On Double AL-Zughair Transform Partial Technique and Solving Telegraphic Equation

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Abstract— *In this paper, we developing one of the new transform, namely (AL-**Z**ughair transform),was introduced through the study of (Double AL-**Z**ughair transform, inverse Double AL-**Z**ughair transform), method is introduced used to solve the one dimensional partial differential equations with variable coefficients and then we discussed some examples to illustrate this concept and properties. On the other hand, (Double AL-**Z**ughair transform) was presented for the partial derivatives with two variables* $(\mathscr{v},\mathfrak{I})$ *and the transport equation and some other partial equations were solved, which indicates the effectiveness of this transformation in solving them in a quick and simplified manner.*

Keywords—AL-*Z*ughair transform, Double AL-*Z*ughair transform, inverse of double AL-*Z*ughair transform, *partial differential equations (PDEs)*.

I. INTRODUCTION

The introduction of the integral transform which by Euler in (1763)was one of the most celebrated events in the mathematics community around the word. The integral transform is one of the best of recognized techniques for solving (PDEs) [1-3]. Therefore, many integral transformations have emerged to solve many applications in different fields such as physics, engineering, etc. In 2017, H. Ali developed a novel transform known as the AL-Zughair transform, which has been utilized to solve several forms of D.E with the with variable coefficients [4-5].A number of researchers were interested in developing the solution of partial differential equations because of their importance in many applied fields. So (in2010)Eltayeb. H and Kilicman. A, they presented (double Sumudu and double Laplace transform)[6].And at the same years they presented note on integral transforms and PDE[7]and other years [8-10]. Discuss both Y. Singh and H. Mandia,(in2012) Relationship between(double Laplace and double mellin transform) in terms of generalized hyper- geometric function with applications[11]. Because of the effectiveness and development of (AL-Zughair transform) in recent years in applied fields[12-14], (Double AL-Zughair transform) has been discussed. The idea of the study, we use the(Double AL-Zughair transform) to solve a first and second PDE, specially one dimensional. Through this method the PDEs is solved without converting it into ordinary differential equation, therefore no need to find complete solution of ordinary differential equation. This is the biggest advantage of this method and showing the speed and effectiveness of this transform

II. BASIC CONCEPTS

Here we will present some important definitions and laws that we will rely on in our research.

2.1Definition[15]

Let $\hbar(\mathscr{v})$ be defined function on interval (a,c),the integral transform is given by the integral

$$\mathcal{H}(\wp)=\int_a^c \mathcal{H}(\mathscr{v},\wp)\hbar(\mathscr{v})\,d\mathscr{v}$$

Such that the above integral is converges, $\mathcal{H}(\mathscr{v},\wp)$ the kernel of the transform a fixed function of two variables and a,c are real numbers or $(\pm\infty)$.

2.2Definition[4]

AL- Zughair transform for the piecewise continuous function $\hbar(\mathscr{v})$, is defined by the following integral

$$\mathcal{H}(\wp)=\int_1^{e}\frac{(ln\mathscr{v})^{\wp}}{\mathscr{v}}\hbar(\mathscr{v})d\mathscr{v}=Z[\hbar(\mathscr{v})]$$

The integral is convergent , $\mathcal{p} > -1$ is a constant

2.3Definition[4]

Let $\hbar(\mathcal{v})$, *be* a function where $\mathcal{v} \in [1, \mathrm{e}]$ and $Z[\hbar(\mathcal{v})] = \mathcal{H}(\mathcal{p})$, $\hbar(\mathcal{v})$ is said to be an inverse for AL $-$ Zughair - Transform and written as $Z^{-1}[\mathcal{H}(\mathcal{p})] = \hbar(\mathcal{v})$.

$$Z^{-1}[\mathcal{H}(\mathcal{p})] = \hbar(ln\mathcal{v}) = \frac{1}{2\pi i}\int_{\epsilon-i\pi}^{\epsilon+i\pi} \frac{(ln\mathcal{v})^{\mathcal{p}}}{\mathcal{v}}\mathcal{H}(\mathcal{p})d\mathcal{p}$$

$$= \frac{1}{2\pi\mathcal{v}i}\int_{\epsilon-i\pi}^{\epsilon+i\pi} (ln\mathcal{v})^{\mathcal{p}}\,\mathcal{H}(\mathcal{p})d\mathcal{p}, \epsilon > 1, i \in \mathbb{C}$$

2.4Table of Select function AL- Zughair transform[4]

Function	$\hbar(\mathcal{v}) = \int_1^{\mathrm{e}} \frac{(ln\mathcal{v})^{\mathcal{p}}}{\mathcal{v}}\hbar(\mathcal{v})$	Regional of convergence
$\mathcal{K}$ = constant	$\frac{\mathcal{K}}{\mathcal{p}+1}$	$\mathcal{p} > -1$
$(ln\mathcal{v})^{\mathfrak{h}}$	$\frac{1}{\mathcal{p}+(\mathfrak{h}+1)}$	$\mathcal{p} > -(\mathfrak{h}+1)$
$(ln(ln\mathcal{v}))^{\mathfrak{h}}$	$\frac{(-1)^{\mathfrak{h}}\mathfrak{h}!}{(\mathcal{p}+1)^{\mathfrak{h}+1}}$	$\mathcal{p} > -1$
$\sin(\gamma\ln(ln\mathcal{v}))$	$\frac{-\gamma}{(\mathcal{p}+1)^2+\gamma^2}$	$\mathcal{p} > -1, \gamma$ *is constant*
$\sinh(\gamma\ln(ln\mathcal{v}))$	$\frac{-\gamma}{(\mathcal{p}+1)^2-\gamma^2}$	$\mathcal{p} > -1, \gamma$ *is constant*
$\cos(\gamma\ln(ln\mathcal{v}))$	$\frac{\mathcal{p}+1}{(\mathcal{p}+1)^2+\gamma^2}$	$\mathcal{p} > -1, \gamma$ *is constant*
$\cosh(\gamma\ln(ln\mathcal{v}))$	$\frac{\mathcal{p}+1}{(\mathcal{p}+1)^2-\gamma^2}$	$\mathcal{p} > -1, \gamma$ *is constant*

III.PARTIAL FORMULA OF DOUBLE AL- ZUGHAIR TRANSFORM

This part , definition of partial(Double AL- Zughair transform), definition(inverse of partial Double AL- Zughair transform)and we review some examples.

3.1Definition(Double AL $-$ Zughair transform)

Let $\psi(\mathcal{v}, \mathfrak{I})$ be a function of two variable ($\mathcal{v}$ and $\mathfrak{I}$), where $(\mathcal{v}, \mathfrak{I}) \in [1, \mathrm{e}] \times [1, \mathrm{e}]$. The Double AL- Zughair transform is defined by :

$$Z^{\mathfrak{I}}Z^{\mathcal{v}}[\,\psi(\mathcal{v}, \mathfrak{I})] = \int_1^{\mathrm{e}} \frac{(ln\mathfrak{I})^{\mathcal{p}}}{\mathfrak{I}}\int_1^{\mathrm{e}} \frac{(ln\mathcal{v})^{\mathcal{q}}}{\mathcal{v}}\,\psi(\mathcal{v}, \mathfrak{I})d\mathcal{v}d\mathfrak{I} = \mathcal{V}(\mathcal{q}, \mathcal{p})$$

The integral is convergent and $\mathcal{p}, \mathcal{q}$ are constants.

3.2Proposition (Linear property)

$$Z^{\mathfrak{I}}Z^{\mathcal{v}}[\mathbb{A}\,\psi_1(\mathcal{v}, \mathfrak{I}) \pm \mathbb{B}\psi_2(\mathcal{v}, \mathfrak{I})\,] = \mathbb{A}Z^{\mathfrak{I}}Z^{\mathcal{v}}[\psi_1(\mathcal{v}, \mathfrak{I})] \pm \mathbb{B}Z^{\mathfrak{I}}Z^{\mathcal{v}}[\psi_2(\mathcal{v}, \mathfrak{I})]$$

Where $\mathbb{A}, \mathbb{B}$ are constants, the functions $\psi_1(\mathcal{v}, \mathfrak{I})$and $\psi_2(\mathcal{v}, \mathfrak{I})$are defined when $(\mathcal{v}, \mathfrak{I}) \in [1, \mathrm{e}] \times [1, \mathrm{e}]$

Proof

Because of the linear property of the integrals, the proof is obvious.

3.3.Illustrative Examples

The following examples show Double AL- Zughair transform

Example1: Let $\boldsymbol{\psi(\mathcal{v}, \mathfrak{I}) = (ln\,\mathcal{v})^2 sinh(\ln(ln\,\mathfrak{I}),}$ find $\boldsymbol{Z^{\mathfrak{I}}Z^{\mathcal{v}}}$.

Solution: To find $\boldsymbol{Z^{\mathfrak{I}}Z^{\mathcal{v}}}$ see the following

$$Z^{\mathfrak{I}}Z^{\mathcal{v}}[(ln\,\mathcal{v})^2 sinh(\ln(ln\,\mathfrak{I})]$$

$$= \int_1^{\mathrm{e}} \frac{(ln\mathfrak{I})^{\mathcal{p}}}{\mathfrak{I}}\int_1^{\mathrm{e}} \frac{(ln\mathcal{v})^{\mathcal{q}}}{\mathcal{v}}(ln\,\mathcal{v})^2 sinh(\ln(ln\,\mathfrak{I})\,d\mathcal{v}d\mathfrak{I}$$

$$= \int_1^{\mathrm{e}} \frac{(ln\mathfrak{I})^{\mathcal{p}}}{\mathfrak{I}}\left(sinh(\ln(ln\,\mathfrak{I}))\int_1^{\mathrm{e}} \frac{(ln\mathcal{v})^{\mathcal{q}+2}}{\mathcal{v}}d\mathcal{v}\right)d\mathfrak{I}$$

$$= \frac{1}{(\mathcal{q}+3)}(\int_1^{\mathrm{e}} \frac{(ln\,\mathfrak{I})^{\mathcal{p}}}{\mathfrak{I}}sinh(ln(ln\,\mathfrak{I})\,d\mathfrak{I}\,)$$

$$= \frac{1}{(\mathcal{q}+3)}\left(\frac{-1}{(\mathcal{p}+1)^2-1}\right) = \frac{-1}{(\mathcal{q}+3)(\mathcal{p}+2)\mathcal{p}}$$

Example2:Let

$\boldsymbol{\psi(\mathcal{v}, \mathfrak{I}) = 2(\ln(ln\mathcal{v})^2\, sin(\ln(ln\,\mathfrak{I}))\, cos(2\ln(ln\,\mathfrak{I})),}$ **find** $\boldsymbol{Z^{\mathfrak{I}}Z^{\mathcal{v}}}$

Solution: To find $\boldsymbol{Z^{\mathfrak{I}}Z^{\mathcal{v}}}$ see the following

$$Z^{\mathfrak{I}}Z^{\mathcal{v}}[2(\ln(ln\mathfrak{I})^2\, sin(\ln(ln\,\mathfrak{I}))\, cos(2\ln(ln\,\mathfrak{I}))]$$

$$= 2\int_1^{\mathrm{e}} \frac{(ln\,\mathfrak{I})^{\mathcal{p}}}{\mathfrak{I}}\int_1^{\mathrm{e}} \frac{(ln\,\mathcal{v})^{\mathcal{q}}}{\mathcal{v}}(\ln(ln\mathcal{v}))^2\, sin(\ln(ln\,\mathfrak{I}))\, cos(2\ln(ln\,\mathfrak{I}))\,d\mathcal{v}d\mathfrak{I}$$

$$= 2\int_1^{\mathrm{e}} \frac{(ln\,\mathfrak{I})^{\mathcal{p}}}{\mathfrak{I}}$$

$$\left(sin(\ln(ln\,\mathfrak{I}))\, cos(2\ln(ln(\mathfrak{I})))\int_1^{\mathrm{e}} \frac{(ln\mathcal{v})^{\mathcal{q}}}{\mathcal{v}}(\ln(ln\mathcal{v}))^2 d\mathcal{v}\right)d\mathfrak{I}$$

$$= \frac{4}{\mathcal{q}^3}\int_1^{\mathrm{e}} \frac{(ln\mathfrak{I})^{\mathcal{p}}}{\mathfrak{I}}sin(\ln(ln\,\mathfrak{I}))\, cos(2\ln(ln\,\mathfrak{I}))\,d\mathfrak{I}$$

$$= \frac{2}{\mathcal{q}^3}\int_1^{\mathrm{e}} \frac{(ln\mathfrak{I})^{\mathcal{p}}}{\mathfrak{I}}sin(3\ln(ln\,\mathfrak{I}))d\mathfrak{I}$$

$$+ \frac{2}{\mathcal{q}^3}\int_1^{\mathrm{e}} \frac{(ln\mathfrak{I})^{\mathcal{p}}}{\mathfrak{I}}cos(\ln(ln\,\mathfrak{I}))d\mathfrak{I}$$

$$= \frac{-6}{\mathcal{q}^3((\mathcal{p}+1)^2+9)} + \frac{2(\mathcal{p}+1)}{\mathcal{q}^3((\mathcal{p}+1)^2+9)}$$

3.4Definition(Inverse Double AL- Zughair transform)

Let $\psi(\mathcal{v},\mathfrak{I})$be a function where$(\mathcal{v},\mathfrak{I}) \in [1,e] \times [1,e]$ and $Z^{\mathfrak{I}}Z^{\mathcal{v}}[\psi(\mathcal{v},\mathfrak{I})] = \mathcal{V}(q,p)$, inverse for double AL-Zughair transform $\mathcal{V}(q,p)$ is $\psi(\mathcal{v},\mathfrak{I})$ and written as $(Z^{\mathfrak{I}}Z^{\mathcal{v}})^{-1}[\mathcal{V}(q,p)] = \psi(\mathcal{v},\mathfrak{I})$.

$$(Z^{\mathfrak{I}}Z^{\mathcal{v}})^{-1}(Z^{\mathfrak{I}}Z^{\mathcal{v}})^{-1}[\mathcal{V}(q,p)] = \psi(ln\mathcal{v}, ln\mathfrak{I})$$

$$= \frac{1}{2\pi i}\int_{\epsilon-i\pi}^{\epsilon+i\pi}\frac{(ln\mathfrak{I})^{p}}{\mathfrak{I}}\int_{\epsilon-i\pi}^{\epsilon+i\pi}\frac{(ln\mathcal{v})^{q}}{\mathcal{v}}\psi(\mathcal{v},\mathfrak{I})dp\,dq$$

$$= \frac{1}{2\pi\mathcal{v}\mathfrak{I}i}\int_{\epsilon-i\pi}^{\epsilon+i\pi}\int_{\epsilon-i\pi}^{\epsilon+i\pi}(ln\mathcal{v})^{q}(ln\mathfrak{I})^{p}\psi(\mathcal{v},\mathfrak{I})dp\,dq,$$

$$\epsilon > 1, i \in \mathbb{C}$$

3.5Note

The inverse for Double AL- Zughair transform is linear property :

$$(Z^{\mathfrak{I}}Z^{\mathcal{v}})^{-1}[\mathbb{A}\mathcal{V}_1(q,p) \pm \mathbb{B}\mathcal{V}_2(q,p)] =$$
$$\mathbb{A}\,(Z^{\mathfrak{I}}Z^{\mathcal{v}})^{-1}[\mathcal{V}_1(q,p)] \pm \mathbb{B}(Z^{\mathfrak{I}}Z^{\mathcal{v}})^{-1}[\mathcal{V}_2(q,p)] =$$
$$\mathbb{A}\psi_1(\mathcal{v},\mathfrak{I}) \pm \mathbb{B}\psi_2(\mathcal{v},\mathfrak{I})$$

Where $\mathbb{A},\mathbb{B}$ are constants, and $\psi_1(\mathcal{v},\mathfrak{I})$, $\psi_2(\mathcal{v},\mathfrak{I})$ be the inverse transform of $\mathcal{V}_1(q,p)$ and $\mathcal{V}_2(q,p)$ respectively.

3.6Illustrative Examples

The following examples show the inverse Double $\mathcal{AL}$-Zughair transform

Example3 ***:*** To find

I. $\boldsymbol{(Z^{\mathfrak{I}}Z^{\mathcal{v}})^{-1}}\left[\frac{1}{q^4(p+4)}\right] = -\frac{1}{6}(ln\,\mathfrak{I})^3(\ln(ln\,\mathcal{v}))^3$

II. $\boldsymbol{(Z^{\mathfrak{I}}Z^{\mathcal{v}})^{-1}}\left[\frac{3}{\frac{(p+1)(1-25q^2)}{(q+2)}}\right] = 3\,cosh(5\ln(ln\,\mathcal{v})) - \frac{3}{5}sinh(5\ln(ln\,\mathcal{v})$

iv. *Double AL- Zughair transform of Partial Derivatives*

In this part, we will explain Double AL- Zughair transform for (second/first) partial derivatives with respect to $\mathcal{v},\mathfrak{I}$, and some proposition and examples.

4.1Note :

From here we will consider the following :

Let $\psi(ln\,\mathcal{v}, ln\mathfrak{I}) = \psi$ be defined for$(\mathcal{v},\mathfrak{I}) \in [1,e] \times [1,e]$ and $\psi_{\mathcal{v}}(ln\,\mathcal{v}, ln\,\mathfrak{I}) = \psi_{\mathcal{v}}, \psi_{\mathfrak{I}}(ln\mathcal{v}, ln\,\mathfrak{I}) = \psi_{\mathfrak{I}}$,

$\psi_{\mathcal{v}\mathcal{v}}(ln\mathcal{v}, ln\,\mathfrak{I}) = \psi_{\mathcal{v}\mathcal{v}},\ \psi_{\mathfrak{I}\mathfrak{I}}(ln\mathcal{v}, ln\,\mathfrak{I}) = \psi_{\mathfrak{I}\mathfrak{I}}$,

$\psi_{\mathcal{v}\mathfrak{I}}(ln\mathcal{v}, ln\,\mathfrak{I}) = \psi_{\mathcal{v}\mathfrak{I}}$

Such that $\psi_{\mathcal{v}}, \psi_{\mathfrak{I}}, \psi_{\mathcal{v}\mathcal{v}}, \psi_{\mathcal{v}\mathfrak{I}}$, are exists.

4.2. The first Double AL- Zughair transform for derivatives with respect to $\mathcal{v}$

$$Z^{\mathfrak{I}}Z^{\mathcal{v}}[ln\,\mathcal{v}\psi_{\mathcal{v}}] = Z^{\mathfrak{I}}[\psi(1, ln\mathfrak{I})] - (q+1)Z^{\mathfrak{I}}Z^{\mathcal{v}}[\psi]$$

Proof

$$Z^{\mathfrak{I}}Z^{\mathcal{v}}[ln\,\mathcal{v}\,\psi_{\mathcal{v}}] = \int_1^e\frac{(ln\mathfrak{I})^{p}}{\mathfrak{I}}\int_1^e\frac{(ln\mathcal{v})^{q}}{\mathcal{v}}ln\,\mathcal{v}\,\psi_{\mathcal{v}}\,d\mathcal{v}d\mathfrak{I}$$

$$= \int_1^e\frac{(ln\mathfrak{I})^{p}}{\mathfrak{I}}\left(\int_1^e\frac{(ln(\mathcal{v}))^{q+1}}{\mathcal{v}}\psi_{\mathcal{v}}\,d\mathcal{v}\right)d\mathfrak{I} =$$

$$\int_1^e\frac{(ln\mathfrak{I})^{p}}{\mathfrak{I}}\left((ln\,\mathcal{v})^{q+1}\,\psi\,\Big|_1^e - (q+1)\int_1^e\frac{(ln\mathcal{v})^{q}}{\mathcal{v}}\psi d\mathcal{v}\right)d\mathfrak{I}$$

$$= \int_1^e\frac{(ln\mathfrak{I})^{p}}{\mathfrak{I}}(\psi(1, ln\mathfrak{I})d\mathfrak{I} - (q+1)\left(\int_1^e\frac{(ln\mathcal{v})^{q}}{\mathcal{v}}\psi d\mathcal{v}\right)d\mathfrak{I}$$

$$= Z^{\mathfrak{I}}[\psi(1, ln\,\mathfrak{I})] - (q+1)Z^{\mathfrak{I}}Z^{\mathcal{v}}[\psi]$$

4.3The first Double AL- Zughair transform for derivatives with respect to t

$$Z^{\mathfrak{I}}Z^{\mathcal{v}}[ln\,\mathfrak{I}\psi_{\mathfrak{I}}] = Z^{\mathcal{v}}[\mathcal{U}(ln\,\mathfrak{I}, 1)] - (q+1)Z^{\mathfrak{I}}Z^{\mathcal{v}}[\psi]$$

Proof

$$Z^{\mathfrak{I}}Z^{\mathcal{v}}[ln\,\mathfrak{I}\,\psi_{\mathfrak{I}}] = \int_1^e\frac{(ln\mathfrak{I})^{p}}{\mathfrak{I}}\int_1^e\frac{(ln\mathcal{v})^{q}}{\mathcal{v}}ln\,\mathcal{v}\,\psi_{\mathfrak{I}}d\mathcal{v}d\mathfrak{I}$$

$$= \int_1^e\frac{(ln\mathcal{v})^{q}}{\mathcal{v}}\left(\int_1^e\frac{(ln\mathfrak{I})^{p+1}}{\mathfrak{I}}\psi_{\mathfrak{I}}d\mathfrak{I}\right)d\mathcal{v} =$$

$$\int_1^e\frac{(\ln\mathcal{v})^{q}}{\mathcal{v}}((ln\,\mathfrak{I})^{p+1}\,\psi)\,\Big|_1^e - (p+1)\int_1^e\frac{(ln\,\mathfrak{I})^{p}}{t}\psi)d\mathfrak{I})d\mathcal{v}$$

$$= \int_1^e\frac{(ln\mathcal{v})^{q}}{\mathcal{v}}\left(\psi(ln\,\mathcal{v}, 1) - (p+1)\left(\int_1^e\frac{(ln\mathfrak{I})^{p}}{\mathfrak{I}}\psi d\mathfrak{I}\right)\right)d\mathcal{v}$$

$$Z^{\mathfrak{I}}Z^{\mathcal{v}}[ln\,\mathfrak{I}\,\psi_{\mathfrak{I}}] = Z^{\mathcal{v}}[\psi(ln\,\mathcal{v}, 1)] - (p+1)Z^{\mathfrak{I}}Z^{\mathcal{v}}[\psi]$$

4.4The second Double AL- Zughair transform for partial derivatives with respect to $\mathcal{v}$

$$Z^{\mathfrak{I}}Z^{\mathcal{v}}[(ln\,\mathcal{v})^2\psi_{\mathcal{v}\mathcal{v}}] = Z^{\mathfrak{I}}[\psi_{\mathcal{v}}(1, ln\,\mathfrak{I})]$$
$$-(q+2)Z^{\mathfrak{I}}[\psi(1, ln\,\mathfrak{I})] + (q+2)(q+1)Z^{\mathfrak{I}}Z^{\mathcal{v}}[\psi]$$

Proof

$$Z^{\mathfrak{I}}Z^{\mathcal{v}}[(ln\mathcal{v})^2\psi_{\mathcal{v}\mathcal{v}}] = \int_1^e\frac{(ln\mathfrak{I})^{p}}{\mathfrak{I}}\int_1^e\frac{(ln\mathcal{v})^{q}}{\mathcal{v}}(ln\,\mathcal{v})^2\psi_{\mathcal{v}\mathcal{v}}d\mathcal{v}d\mathfrak{I}$$

$$= \int_1^e\frac{(ln\mathfrak{I})^{p}}{\mathfrak{I}}\left(\int_1^e\frac{(ln\mathcal{v})^{q+2}}{\mathcal{v}}\psi_{\mathcal{v}\mathcal{v}}d\mathcal{v}\right)d\mathfrak{I}$$

$$=$$

$$\int_1^e\frac{(ln\,\mathfrak{I})^{p}}{\mathfrak{I}}((ln\,\mathcal{v})^{q+2}\,\psi_{\mathcal{v}}\,\Big|_1^e - (q+2)(\int_1^e\frac{(\ln\mathcal{v})^{q+1}}{\mathcal{v}}\psi_{\mathcal{v}}\,d\mathcal{v})d\mathfrak{I}$$

$$= Z^{\mathfrak{I}}[\psi_{\mathcal{v}}(1, ln\,\mathcal{v})] - (q+2)Z^{\mathfrak{I}}[\psi(1, ln\,\mathfrak{I})]$$
$$+(q+2)(q+1)Z^{\mathfrak{I}}Z^{\mathcal{v}}[\psi]$$

4.5The second Double AL- Zughair transform for derivatives with respect to $\mathfrak{I}$

$$Z^{\mathfrak{I}}Z^{\mathcal{v}}[(ln\,\mathfrak{I})^2\psi_{\mathfrak{I}\mathfrak{I}}] = Z^{\mathcal{v}}[\psi_{\mathfrak{I}}(ln\,\mathcal{v}, 1)]$$

$-(p+2)Z^{v}[\psi(\ln v,1)]+(p+2)(p+1)Z^{\Im}Z^{v}[\psi]$

Proof

$$Z^{\Im}Z^{v}[(\ln\Im)^2\psi_{\Im\Im}]=\int_1^e\frac{(\ln v)^{q}}{v}\int_1^e\frac{(\ln v)^{p}}{\Im}(\ln\Im)^2\psi_{\Im\Im}d\Im dv$$

$$=\int_1^e\frac{(\ln v)^{q}}{v}\left(\int_1^e\frac{(\ln\Im)^{p+2}}{\Im}\psi_{\Im\Im}d\Im\right)dv$$

$$=\int_1^e\frac{(\ln v)^{q}}{v}((\ln\Im)^{p+2}\psi_{\Im}\Big|_1^e$$

$$-(p+2)\int_1^e\frac{(\ln\Im)^{p+1}}{\Im}\psi_{\Im}d\Im)dv$$

$$=Z^{v}[\psi_{\Im}(\ln v,1)]-(p+2)Z^{v}[\psi(\ln v,1)]$$

$$+(p+2)(p+1)Z^{\Im}Z^{v}[\psi]$$

4.6The first Double AL- Zughair transform for derivatives with respect to $(\boldsymbol{v\ and\ \Im})$

$$Z^{\Im}Z^{v}[\ln v\ln\Im\psi_{v\Im}]=\psi(1,1)-(p+1)Z^{\Im}[\psi(1,\ln\Im]-(q+1)Z^{v}[\psi(\ln v,1)]+(p+1)(q+1)Z^{\Im}Z^{v}[\psi]$$

Proof

$Z^{\Im}Z^{v}[\ln v\ln\Im\psi_{v\Im}]$

$$=\int_{\Im=1}^{e}\frac{(\ln\Im)^{p}}{\Im}\int_{v=1}^{e}\frac{(\ln v)^{q}}{v}\ln v\ln\Im\psi_{v\Im}dvd\Im$$

$$=\int_{\Im=1}^{e}\frac{(\ln\Im)^{p}}{\Im}\left(\ln\Im\int_{v=1}^{e}\frac{(\ln v)^{q+1}}{v}\psi_{v\Im}dv\right)d\Im$$

$$=\int_{\Im=1}^{e}\frac{(\ln\Im)^{p}}{t}\ln\Im\left((\ln v)^{\delta+1}\psi_{\Im}\Big|_{v=1}^{e}\right.$$

$$\left.-(p+1)\int_{v=1}^{e}\frac{(\ln v)^{q}}{v}\psi_{\Im}dv\right)d\Im$$

$$=\int_{\Im=1}^{e}\frac{(\ln\Im)^{p}}{t}\ln\Im(\psi_{\Im}(1,\ln\Im)-$$

$$(q+1)(\int_{v=1}^{e}\frac{(\ln\Im)^{q}}{v}\psi_{\Im}dv))d\Im$$

$$=\int_{\Im=1}^{e}\frac{(\ln\Im)^{p+1}}{\Im}\psi_{\Im}(1,\ln\Im)-$$

$$(q+1)\int_{v=1}^{e}\frac{(\ln v)^{q}}{v}\left(\int_{\Im=1}^{e}\frac{(\ln\Im)^{p+1}}{\Im}\psi_{\Im}d\Im\right)dv$$

$$=(\ln\Im)^{\frac{1}{p}}\psi(1,\ln\Im)\Big|_{t=1}^{e}-$$

$$(p+1)\int_{\Im=1}^{e}\frac{(\ln\Im)^{p+1}}{\Im}\psi(1,\ln\Im)d\Im-$$

$$(q+1)\int_{v=1}^{e}\frac{(\ln v)^{q}}{v}((\ln\Im)^{(p+1)}\psi\Big|_{\Im=1}^{e}$$

$$-(p+1)\int_{\Im=1}^{e}\frac{(\ln\Im)^{p}}{\Im}\psi d\Im)dv$$

$$Z^{\Im}Z^{v}[\ln v\ln\Im\psi_{v\Im}]=\psi(1,1)-(p+1)Z^{\Im}[\psi(1,\ln\Im)]$$

$$-(q+1)Z^{v}[\psi(\ln v,1)]+(p+1)(q+1)Z^{\Im}Z^{v}[\psi]$$

4.7Illustrative Examples

The following examples show Double AL-Zughair transform for solving partial differential equations.

Examples4: Solve the following (PDE)

$$\boldsymbol{\ln v\psi_{v}=2\ln\Im\psi_{\Im}+\psi}$$

With the conditions:

$$\boldsymbol{\psi(\ln v,1)=-\ln v,\psi(1,\ln\Im)=-1}$$

By Double AL- Zughair transform.

Solution:

By taking $Z^{\Im}Z^{v}$ to both sides, we get

$Z^{\Im}[\psi(1,\ln\Im)]-(q+1)Z^{\Im}Z^{v}[\psi]=2Z^{v}[\psi(\ln v,1)]-2(p+1)Z^{\Im}Z^{v}[\psi]+Z^{\Im}Z^{v}[\psi]$

$$Z^{\Im}[\psi(1,\ln\Im)]=\frac{-1}{(p+1)},\qquad Z^{v}[\psi(\ln v,1)]=\frac{-1}{(q+2)}$$

$$(2p-q)Z^{\Im}Z^{v}[\psi]=\frac{-2}{(q+2)}+\frac{1}{(p+1)}$$

$$Z^{\Im}Z^{v}[\psi]=\frac{-2}{(2p-q)(q+2)}+\frac{1}{(2p-q)(p+1)}$$

$$Z^{\Im}Z^{v}[\psi]=\frac{-2(p+1)+(q+2)}{(2p-q)(p+2)(q+1)}=\frac{1}{(q+2)(p+1)}$$

By taking$(Z^{\Im}Z^{v})^{-1}$ to both sides, we get $\psi=\ln v$

Examples5: Consider the following Telegrapher equation, solve (PDE):$\boldsymbol{(\ln v)^2\psi_{vv}-(\ln\Im)^2\psi_{\Im\Im}-2\psi=0}$

With the conditions:

$$\boldsymbol{\psi(\ln v,1)=\frac{1}{3}(\ln v)^{-1}-\frac{1}{3}(\ln v)^2,\psi_{v}(1,\ln\Im)=1}$$

By Double AL- Zughair transform.

Solution:

By taking $Z^{\Im}Z^{v}$ to both sides, we get

$Z^{\Im}[\psi_{v}(1,\ln\Im)]-(q+2)Z^{\Im}[\psi(1,\ln\Im)]$

$+(q+2)(q+1)Z^{\Im}Z^{v}[\psi]-Z^{v}[\psi_{\Im}(\ln\Im,1)]+$

$$(p+2)Z^{v}[\psi(\ln v,1)]-(p+2)(p+1)Z^{\Im}Z^{v}[\psi]-2Z^{\Im}Z^{v}[\psi]=0$$

$$Z^{\Im}[\psi_{v}(1,\ln\Im)]=\frac{1}{(p+1)}\ ,Z^{v}[\psi(\ln v,1)]=\frac{1}{q(q+3)}$$

$$\big((q+2)(q+1)-(p+2)(p+1)-2\big)Z^{\Im}Z^{v}[\psi]=\frac{1}{(p+1)}-\frac{(p+2)}{q(q+3)}$$

$$(q^2+3q-(p^2+3p+2))Z^{\Im}Z^{v}[\psi]=\frac{1}{(p+1)}-\frac{(p+2)}{q(q+3)}$$

$$Z^{\Im}Z^{v}[\psi]=\frac{1}{(q^2+3q-(p^2+3p+2))(p+1)}-\frac{(p+2)}{(q^2+3q-(p^2+3p+2))q(q+3)}$$

$$=\frac{q(q+3)-(p+1)(p+2)}{(q^2+3q-p^2+3p)(p+1)q(q+3)}$$

$$=\frac{1}{(p+1)q(q+3)}$$

By taking $(Z^{\Im}Z^{v})^{-1}$ to both sides, we get

$$\psi=\frac{1}{3}(\ln v)^{-1}-\frac{1}{3}(\ln v)^{2}$$

Examples6: Solve the following (PDE*)*:

$$\boldsymbol{\ln v\ \ln\Im\ \psi_{v\Im}+\ln\Im\ \psi_{\Im}=2sinh(\pi(\ln(\ln\Im))+3\ln\Im(\ln v)^2}$$

With the conditions:

$$\boldsymbol{\psi(\ln v,1)=0,\psi_{v}(1,\ln\Im)=0,\psi(1,1)=1}$$

By Double AL- Zughair transform

Solution:

By taking $Z^{\Im}Z^{v}$ to both sides, we get

$$\psi(1,1)-(p+1)Z^{\Im}[\psi(1,\ln\Im)]-(q+1)Z^{v}[\psi(\ln v,1)]+(p+1)(q+1)Z^{\Im}Z^{v}[\psi]+Z^{v}[\psi(\ln v,1)]-(p+1)Z^{\Im}Z^{v}[\psi]=-\frac{2\pi}{(p+1)((q+1)^2-\pi^2)}+\frac{3}{(p+2)(q+3)}-1$$

$$\big((p+1)(q+1)-(p+1)\big)Z^{\Im}Z^{v}[\psi]=\frac{-2\pi}{(p+1)((q+1)^2-\pi^2)}+\frac{3}{(p+2)(q+3)}-1$$

$$(q(p+1))Z^{\Im}Z^{v}[\psi]=\frac{-2\pi}{(p+1)((q+1)^2-\pi^2)}+\frac{3}{(p+2)(q+3)}-1$$

$$Z^{\Im}Z^{v}[\psi]=\frac{-2\pi}{(p+1)^2q((q+1)^2-\pi^2)}+\frac{3}{(p+1)(p+2)q(q+3)}-\frac{1}{q(p+1)}$$

By taking $(Z^{\Im}Z^{v})^{-1}$ to both sides, we get

$$\psi=\frac{1}{\pi+1}\ln(\ln\Im)\,(\ln\Im)^{\pi}-1)+\frac{1}{-\pi+1}\ln(\ln\Im)\,(\ln v)^{-\pi}-1)+(1-\ln\Im)(1+\ln\Im)-(\ln v)^{-1}$$

v. Conclusion

In this work , we developed (Double AL-Zughair transform) which is considered a tool for solving (first / second) PDEs with variable coefficients, specially one dimensional. We also explained some examples and some properties which can be used in the future to solve many important applications.

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