[\varpi(\ln(lng))]$$

3) $\boldsymbol{\mathcal{Z}[\varpi'''(ln(lng))] = \varpi''(0) - (\ell + 1)\varpi'(0) + (\ell + 1)^2\varpi(0) - (\ell + 1)^3\mathcal{Z}[\varpi(ln(lng))]}$

Proof:

$$\mathcal{Z}[\varpi'''(\ln(ln\mathscr{g}))] = \int_1^e \frac{(ln\mathscr{g})^{\ell}}{\mathscr{g}} \varpi'''(\ln(ln\mathscr{g}))\, d\mathscr{g}$$
$$= \int_1^e \frac{(ln\mathscr{g})^{\ell+1}}{\mathscr{g}(ln\mathscr{g})} \varpi'''(\ln(ln\mathscr{g}))\, d\mathscr{g}$$

Integrate by part:

Let , $u = (ln\mathscr{g})^{\ell+1} \Rightarrow du = (\ell+1)\frac{(ln\mathscr{g})^{\ell}}{\mathscr{g}} d\mathscr{g}$

and $dv = \frac{\varpi'''(\ln(ln\mathscr{g}))}{\mathscr{g}(ln\mathscr{g})} \Rightarrow v = \varpi''(\ln(ln\mathscr{g}))$

$$so, \int_1^e \frac{(ln\mathscr{g})^{\ell+1}}{\mathscr{g}(ln\mathscr{g})} \varpi'''(\ln(ln\mathscr{g}))\, d\mathscr{g}$$
$$= (ln\mathscr{g})^{\ell+1}.\varpi''(\ln(ln\mathscr{g}))|_1^e - (\ell+1)\int_1^e \frac{(ln\mathscr{g})^{\ell}}{\mathscr{g}} \varpi''(\ln(ln\mathscr{g}))\, d\mathscr{g}$$
$$= \varpi''(0) - (\ell+1)\int_1^e \frac{(ln\mathscr{g})^{\ell}}{\mathscr{g}} \varpi''(\ln(ln\mathscr{g}))\, d\mathscr{g}$$
$$= \varpi''(0) - (\ell+1)\left[\varpi'(0) - (\ell+1)\varpi(0) + (\ell+1)^2\mathcal{Z}[\varpi(\ln(ln\mathscr{g}))]\right]$$
$$= \varpi''(0) - (\ell+1)\varpi'(0) + (\ell+1)^2\varpi(0) - (\ell+1)^3\mathcal{Z}[\varpi(\ln(ln\mathscr{g}))]$$

4) $\boldsymbol{\mathcal{Z}[\varpi^{iv}(ln(ln\mathscr{g}))] = \varpi'''(0) - (\ell+1)\varpi''(0) + (\ell+1)^2\varpi'(0) - (\ell+1)^3\varpi(0) + (\ell+1)^4\mathcal{Z}[\varpi(ln(ln\mathscr{g}))]}$

Proof:

$$\mathcal{Z}[\varpi^{iv}(\ln(ln\mathscr{g}))] = \int_1^e \frac{(ln\mathscr{g})^{\ell}}{\mathscr{g}} \varpi^{iv}(\ln(ln\mathscr{g}))\, d\mathscr{g}$$
$$= \int_1^e \frac{(ln\mathscr{g})^{\ell+1}}{\mathscr{g}(ln\mathscr{g})} \varpi^{iv}(\ln(ln\mathscr{g}))\, d\mathscr{g}$$

Integrate by part:

Let, $u = (ln\mathscr{g})^{\ell+1} \Rightarrow du = (\ell+1)\frac{(ln\mathscr{g})^{\ell}}{\mathscr{g}} d\mathscr{g}$

and $dv = \frac{\varpi^{iv}(\ln(ln\mathscr{g}))}{\mathscr{g}(ln\mathscr{g})} \Rightarrow v = \varpi'''(\ln(ln\mathscr{g}))$

$$so, \int_1^e \frac{(ln\mathscr{g})^{\ell+1}}{\mathscr{g}(ln\mathscr{g})} \varpi^{iv}(\ln(ln\mathscr{g}))\, d\mathscr{g} = (ln\mathscr{g})^{\ell+1}.\varpi'''(ln(ln\mathscr{g}))|_1^e - (\ell+1)\int_1^e \frac{(ln\mathscr{g})^{\ell}}{\mathscr{g}} \varpi'''(\ln(ln\mathscr{g}))\, d\mathscr{g}$$
$$= \varpi'''(0) - (\ell+1)\int_1^e \frac{(ln\mathscr{g})^{\ell}}{\mathscr{g}} \varpi'''(\ln(ln\mathscr{g}))\, d\mathscr{g}$$
$$= \varpi'''(0) - (\ell+1)\left[\varpi''(0) - (\ell+1)\varpi'(0) + (\ell+1)^2\varpi(0) - (\ell+1)^3\mathcal{Z}[\varpi(\ln(ln\mathscr{g}))]\right]$$
$$= \varpi'''(0) - (\ell+1)\varpi''(0) + (\ell+1)^2\varpi'(0) - (\ell+1)^3\varpi(0) + (\ell+1)^4\mathcal{Z}[\varpi(\ln(ln\mathscr{g}))]$$

3.1 The general case of Al-Zughair transform for derivatives

From the above section, we obtain:

$$\boldsymbol{\mathcal{Z}[\varpi^{(n)}(\ln(ln\mathscr{g}))] = \varpi^{n-1}(0) - (\ell+1)\varpi^{n-2}(0) + (\ell+1)^2\varpi^{n-3}(0) - (\ell+1)^3\varpi^{n-4}(0) + \cdots + (-1)^{n-1}(\ell+1)^{n-1}\,\varpi(0) + (-1)^n(\ell+1)^n\mathcal{Z}[\varpi(\ln(ln\mathscr{g}))]}$$

Note: In the following section, we will present some examples on ordinary differential equations with constant coefficients subject to some initial conditions and how the Al-Zughair transform and its inverse solve it.

Example (1):

$$\varpi'''(\ln(ln\mathscr{g})) - \varpi''(\ln(ln\mathscr{g})) = sinh(2\ln(ln\mathscr{g})) \quad ;\varpi''(0) = \varpi'(0) = \varpi(0) = 0$$

Solution:

Take Al-Zughair transform for both sides of above equation to obtain:

$$\mathcal{Z}[\varpi'''(\ln(ln\mathscr{g}))] - \mathcal{Z}[\varpi''(\ln(ln\mathscr{g}))] = \mathcal{Z}[sinh(2\ln(ln\mathscr{g}))]$$
$$\varpi''(0) - (\ell+1)\varpi'(0) + (\ell+1)^2\varpi(0) - (\ell+1)^3\mathcal{Z}[\varpi(\ln(ln\mathscr{g}))] - \varpi'(0) + (\ell+1)\varpi(0) - (\ell+1)^2\mathcal{Z}[\varpi(\ln(ln\mathscr{g}))] = \frac{-2}{[(\ell+1)^2-4]}$$
$$\Rightarrow -(\ell+1)^2\mathcal{Z}[\varpi(\ln(ln\mathscr{g}))][(\ell+1)+1] = \frac{-2}{(\ell+3)(\ell-1)}$$
$$\Rightarrow -(\ell+1)^2\mathcal{Z}[\varpi(\ln(ln\mathscr{g}))][(\ell+2)] = \frac{-2}{(\ell+3)(\ell-1)}$$
$$\Rightarrow \mathcal{Z}[\varpi(\ln(ln\mathscr{g}))] = \frac{2}{(\ell+3)(\ell+2)(\ell+1)^2(\ell-1)} \quad \ldots(1)$$

Take $\mathcal{Z}^{-1}$ to both sides of equation (1) to obtain:

$$\varpi(\ln(ln\mathscr{g})) = \mathcal{Z}^{-1}\left[\frac{A}{(\ell+3)} + \frac{B}{(\ell+2)} + \frac{C}{(\ell+1)} + \frac{D}{(\ell+1)^2} + \frac{E}{(\ell-1)}\right]$$

$A + B + C + E = 0$
$3A + 4B + 5C + D + 7E = 0$
$A + 2B + 5C + 4D + 17E = 0$
$-3A - 4B - 5C + D + 17E = 0$
$-2A - 3B - 6C - 6D + 6E = 2$

Hence,

$A = \frac{1}{8} \quad , B = \frac{-2}{3} \quad , C = \frac{1}{2} \quad , D = \frac{-1}{2} \quad , E = \frac{1}{24}$

And hence the solution is given by:

$$\varpi(\ln(ln\mathcal{g})) = \mathcal{Z}^{-1}\left[\frac{1}{8}.\frac{1}{(\ell+3)} - \frac{2}{3}.\frac{1}{(\ell+2)} + \frac{1}{2}.\frac{1}{(\ell+1)} - \frac{1}{2}.\frac{1}{(\ell+1)^2} + \frac{1}{24}.\frac{1}{(\ell-1)}\right]$$

$$\varpi(\ln(ln\mathcal{g})) = \frac{1}{8}(ln\mathcal{g})^2 - \frac{2}{3}(ln\mathcal{g}) + \frac{1}{2} + \frac{1}{2}(\ln(ln\mathcal{g})) + \frac{1}{24}(ln\mathcal{g})^{-2}$$

Example (2):

$-2 + \varpi'(\ln(ln\mathcal{g})) = (ln\mathcal{g})^2 \quad ; \varpi(0) = 2$

Solution:

Take Al-Zughair transform for both sides of above equation to obtain:

$$-\mathcal{Z}[2] + \mathcal{Z}[\varpi'(\ln(ln\mathcal{g}))] = \mathcal{Z}[(ln\mathcal{g})^2]$$

$$\frac{-2}{(\ell+1)} + \varpi(0) - (\ell+1)\,\mathcal{Z}[\varpi(\ln(ln\mathcal{g}))] = \frac{1}{(\ell+3)}$$

$$\frac{-2}{(\ell+1)} + 2 - (\ell+1)\,\mathcal{Z}[\varpi(\ln(ln\mathcal{g}))] = \frac{1}{(\ell+3)}$$

$$\frac{2\ell}{(\ell+1)} - (\ell+1)\,\mathcal{Z}[\varpi(\ln(ln\mathcal{g}))] = \frac{1}{(\ell+3)}$$

$$-(\ell+1)\,\mathcal{Z}[\varpi(\ln(ln\mathcal{g}))] = \frac{1}{(\ell+3)} - \frac{2\ell}{(\ell+1)}$$

$$-(\ell+1)\,\mathcal{Z}[\varpi(\ln(ln\mathcal{g}))] = \frac{-2\ell^2 - 5\ell + 1}{(\ell+3)(\ell+1)}$$

$$\mathcal{Z}[\varpi(\ln(ln\mathcal{g}))] = \frac{2\ell^2 + 5\ell - 1}{(\ell+3)(\ell+1)^2} \quad \ldots(2)$$

Take $\mathcal{Z}^{-1}$ to both sides of equation (2) to obtain:

$$\varpi(\ln(ln\mathcal{g})) = \mathcal{Z}^{-1}\left[\frac{A}{(\ell+3)} + \frac{B}{(\ell+1)} + \frac{C}{(\ell+1)^2}\right]$$

$A + B = 2$

$2A + 4B + C = 5$

$A + 3B + 3C = -1$

Hence,

$A = \frac{1}{2} \quad , \quad B = \frac{3}{2} \quad , \quad C = -2$

And hence the solution is given by:

$$\varpi(\ln(ln\mathcal{g})) = \mathcal{Z}^{-1}\left[\frac{1/2}{(\ell+3)} + \frac{3/2}{(\ell+1)} - \frac{2}{(\ell+1)^2}\right]$$

$$\varpi(\ln(ln\mathcal{g})) = \frac{1}{2}(ln\mathcal{g})^2 + \frac{3}{2} + 2(\ln(ln\mathcal{g}))$$

Example (3):

$\varpi''(\ln(ln\mathcal{g})) + 3\varpi'(\ln(ln\mathcal{g})) = 3 \quad ; \varpi'(0) = 0, \varpi(0) = 1$

Solution:

Take Al-Zughair transform for both sides of above equation to obtain:

$$\mathcal{Z}[\varpi''(\ln(ln\mathcal{g}))] + 3\mathcal{Z}[\varpi'(\ln(ln\mathcal{g}))] = \mathcal{Z}[3]$$

$$\varpi'(0) - (\ell+1)\varpi(0) + (\ell+1)^2\mathcal{Z}[\varpi(\ln(ln\mathcal{g}))] + 3\varpi(0) - 3(\ell+1)\,\mathcal{Z}[\varpi(\ln(ln\mathcal{g}))] = \frac{3}{(\ell+1)}$$

$$\Rightarrow -(\ell-2) + (\ell+1)\mathcal{Z}[\varpi(\ln(ln\mathcal{g}))][(\ell+1) - 3] = \frac{3}{(\ell+1)}$$

$$\Rightarrow (\ell+1)\mathcal{Z}[\varpi(\ln(ln\mathcal{g}))][(\ell-2)] = \frac{3}{(\ell+1)} + (\ell-2)$$

$$\Rightarrow (\ell+1)(\ell-2)\mathcal{Z}[\varpi(\ln(ln\mathcal{g}))] = \frac{\ell^2 - \ell + 1}{(\ell+1)}$$

$$\Rightarrow \mathcal{Z}[\varpi(\ln(ln\mathcal{g}))] = \frac{\ell^2 - \ell + 1}{(\ell+1)^2(\ell-2)} \quad \ldots(3)$$

Take $\mathcal{Z}^{-1}$ to both sides of equation (3) to obtain:

$$\varpi(\ln(ln\mathcal{g})) = \mathcal{Z}^{-1}\left[\frac{A}{(\ell+1)} + \frac{B}{(\ell+1)^2} + \frac{C}{(\ell-2)}\right]$$

$A + C = 1$

$-A + B + 2C = -1$

$-2A - 2B + C = 1$

Hence,

$A = \frac{2}{3} \quad , \quad B = -1 \quad , \quad C = \frac{1}{3}$

And hence the solution is given by:

$$\varpi(\ln(ln\mathcal{g})) = \mathcal{Z}^{-1}\left[\frac{2/3}{(\ell+1)} - \frac{1}{(\ell+1)^2} + \frac{1/3}{(\ell-2)}\right]$$

$$\varpi(\ln(ln\mathcal{g})) = \frac{2}{3} + (\ln(ln\mathcal{g})) + \frac{1}{3}(ln\mathcal{g})^{-3}$$

iv. Conclusion

This research aim is to solve linear ordinary differential equations with constant coefficients by us Al-Zughair transform, as no one has previously solved ordinary equations with constant coefficients through using this transform. There for the Al-Zughair method with constants can be exploited to open the door for solving related problems in different scientific and industrial fields.

REFERENCES

[1] James C.R, "**An Introduction to Ordinary Differential Equations** ", Cambridge University Press, New York, 2004J. Clerk Maxwell, A Treatise on Electricity and Magnetism, 3rd ed., vol. 2. Oxford: Clarendon, 1892, pp.68-73.

[2] Richard B., " **Differential Equations** " Second Edition, McGraw-Hill Companies, Inc.,New York, 2003.

[3] Hazewinkel M.. " Differential Equations Ordinary " Encyclopedia of Mathematics, springer, ISBN 978-1-55608-010-4, 2001.

[4] Smith D.E., "**History of Mathematics** ", V:1, Dover publications, INC., New York, 1958.

[5] George C., " **Partail Differentail Equation** " Dover, New York, 1952.

[6] Elfving G., " **The History of Mathematics in Finland** " Frenckell, Helsinki, 1981.

[7] Mohammed A.H., A.N., Kathem " **Solving Euler's Equation Using New Transformation** " Karbala university magazine for completely sciences, V.:6, no.: 4, 2008.

[8] Mohammed A.H., Aymen, M.H. " **Al-Tememe Transformation for Solving Ordinary Differential Equations "** A thesis of MSc. Submitted to the university of Kufa, Faculty of computer sciences and mathematics, 2017

[9] Mohammed A.H., Sadiq B.A '' **New Integral Transform and its Uses ''** thesis of MSc. University of Kufa, Faculty of Education for Girls, 2017.

[10] Mohammed A.H., Sadiq B.A '' **AL-Zughair transform '',** LAPLAMBERT Academic Publication ISBN: 978-3-659-53070-8, 2017.

[11] Mohammed A.H., Abdullah, N.G**., "AL-Zughair and Al-Zughair Expansion Transformations and Some Uses"**, International Journal of Mechanical and Production Engineering Research and Development (IJMPERD) ISSN: 2249-8001, 12 September 2018.

[12] Mohammed A.H., Majde A.Q., **"An Extension of Al-Zughair Integral Transform"**, a thesis submitted to the council of the college of Computer Science and Mathematics 2020.

[13] A.H. Battor, E.A. Kuffi, A.H. Mohammed**," On Some Integral Transforms with Applications"**, A thesis submitted to the council of the faculty of Education for Girls 2022 A.D

[14] Gabriel Nagy, "ordinary Differential Equations " Mathematic Department, Mishgan State University, East Lansing, MI, 48824.October 14,2014.