A Study of The Neighborhood of A New Class of Analytic Functions Containing The q-Symmetric Multiplier Transform Operator

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DOI: http://dx.doi.org/10.31642/JoKMC/2018/120206
Received Mar.24, 2025. Accepted for publication Jul. 4, 2025

Abstract*— *In this paper I have defined new subclasses of analytic functions characterized by negativity of their coefficients by the q-symmetric multiplier transform operator. In fact, they are subclasses of the class of starlike and convex functions of complex order, and through them I studied the* $(\boldsymbol{n},\boldsymbol{\xi})$ *−neighborhood of these subclasses and the inclusion relations between them. So I put forward the conditions through which these subclasses belong to the neighborhood or vice versa, the neighborhood belongs to them, and I explained the special cases of them.

Keywords— ***Analytic Functions, Multiplier Transform Operator, Neighborhood, q- Derivative, Convex Function, Starlike Function.***

I. Introduction

I begin my research by assuming the class $\mathcal{A}(n)$ which is the class of all normalized analytic functions inside the unit disc

$\mathcal{U}=\{z\in\mathbb{C}:|z|<1\}$, and each element of which is written in the form

$$f(z)=z-\sum_{k=n+1}^{\infty}a_k\,z^k\,,(a_k\geq 0,\ n\in\mathbb{N}=\{1,2,3,\dots\})\,. \tag{1}$$

Now I need to introduce the concept of (n,ξ) −neighborhood of the function $f(z)$ in the basic class $\mathcal{A}(n)$, which I got from the two sources [1], [2] as follows:

$$\mathcal{N}_{n,\xi}(f)=\left\{h\in\mathcal{A}(n):h(z)=z-\sum_{k=n+1}^{\infty}c_k\,z^k\,,\sum_{k=n+1}^{\infty}k|a_k-c_k\,|\leq\xi\right\}. \tag{2}$$

If we take this definition for the identity function $I(z)=z$, we get the definition

$$\mathcal{N}_{n,\xi}(I)=\left\{h\in\mathcal{A}(n):h(z)=z-\sum_{k=n+1}^{\infty}c_k\,z^k\,,\sum_{k=n+1}^{\infty}k|c_k\,|\leq\xi\right\}, \tag{3}$$

which is a special case of the definition of the general neighborhood.

Now I will recall the two definitions of the starlike function of complex order η and the convex function of complex order η, which are the cornerstone of the research and have been proposed or rather studied by the researchers referred in the sources [3], [4], [5],] 6] as follows:

Let f be a function in the defined class $\mathcal{A}(n)$. Then f has the starlike property of complex order η if the following condition is satisfied:

$$\mathcal{R}e\left\{1+\frac{1}{\eta}\left(\frac{zf'(z)}{f(z)}-1\right)\right\}>0,\quad(z\in\mathcal{U},\eta\in\mathbb{C}-\{0\}). \tag{4}$$

The set of all starlike functions of complex order η is denoted by the symbol St_n.

In the same way for the function f in the class $\mathcal{A}(n)$, it is characterized by the property of convexity of complex order η if the following condition is satisfied:

$$\mathcal{R}e\left\{1+\frac{1}{\eta}\left(\frac{zf''(z)}{f'(z)}\right)\right\}>0,\quad (z\in\mathcal{U},\eta\in\mathbb{C}-\{0\}). \qquad (5)$$

The set of all convex functions of complex order η is denoted by the symbol Ct_n.

Let's start by defining the quantum derivative or the derivative q of the function f which is considered one of the operators that has attracted many researches say [7], [8], [9], [10] to study it recently due its great importance in the field of mathematical and physical sciences and its important applications in the topics of quantum transform analysis, quantum integral equation, special functions, and many other important topics.

As for the derivative $\mathrm{q}, 0<\mathrm{q}<1$, it has been studied by the most important researchers referred to in the sources [11], [12] in light of which the function f in the important set $\mathcal{A}(n)$ is defined, and its formula (1) is as follows:

$$\mathfrak{D}t_{\mathrm{q},n}f(z)=\begin{cases}\dfrac{f(z)-f(\mathrm{q}z)}{(1-\mathrm{q})z} & if\ z\neq 0\\ f'(z) & if\ z=0\end{cases}$$

$$\mathfrak{D}t_{\mathrm{q},n}f(z)=z-\sum_{k=n+1}^{\infty}(k)_{\mathrm{q}}a_k z^k,\ (a_k\geq 0,\ n\in\mathbb{N}=\{1,2,3,\dots\}). \qquad (6)$$

$$(k)_{\mathrm{q}}=\frac{1-\mathrm{q}^k}{1-\mathrm{q}}=1+\mathrm{q}+\mathrm{q}^2+\dots+\mathrm{q}^{k-1} \qquad (7)$$

$$f'(z)=\lim_{\mathrm{q}\to 1^-}\mathfrak{D}t_{\mathrm{q},n}f(z)=\lim_{\mathrm{q}\to 1^-}\frac{f(z)-f(\mathrm{q}z)}{(1-\mathrm{q})z}$$

$$\mathfrak{D}t_{\mathrm{q},1}f(z)=\mathfrak{D}t_{\mathrm{q}}f(z).$$

In the same vein, some researchers in sources [13], [14] have discussed the concept of q integration for a complex function f that is differentiable and under the condition that the series contained in the definition is convergent in the following form:

$$\int_0^{\infty}f(s)d_{\mathrm{q}}s=z(1-\mathrm{q})\sum_{k=0}^{\infty}\mathrm{q}^k f(z\mathrm{q}^k).$$

Now we come to our function f defined by equation (1). The integral of q for it is as follows:

$$\int_0^{\infty}f(s)d_{\mathrm{q}}s=\frac{z^2}{|2|_{\mathrm{q}}}+\sum_{k=n+1}^{\infty}\frac{a_k z^k}{|k+1|_{\mathrm{q}}},$$

$$\lim_{\mathrm{q}\to 1^-}\int_0^{\infty}f(s)d_{\mathrm{q}}s=\frac{z^2}{2}+\sum_{k=n+1}^{\infty}\frac{a_k z^k}{k+1},$$

In fact equal to the natural integral.

Also, some researchers in recent years have studied the q-symmetric multiplier transform operator in their research, such as [15], [16] and they defined it as follows:

$$\mathfrak{R}^{\ell}_{\mathrm{q},n}(\mathfrak{p})f(z)=\frac{z^{\ell-\mathrm{q}}}{|\mathfrak{p}+1\}_{\mathrm{q}}}\mathfrak{D}t_{\mathrm{q},n}(z^{\mathfrak{p}}\mathfrak{R}^{\ell-1}_{\mathrm{q},n}(\mathfrak{p})f(z)m\ (\mathfrak{p}>-1,\ell\in\mathbb{Z},z\in\mathcal{U}),$$

$$\mathfrak{R}^{\ell}_{\mathrm{q},n}(\mathfrak{p})f(z)=z+\sum_{k=n+1}^{\infty}\left[\frac{(\mathfrak{p}+k)_{\mathrm{q}}}{(\mathfrak{p}+1)_{\mathrm{q}}}\right]^{\ell}a_k z^k,(\mathfrak{p}>-1,0<\mathrm{q}<1,\ell\in\mathbb{Z},z\in\mathcal{U}), \qquad (8)$$

If we notice this interesting operator, we may see that it is a generalization of many of the operators are known and studied by distinguished researchers interested in this interesting and important field at the same time. Now I will review some of these operators, for example $\mathfrak{R}^{\ell}_{\mathrm{q},1}(0)f(z)=\mathfrak{D}^{\ell}$[16], also $\mathfrak{R}^{\ell}_{\mathrm{q},1}(1)f(z)=\mathfrak{D}^{\ell}f(z)$ [17], and $\mathfrak{R}^{-1}_{\mathrm{q},1}(0)f(z)=\mathfrak{I}^{\ell}f(z)$ [18].

Now let's get to the heart of the matter, which is using this operator to define the following two classes.

Definition 1.1. Let f be in $\mathcal{A}(n)$ defined by formula (1). We say that f is an element of the new class $\mathfrak{G}^{\ell}_{n,\mathrm{q}}(\mathfrak{p},\varrho,\kappa,\upsilon)$

If the following condition is satisfied:

$$\left|\frac{1}{\varrho}\left(\frac{z\mathfrak{D}t_{\mathrm{q},n}\left(\mathfrak{R}^{\ell}_{\mathrm{q},n}(\mathfrak{p})f(z)\right)+\kappa z^2\mathfrak{D}t^2_{\mathrm{q},n}(\mathfrak{R}^{\ell}_{\mathrm{q},n}(\mathfrak{p})f(z))}{\kappa z\mathfrak{D}t_{\mathrm{q},n}(\mathfrak{R}^{\ell}_{\mathrm{q},n}(\mathfrak{p})f(z)+(1-\kappa)\mathfrak{R}^{\ell}_{\mathrm{q},n}(\mathfrak{p})f(z)}-1\right)\right|<\upsilon, \qquad (9)$$

$$(0\leq\kappa\leq 1,0<\upsilon\leq 1,\quad \varrho\in\mathbb{C}-\{0\},z\in\mathcal{U}).$$

Definition 1.2. We take f in $\mathcal{A}(n)$ defined by formula (1). Thus f belongs to the class $\mathfrak{A}^{\ell}_{n,\mathrm{q}}(\mathfrak{p},\varrho,\kappa,\upsilon)$

If the following inequality is satisfied:

$$\left|\frac{1}{\varrho}\left(\mathfrak{D}t_{\mathrm{q},n}\left(\mathfrak{R}^{\ell}_{\mathrm{q},n}(\mathfrak{p})f(z)\right)+\kappa z\mathfrak{D}t^2_{\mathrm{q},n}(\mathfrak{R}^{\ell}_{\mathrm{q},n}(\mathfrak{p})f(z))-1\right)\right|<\upsilon, \qquad (10)$$

$$(0\leq\kappa\leq 1,0<\upsilon\leq 1,\quad \varrho\in\mathbb{C}-\{0\},z\in\mathcal{U}).$$

In addition, if we look closely at previous researches in this field, we note that some researchers studied subclasses that are considered special cases of these two known classes by taking $\kappa=\ell=0,\ \upsilon=1,\mathrm{q}\to 1^-$ mentioned in the sources [19], [20].

$$\mathfrak{G}^{\ell}_{n,\mathrm{q}}(\mathfrak{p},\varrho,0,1)\subset St_n,$$

$$\mathfrak{A}^{\ell}_{n,\mathrm{q}}(\mathfrak{p},\varrho,0,1)\subset Ct_n.$$

2. Basic Results and Containment Relations of The Classes $\mathfrak{G}_{n,\mathfrak{q}}^{\ell}(\mathfrak{p},\varrho,\kappa,\upsilon)$ and $\mathfrak{A}_{n,\mathfrak{q}}^{\ell}(\mathfrak{p},\varrho,\kappa,\upsilon)$ With The Neighborhood of The Identity Function $\mathcal{N}_{n,\xi}(I)$:

First we need to prove the two basic results which are considered the essence of the subject and on the basis of which we obtain the containment results of the (n,ξ) −neighborhood of the identity function $\mathcal{N}_{n,\xi}(I)$ with the two new classes $\mathfrak{G}_{n,\mathfrak{q}}^{\ell}(\mathfrak{p},\varrho,\kappa,\upsilon)$ and $\mathfrak{A}_{n,\mathfrak{q}}^{\ell}(\mathfrak{p},\varrho,\kappa,\upsilon)$ defined above.

Theorem 2.1. The function $f(z)$ in the hypothetical fundamental class $\mathcal{A}(n)$ is an element of the class $\mathfrak{G}_{n,\mathfrak{q}}^{\ell}(\mathfrak{p},\varrho,\kappa,\upsilon)$ if and only if

$$\sum_{k=n+1}^{\infty}\left[\frac{(\mathfrak{p}+k)_{\mathfrak{q}}}{(\mathfrak{p}+1)_{\mathfrak{q}}}\right]^{\ell}\left(\kappa((k)_q-1)+1\right)\left((k)_q+\upsilon|\varrho|-1\right)a_k \le \upsilon|\varrho|. \qquad (11)$$

Proof. First we assume that the function $f(z)$ is an element of the defined class $\mathfrak{G}_{n,\mathfrak{q}}^{\ell}(\mathfrak{p},\varrho,\kappa,\upsilon)$. In light of this fact, and by applying Definition 1.1 to the new class $\mathfrak{G}_{n,\mathfrak{q}}^{\ell}(\mathfrak{p},\varrho,\kappa,\upsilon)$, we obtain

$$\mathcal{R}e\left\{\frac{z\mathfrak{D}t_{\mathfrak{q},n}\left(\mathfrak{R}_{\mathfrak{q},n}^{\ell}(\mathfrak{p})f(z)\right)+\kappa z^2\mathfrak{D}t_{\mathfrak{q},n}^2(\mathfrak{R}_{\mathfrak{q},n}^{\ell}(\mathfrak{p})f(z))}{\kappa z\mathfrak{D}t_{\mathfrak{q},n}(\mathfrak{R}_{\mathfrak{q},n}^{\ell}(\mathfrak{p})f(z)+(1-\kappa)\,\mathfrak{R}_{\mathfrak{q},n}^{\ell}(\mathfrak{p})f(z)}-1\right\} > -\upsilon\,|\varrho|, \qquad (z\in\mathcal{U})$$

Or, rather it can be simplified as

$$\mathcal{R}e\left\{\frac{-\sum_{k=n+1}^{\infty}\left[\frac{(\mathfrak{p}+k)_{\mathfrak{q}}}{(\mathfrak{p}+1)_{\mathfrak{q}}}\right]^{\ell}\left(\kappa((k)_q-1)+1\right)\left((k)_q+\upsilon|\varrho|-1\right)a_k z^k.}{1-\sum_{k=n+1}^{\infty}\left[\frac{(\mathfrak{p}+k)_{\mathfrak{q}}}{(\mathfrak{p}+1)_{\mathfrak{q}}}\right]^{\ell}\left(\kappa((k)_q-1)+1\right)a_k z^k}\right\} > -\upsilon\,|\varrho|, \qquad (z\in\mathcal{U}), \quad (12)$$

So we get what is required for the first side of the theorem for taking the real values of the variables z on the real axis and approaching one form the left side.

To complete the proof of the theorem, we to the other opposite side by assuming that condition (11) is satisfied. We have:

$$\left|\frac{z\mathfrak{D}t_{\mathfrak{q},n}\left(\mathfrak{R}_{\mathfrak{q},n}^{\ell}(\mathfrak{p})f(z)\right)+\kappa z^2\mathfrak{D}t_{\mathfrak{q},n}^2(\mathfrak{R}_{\mathfrak{q},n}^{\ell}(\mathfrak{p})f(z))}{\kappa z\mathfrak{D}t_{\mathfrak{q},n}(\mathfrak{R}_{\mathfrak{q},n}^{\ell}(\mathfrak{p})f(z)+(1-\kappa)\,\mathfrak{R}_{\mathfrak{q},n}^{\ell}(\mathfrak{p})f(z)}-1\right|$$

$$=\left|\frac{\sum_{k=n+1}^{\infty}\left[\frac{(\mathfrak{p}+k)_{\mathfrak{q}}}{(\mathfrak{p}+1)_{\mathfrak{q}}}\right]^{\ell}\left(\kappa((k)_q-1)+1\right)\left((k)_q+\upsilon|\varrho|-1\right)a_k z^k.}{z-\sum_{k=n+1}^{\infty}\left[\frac{(\mathfrak{p}+k)_{\mathfrak{q}}}{(\mathfrak{p}+1)_{\mathfrak{q}}}\right]^{\ell}\left(\kappa((k)_q-1)+1\right)a_k z^k}\right|$$

$$\le\frac{\upsilon|\varrho|\left(1-\sum_{k=n+1}^{\infty}\left[\frac{(\mathfrak{p}+k)_{\mathfrak{q}}}{(\mathfrak{p}+1)_{\mathfrak{q}}}\right]^{\ell}\left(\kappa((k)_q-1)+1\right)\left((k)_q+\upsilon|\varrho|-1\right)a_k\right)}{1-\sum_{k=n+1}^{\infty}\left[\frac{(\mathfrak{p}+k)_{\mathfrak{q}}}{(\mathfrak{p}+1)_{\mathfrak{q}}}\right]^{\ell}\left(\kappa((k)_q-1)+1\right)a_k}\le\upsilon|\varrho|.$$

By taking the value of z on the boundaries of the unit circle $\mathcal{U}$, then using the maximum modulus theorem, we obtain the required result, which is

$$f\in\mathfrak{G}_{n,\mathfrak{q}}^{\ell}(\mathfrak{p},\varrho,\kappa,\upsilon).$$

Thus, we complete the proof of the theorem.

Then we come to the following important basic result which can be proven in a similar way to the previous result and which relates to the second new knowledge class $\mathfrak{A}_{n,\mathfrak{q}}^{\ell}(\mathfrak{p},\varrho,\kappa,\upsilon)$ as follows:

Theorem 2.2. The function $f(z)$ in the hypothetical fundamental class $\mathcal{A}(n)$ is an element of the class $\mathfrak{A}_{n,\mathfrak{q}}^{\ell}(\mathfrak{p},\varrho,\kappa,\upsilon)$ if and only if

$$\sum_{k=n+1}^{\infty}\left[\frac{(\mathfrak{p}+k)_{\mathfrak{q}}}{(\mathfrak{p}+1)_{\mathfrak{q}}}\right]^{\ell}(k)_{\mathfrak{q}}\left(\kappa((k)_q-1)+1\right)a_k \le \upsilon|\varrho|. \qquad (13)$$

Remark 2. 1: In Theorem 2.1 we note that if we take specific values for example $\ell=0,\varrho=1,\upsilon=1-\alpha,\mathfrak{q}\to1^-$, we get a special case taken and studied by the distinguished researcher in source no [19] and presented it as a basic result of his research.

We will prove that the define class $\mathfrak{G}_{n,\mathfrak{q}}^{\ell}(\mathfrak{p},\varrho,\kappa,\upsilon)$ is included in the class of the (n,ξ) −neighborhood with respect to the identity function $\mathcal{N}_{n,\xi}(I)$ and state the condition through which the amazing result is reached.

Theorem 2.3. The new defined class $\mathfrak{G}_{n,\mathfrak{q}}^{\ell}(\mathfrak{p},\varrho,\kappa,\upsilon)$ is a subset of the (n,ξ) −neighborhood with respect to the identity function $\mathcal{N}_{n,\xi}(I)$, or

$$\mathfrak{G}_{n,\mathfrak{q}}^{\ell}(\mathfrak{p},\varrho,\kappa,\upsilon)\subset\mathcal{N}_{n,\xi}(I)$$

If

$$\xi=\frac{\upsilon|\varrho|(n+1)\mathfrak{q}[(\mathfrak{p}+1)\mathfrak{q}]^{\ell}}{\left(\kappa((n+1)_q-1)+1\right)\left((n+1)_q+\upsilon|\varrho|-1\right)[(\mathfrak{p}+1)\mathfrak{q}]^{\ell}},$$

$(|\varrho|<1)$.

Proof. Of, course, in order to prove the theorem, we must take an element in the new class $\mathfrak{G}_{n,\mathfrak{q}}^{\ell}(\mathfrak{p},\varrho,\kappa,\upsilon)$, i.e.

$$f\in\mathfrak{G}_{n,\mathfrak{q}}^{\ell}(\mathfrak{p},\varrho,\kappa,\upsilon)$$

From it we get

$$\left[\frac{(\mathfrak{p}+n+1)_{\mathfrak{q}}}{(\mathfrak{p}+1)_{\mathfrak{q}}}\right]^{\ell}\left(\kappa((n+1)_q-1)+1\right)\left((n+1)_q+\upsilon|\varrho|-1\right)\sum_{k=n+1}^{\infty}a_k \le \upsilon|\varrho|, \qquad (14)$$

or

$$\sum_{k=n+1}^{\infty}a_k \le \frac{\upsilon|\varrho|[(\mathfrak{p}+1)\mathfrak{q}]^{\ell}}{\left(\kappa((n+1)_q-1)+1\right)\left((n+1)_q+\upsilon|\varrho|-1\right)[(\mathfrak{p}+1)\mathfrak{q}]^{\ell}}. \qquad (15)$$

By referring to the basic condition in Theorem 1.1, the intended inequality 10 and the fact found in the previous inequality 15, we conclude the following:

$$\left[\frac{(\mathfrak{p}+n+1)_{\mathfrak{q}}}{(\mathfrak{p}+1)_{\mathfrak{q}}}\right]^{\ell}\left(\kappa((n+1)_q-1)+1\right)\sum_{k=n+1}^{\infty}(k)_{\mathfrak{q}}a_k \le \upsilon|\varrho|+(1-\upsilon|\varrho|)\left(\kappa(n+1)_q+\upsilon|\varrho|-1\right)) \times\left[\frac{(\mathfrak{p}+n+1)_{\mathfrak{q}}}{(\mathfrak{p}+1)_{\mathfrak{q}}}\right]^{\ell}\sum_{k=n+1}^{\infty}a_k$$

$$\le \upsilon|\varrho|+(1-\upsilon|\varrho|)\left(\kappa(n+1)_q-1\right)+1))\left[\frac{(\mathfrak{p}+n+1)_{\mathfrak{q}}}{(\mathfrak{p}+1)_{\mathfrak{q}}}\right]^{\ell} \times\frac{\upsilon|\varrho|[(\mathfrak{p}+1)\mathfrak{q}]^{\ell}}{\left(\kappa((n+1)_q-1)+1\right)\left((n+1)_q+\upsilon|\varrho|-1\right))[(\mathfrak{p}+1)\mathfrak{q}]^{\ell}},$$

$$\sum_{k=n+1}^{\infty}(k)_{\mathfrak{q}}a_k \le \frac{\upsilon|\varrho|(n+1)\mathfrak{q}[(\mathfrak{p}+1)\mathfrak{q}]^{\ell}}{\left(\kappa((n+1)_q-1)+1\right)\left((n+1)_q+\upsilon|\varrho|-1\right))[(\mathfrak{p}+1)\mathfrak{q}]^{\ell}}, = \xi. \quad (16)$$

So, f is an element of (n,ξ) −neighborhood with respect to the identity function $\mathcal{N}_{n,\xi}(I)$, in light of or relying on equation (3).

And thus the proof is successfully completed.

We also show the containment relationship between the new defined class $\mathfrak{A}^{\ell}_{n,\mathfrak{q}}(\mathfrak{p},\varrho,\kappa,\upsilon)$ and the class of (n,ξ) −neighborhood with respect to the identity function $\mathcal{N}_{n,\xi}(I)$ by determining the value of ξ which can be deduced through the proof of the following theorem. We prove it in the same way as the previous theorem, with the difference of using Theorem 2.2 instead of Theorem 2.1.

Theorem 2.4. The new defined class $\mathfrak{A}^{\ell}_{n,\mathfrak{q}}(\mathfrak{p},\varrho,\kappa,\upsilon)$ is a subset of the (n,ξ) −neighborhood with respect to the identity function $\mathcal{N}_{n,\xi}(I)$, or

$$\mathfrak{A}^{\ell}_{n,\mathfrak{q}}(\mathfrak{p},\varrho,\kappa,\upsilon)\subset \mathcal{N}_{n,\xi}(I)$$

If

$$\xi=\frac{\upsilon|\varrho|[(\mathfrak{p}+1)\mathfrak{q}]^{\ell}}{\left(\kappa((n+1)_q-1)+1\right)[(\mathfrak{p}+n+1)\mathfrak{q}]^{\ell}},\qquad (|\varrho|<1).$$

Remark 2. 2: In Theorem 2.3 we note that if we take specific values for example $\kappa=0,\ell=0,\varrho=1,\upsilon=1-\alpha\,(0<\alpha\le 1)$, $\mathfrak{q}\to 1^-$, we get a special case taken and studied by the distinguished researcher in source no [19] and presented it as a basic result of his research.

3. Containment Relations For The Distinct Classes $\mathfrak{G}^{\ell,\omega}_{n,\mathfrak{q}}(\mathfrak{p},\varrho,\kappa,\upsilon)$ and $\mathfrak{A}^{\ell,\omega}_{n,\mathfrak{q}}(\mathfrak{p},\varrho,\kappa,\upsilon)$ and The (n,ξ) −neighborhood of The Conditional Function $\mathcal{N}_{n,\xi}(I)$

First we define two classes $\mathfrak{G}^{\ell,\omega}_{n,\mathfrak{q}}(\mathfrak{p},\varrho,\kappa,\upsilon)$ and $\mathfrak{A}^{\ell,\omega}_{n,\mathfrak{q}}(\mathfrak{p},\varrho,\kappa,\upsilon)$ by means of the previously defined classes $\mathfrak{G}^{\ell}_{n,\mathfrak{q}}(\mathfrak{p},\varrho,\kappa,\upsilon)$ and $\mathfrak{A}^{\ell}_{n,\mathfrak{q}}(\mathfrak{p},\varrho,\kappa,\upsilon)$. Later, we show the containment relations for these two classes with the class (n,ξ) −neighborhood of the conditional function $\mathcal{N}_{n,\xi}(h)$ by determining or distinguishing the value of ω for each of them.

We take a function f in the class $\mathcal{A}(n)$, so it is an element of the distinct class $\mathfrak{G}^{\ell,\omega}_{n,\mathfrak{q}}(\mathfrak{p},\varrho,\kappa,\upsilon)$ if there is a function $h\in\mathfrak{G}^{\ell}_{n,\mathfrak{q}}(\mathfrak{p},\varrho,\kappa,\upsilon)$ that satisfies the following condition:

$$\left|\frac{f(z)}{h(z)}-1\right|<1-\omega,\qquad (0\le\omega<1, z\in\mathcal{U})\qquad (17)$$

Now we define the class $\mathfrak{A}^{\ell,\omega}_{n,\mathfrak{q}}(\mathfrak{p},\varrho,\kappa,\upsilon)$ by the class $\mathfrak{A}^{\ell}_{n,\mathfrak{q}}(\mathfrak{p},\varrho,\kappa,\upsilon)$ in the same previous way:

We take a function f in the basic set $\mathcal{A}(n)$, so it is an element of the distinct class $\mathfrak{A}^{\ell,\omega}_{n,\mathfrak{q}}(\mathfrak{p},\varrho,\kappa,\upsilon)$ if there is a function $h\in\mathfrak{A}^{\ell}_{n,\mathfrak{q}}(\mathfrak{p},\varrho,\kappa,\upsilon)$ that satisfies the following condition:

$$\left|\frac{f(z)}{h(z)}-1\right|<1-\omega,\qquad (0\le\omega<1, z\in\mathcal{U})\qquad (18)$$

Theorem 3.1. The (n,ξ) −neighborhood with respect to the conditional function $\mathcal{N}_{n,\xi}(h)$, where $h\in\mathfrak{G}^{\ell}_{n,\mathfrak{q}}(\mathfrak{p},\varrho,\kappa,\upsilon)$ is a subset of the distinct defined class $\mathfrak{G}^{\ell,\omega}_{n,\mathfrak{q}}(\mathfrak{p},\varrho,\kappa,\upsilon)$, or

$$\mathcal{N}_{n,\xi}(h)\subset\mathfrak{G}^{\ell,\omega}_{n,\mathfrak{q}}(\mathfrak{p},\varrho,\kappa,\upsilon)$$

If

$$\omega=1-\frac{\xi}{(n+1)_{\mathfrak{q}}}\times \left\{\left(\kappa((n+1)_q-1)+1\right)\left((n+1)_q+\upsilon|\varrho|-1\right))[(\mathfrak{p}+n+1)\mathfrak{q}]^{\ell}\right\} /\left\{\left(\kappa((n+1)_q-1)+1\right)\left((n+1)_q+\upsilon|\varrho|-1\right))[(\mathfrak{p}+n+1)\mathfrak{q}]^{\ell}-\upsilon|\varrho|(n+1)_q[(\mathfrak{p}+1)\mathfrak{q}]^{\ell}\right\}\right\}$$

Proof. In order to prove the required containment relation, we take $f\in\mathcal{N}_{n,\xi}(h)$ and directly we obtain the truth

$$\sum_{k=n+1}^{\infty}k|a_k-c_k|\le\xi,$$

i.e. the following inequality of coefficients

$$\sum_{k=n+1}^{\infty} k|a_k - c_k| \leq \frac{\xi,}{(n+1)_q}, \qquad (n \in \mathbb{N})$$

Again we recall condition 14 because $h \in \mathfrak{G}_{n,\mathfrak{q}}^{\ell}(\mathfrak{p}, \varrho, \kappa, \upsilon)$

$$\sum_{k=n+1}^{\infty} a_k \leq \frac{\upsilon|\varrho|[(\mathfrak{p}+1)\mathfrak{q}]^{\ell}}{(\kappa((n+1)_q - 1) + 1)\left((n+1)_q + \upsilon|\varrho| - 1\right)[(\mathfrak{p}+n+1)\mathfrak{q}]^{\ell}}. \quad (19)$$

$$\left|\frac{f(z)}{h(z)} - 1\right| < \frac{\sum_{k=n+1}^{\infty} |a_k - c_k|}{1 - \sum_{k=n+1}^{\infty} c_k}$$

$$\leq \frac{\xi,}{(n+1)_q} \times$$

$$\left\{\left(\kappa((n+1)_q - 1) + 1\right)\left((n+1)_q + \upsilon|\varrho| - 1\right)[(\mathfrak{p}+n+1)\mathfrak{q}]^{\ell}\right\} / \left\{\left(\kappa((n+1)_q - 1) + 1\right)\left((n+1)_q + \upsilon|\varrho| - 1\right)[(\mathfrak{p}+n+1)\mathfrak{q}]^{\ell} - \upsilon|\varrho|(n+1)_q[(\mathfrak{p}+1)\mathfrak{q}]^{\ell}\right\}$$

$$= 1 - \omega.$$

Thus we have deduced the value of ω, in light of which the function f belongs to the special class $\mathfrak{G}_{n,\mathfrak{q}}^{\ell,\omega}(\mathfrak{p}, \varrho, \kappa, \upsilon)$,

which in turn ends the proof of the theorem.

Now we come to the second distinct class $\mathfrak{A}_{n,\mathfrak{q}}^{\ell,\omega}(\mathfrak{p}, \varrho, \kappa, \upsilon)$ and the containment relation with class (n, ξ) −neighborhood with respect to the conditional function $\mathcal{N}_{n,\xi}(h)$ and the statement of the condition on the basis of which the relation is true. Honestly, it is similar to the previous theorem.

Theorem 3.2. The (n, ξ) −neighborhood with respect to the conditional function $\mathcal{N}_{n,\xi}(h)$, where $h \in \mathfrak{A}_{n,\mathfrak{q}}^{\ell}(\mathfrak{p}, \varrho, \kappa, \upsilon)$ is a subset of the distinct defined class $\mathfrak{A}_{n,\mathfrak{q}}^{\ell,\omega}(\mathfrak{p}, \varrho, \kappa, \upsilon)$, or

$$\mathcal{N}_{n,\xi}(h) \subset \mathfrak{A}_{n,\mathfrak{q}}^{\ell,\omega}(\mathfrak{p}, \varrho, \kappa, \upsilon)$$

If

$$\omega = 1 - \frac{\xi}{(n+1)_q} \times \frac{\left(\kappa((n+1)_q - 1) + 1\right)(n+1)_q[(\mathfrak{p}+n+1)\mathfrak{q}]^{\ell}}{\left(\kappa((n+1)_q - 1) + 1\right)(n+1)_q[(\mathfrak{p}+n+1)\mathfrak{q}]^{\ell} - \upsilon|\varrho|[(\mathfrak{p}+1)\mathfrak{q}]^{\ell}}, \quad (|\varrho| < 1|).$$

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