Harnessing the Iman Transform for Efficient Solutions to Heat and Wave Problems

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Abstract**—** *The heat and wave equations are prominent partial differential equations with extensive applications in fundamental sciences and engineering. Integral transform methods provide effective solutions to various challenges in these fields. This chapter explores the Iman transform technique as a tool for solving heat and wave equations represented by partial differential equations.*

Keywords**— *Iman Transform, Heat equation, Wave equation, Differential Equations.***

I. INTRODUCTION

Partial differential equations are used extensively in chemistry, physics, and a variety of other disciplines. As a result, the literature includes a variety of approaches for solving partial differential equations with the goal of establishing symmetry. These equations are used to analyze practical challenges that people face, such as biological growth, tumour size expansion, heat transmission, carbon dating, compound interest calculations, chemical reactions, mixing problems, compartment problems, electric circuits, and trajectory problems [1]. Partial differential equations in mathematics are commonly employed to address a variety of problems, such as boundary value problems, initial value problems, heat and wave equations, and other differential equations. Various methods exist for solving these equations, including numerical approaches, decomposition techniques, separation methods, integral transforms, and more. Nowadays, integral transforms are considered the most convenient and straightforward mathematical tools for solving advanced problems related to initial-value problems, boundary-value problems, differential equations, and integral equations that are encountered in various fields such as technology, science, social sciences, commerce, economics, and engineering. One important feature of integral transformations is that they provide an exact solution to problems without requiring large amounts of computation. Owing to this significant characteristic, a large number of scholars work in this area and become acquainted with different integral transforms. Yang [2] used a Fourier-like integral transformation to find accurate solutions for the steady heat transfer problem. Su et al. [3] applied the Fourier-like integral transform to wave and heat-transfer problems. Vaidya et al. [4] used various approaches to solve the partial differential heat equation. Peker et al. [5] solved heat transfer problems using the Kashuri Fundo transform. Using the Rohit Transform, Gupta et al. [6] solved the wave and heat equations. Gupta Transform was utilized by Gupta et al. [7] to solve the one-dimensional heat and wave equation. Mulugeta et al. [8] used Andualem and Khan Transform to achieve exact solutions for wave and heat equations. Iman Ahmed Almardy [9] employed Iman transform to solve Ordinary Differential Equations. Dinesh and Prakash [10] solved the linear second-kind Volterra integral problem by applying the Upadhyaya transform. Dinesh and Kuffi [11] utilized the Upadhyaya transform in order to derive precise solutions for cardiovascular models. Rania et al. [12] recently employed the Mohanad transform to solve a few ordinary differential equation systems.

II. MATERIAL AND METHOD

IMAN TRANSFORM TECHNIQUE [9]:

Definition 2.1. For an exponential order function, the Iman Transform is defined as:

$$I = \left\{ f(t) : \exists\ K,\ \lambda_1,\ \lambda_2 > 0, \left| f(t) \right| < Ke^{-\lambda_j^2 t} \right\} \tag{1}$$

$$\left.\begin{array}{l} \text{where, } f(t) \text{ be the function in the set I} \\ \quad \text{K be the finite constant number and} \\ \quad \lambda_1, \lambda_2 \text{ may be finite or infinite number} \\ \quad I - \text{Iman Operator.} \end{array}\right\}$$

Definition 2.2. The kernel function of Iman Transform symbolized by I(.), written in the integral form as:

$$\left.\begin{array}{l} I[f(t)] = \frac{1}{v^2} \int_0^\infty \exp\left(-v^2 t\right) f(t)\, dt = B(v),\ t \geq 0,\ \lambda_1 < v < \lambda_2;\ v \geq 0 \\ \qquad \text{and} \quad f(t) = I^{-1}[B(v)], \quad t \geq 0 \end{array}\right\} \tag{2}$$

The inverse of Iman Transform is denoted by , be the factor of variable.

Property 2.3. Linearity of Iman transform:

If and respectively, are the Iman transform of functions , therefore, Iman transform of

$$I\left[p\, f_1(t) + q\, f_2(t)\right] = \left[p\, B_1(v) + q\, B_2(v)\right] \tag{3}$$

Property 2.4. Ordinary derivative properties of Iman transform:

$$\left.\begin{array}{ll} \text{First derivative:} & I\left[f'(t)\right] = v^2 I(v) - \frac{1}{v^2} f(0) = I\left[\frac{df(t)}{dt}\right] \\ \text{Second derivative:} & I\left[f''(t)\right] = v^4 I(v) - f(0) - \frac{1}{v^2} f'(0) = I\left[\frac{d^2 f(t)}{dt^2}\right] \\ \text{nth derivative:} & I\left[f^n(t)\right] = v^n I(v) - \sum_{k=0}^{n-1} \frac{1}{v^{4-2n+2k}} f^k(0) = I\left[\frac{d^n f(t)}{dt^n}\right] \end{array}\right\}$$

(4)

Property 2.5. Partial derivative properties of Iman transform:

$$\left.\begin{array}{ll} \text{First derivative:} & I\left[\frac{\partial f(x,t)}{\partial t}\right] = v^2 I(x,v) - \frac{1}{v^2} f(x,0) = \frac{d}{dx}\left[I(x,v)\right] \\ \text{Second derivative:} & I\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right] = v^4 I(x,v) - f(x,0) - \frac{1}{v^2}\left(\frac{\partial f(x,0)}{\partial t}\right) = \frac{d^2}{dx^2}\left[I(x,v)\right] \\ \text{nth derivative:} & I\left[\frac{\partial^3 f(x,t)}{\partial t^3}\right] = v^6 I(x,v) - v^2 f(x,0) - \frac{\partial f(x,0)}{\partial t} = \frac{d^3}{dx^3}\left[I(x,v)\right] \end{array}\right\} \tag{5}$$

III. TABULATED VALUES

Tabulated values of Iman transform and Inverse of Iman transform for some encountered functions $f(t)$ .Mathematically, Iman transform in integral form expressed as:

$I[f(t)] = \frac{1}{v^2} \int_0^\infty \exp\left(-v^2 t\right) f(t)\, dt = B(v),\ t \geq 0$ and if $I\left[f(t)\right] = B(v)$, then the function $f(t)$ is called the inverse Iman transform of $B[v]$ and is expressible as $f(t) = I^{-1}[B(v)]$, where, I^{-1} called the inverse Iman transform operator.

3(a) Iman Transform of Some functions

$$\left.\begin{array}{lclcl} f(t) & I\left[f(t)\right] = B(v) & f(t) & & I\left[f(t)\right] = B(v) \\ I(1) & = \frac{1}{v^4}, & I(\exp(at)) & = & \frac{1}{v^4 - av^2} \\ I(t) & = \frac{1}{v^6}, & I(\exp(-at)) & = & \frac{1}{v^4 + av^2} \\ I\left(t^n\right) & = \frac{m!}{v^{2m+4}},\ m \in N, & & & \end{array}\right\} a$$

3(b) Inverse of Iman Transform

$$\left.\begin{array}{lclc} I^{-1}[B(v)] & f(t) & I^{-1}[B(v)] & f(t) \\ I^{-1}\left(\frac{1}{v^4}\right) & = 1, & I^{-1}\left(\frac{1}{v^4 - av^2}\right) & = \exp(at) \\ I^{-1}\left(\frac{1}{v^6}\right) & = t, & I^{-1}\left(\frac{1}{v^4 + av^2}\right) & = \exp(-at) \\ I^{-1}\left(\frac{m!}{v^{2m+4}}\right) & = t^n,\ m \in N, & & \end{array}\right\}$$

IV. METHODOLOGY:

Methodology for Solving the Applications of Heat and Wave Equations via Iman Transform Technique:

This chapter utilizes the Iman transform technique to solve heat and wave equations formulated as differential equations, shedding light on their applications in basic sciences, material sciences, and engineering.

Application-4.1. [HEAT EQUATION]:

Consider the heat equation [6]

$$\frac{\partial f(x,t)}{\partial t} = \frac{\partial^2 f(x,t)}{\partial x^2} \tag{6}$$

with the condition

$$f(x,0) = 3\sin 2\pi x,\ f(0,t) = 0,\ f(2,t) = 0,\ 0 < x < 2 \quad and\ t > 0. \tag{7}$$

Relating the Iman transform to the equation (6), both the sides, we have:

$$I\left(\frac{\partial f(x,t)}{\partial t}\right)=I\left(\frac{\partial^2 f(x,t)}{\partial x^2}\right) \qquad \mathbf{(8)}$$

Using the results of partial derivative properties of Iman Transform from equation (5), then equation (8), becomes

$$v^2B(x,v)-\frac{f(x,0)}{v^2}=I\left(\frac{\partial^2 f(x,t)}{\partial x^2}\right) \qquad (9)$$

Using given condition $f(x,0)=3\sin 2\pi x$, in equation (9), we get

$$v^2B(x,v)-\frac{3\sin 2\pi x}{v^2}=\frac{\partial^2}{\partial x^2}B(x,v) \qquad (10)$$

$$\frac{\partial^2}{\partial x^2}B(x,v)-v^2B(x,v)=\frac{3\sin 2\pi x}{v^2}$$

$$\left(D^2-v^2\right)B(x,v)=\frac{3\sin 2\pi x}{v^2} \qquad (11)$$

Equating to zero left side of equation (11) as $\left(D^2-v^2\right)B(x,v)=0$, which is a homogeneous equation. Therefore, the solution of this homogeneous equation written as

$$B(x,v)=C_1e^{vx}+C_2e^{-vx} \qquad (12)$$

where, C_1 and C_2 are constants and given initial conditions as

$$f(0,t)=0,\ f(2,t)=0,\ 0<x<2 \quad and \quad t>0.$$

By using $B(0,v)=0$, equation (12) becomes as

$$0=C_1e^{v(0)}+C_2e^{-v(0)}$$

$$C_1=-C_2$$

Again, using $B(2,v)=0$, in equation (12), we get

$$0=C_2\left(e^{-v(2)}-e^{v(2)}\right)$$

But $e^{-2v}-e^{2v}\neq 0$, therefore, we can write $C_2=0$

Therefore, complementary solution of equation (11) is Zero.

For determining the particular solution of equation (11)

Particular Integral P.I. = $\frac{1}{D^2-v^2}\left(\frac{-3\sin 2\pi x}{v^2}\right)$ (13)

After, simplification, we get

$$B(x,v)=P\cos 2\pi x+Q\sin 2\pi x \qquad (14)$$

On the way to determine the value of P and Q

By using the Technique of undetermined coefficients, we get

$$\frac{\partial}{\partial x}(B(x,v))=-2\pi P\sin 2\pi x+2\pi Q\cos 2\pi x \qquad (15)$$

$$\frac{\partial^2}{\partial x^2}(B(x,v))=-4\pi P\cos 2\pi x-4\pi Q\sin 2\pi x \qquad (16)$$

Substituting the value of $B(x,v)$ and $\frac{\partial^2}{\partial x^2}(B(x,v))$ from equation (14) and (16), in equation (11),

we get

$$-\left(4\pi^2P\cos 2\pi x\right)-\left(4\pi^2Q\sin 2\pi x\right)-v^2(P\cos 2\pi x+Q\sin 2\pi x)=-\frac{3\sin 2\pi x}{v^2}$$

$$-\left(4\pi^2+v^2\right)P\cos 2\pi x-\left(4\pi^2+v^2\right)Q\sin 2\pi x=-\frac{3\sin 2\pi x}{v^2} \qquad (17)$$

Equating the coefficients of like term to both the sides of equation (17)

$$-\left(4\pi^2+v^2\right)P=0\ ; \qquad \text{and} \quad \left(4\pi^2+v^2\right)Q=\frac{3}{v^2};$$

$$P=0 \qquad \text{and} \qquad Q=\frac{3}{v^2\left(4\pi^2+v^2\right)}$$

Therefore, the solution of equation (11), written as

$$B(x,v)=0(\cos 2\pi x)+\frac{3}{v^2\left(4\pi^2+v^2\right)}\sin 2\pi x$$

$$B(x,v)=\frac{3}{v^2\left(4\pi^2+v^2\right)}\sin 2\pi x \qquad (18)$$

Relating the inverse Iman transform to equation (18), we have

$$I^{-1}(B(x,v))=\sin 2\pi x\, I^{-1}\left(\frac{3}{v^2\left(4\pi^2+v^2\right)}\right)$$

$$f(x,t)=3e^{-4\pi^2 t}(\sin 2\pi x)= P.I \tag{19}$$

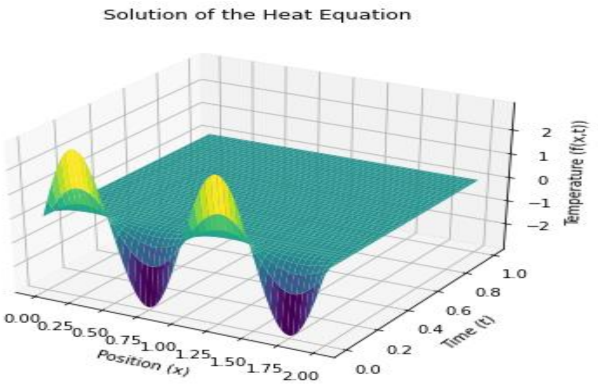
which is the particular solution of (11) and the solution of equation (6), we get

$$f(x,t)=3e^{-4\pi^2 t}(\sin 2\pi x)$$

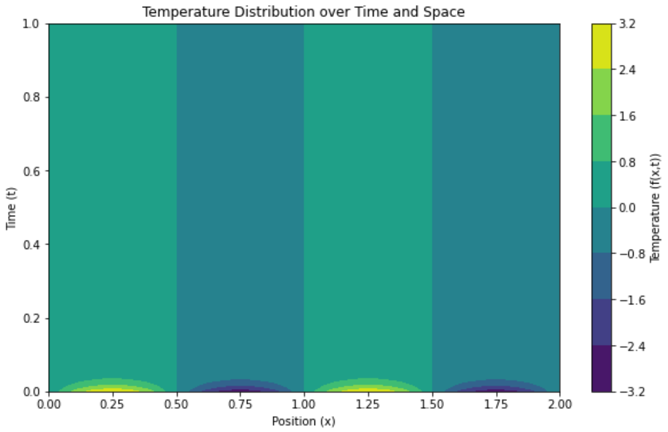
which is the solution of the heat equation, as the given condition $t=0$, $f(x,0)=3\sin 2\pi x$.

Heat Equation:

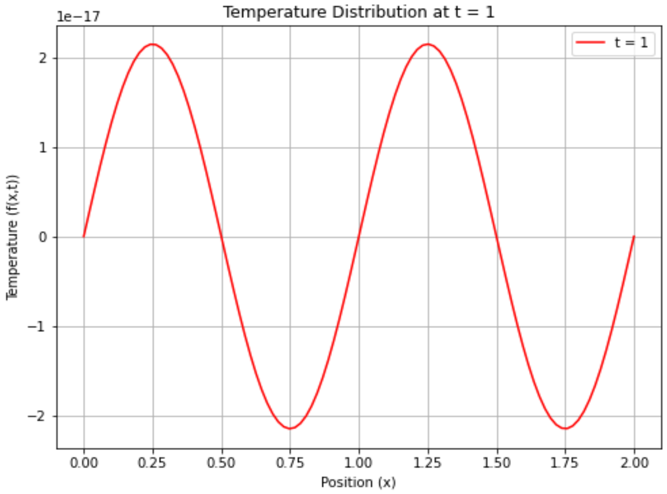
The provided graphs depict the temperature distribution over time and space according to the heat equation and given initial conditions.



Solution of the Heat Equation (Application 4.1)



Temperature Distribution Over Time and Space

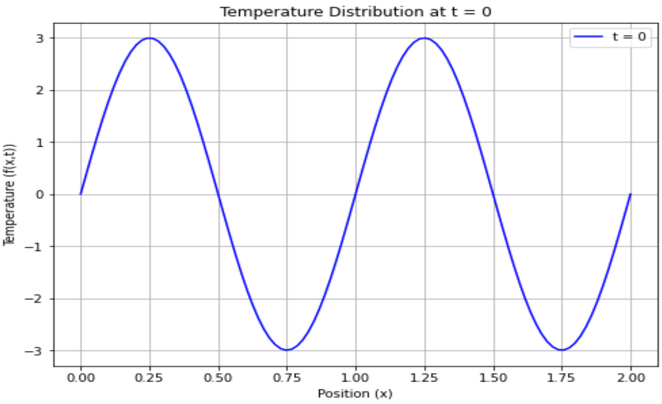


The 3D surface plot illustrates how the temperature f(x,t) evolves across both the spatial domain x and time t. In this plot, the x-axis represents the positionx, the y-axis represents time t, and the z-axis represents the temperaturef(x,t). The color gradient on the surface indicates the temperature at each point in space and time. Cooler temperatures are depicted in blue, while warmer temperatures are shown in yellow. This visualization provides a comprehensive view of how the temperature changes over both space and time.

The contour plot offers a different perspective on the temperature distribution, representing it with contour lines. Similar to the 3D plot, the x-axis represents the positionx, the y-axis represents timet, and the color indicates the temperaturef(x,t). The contour lines connect points of equal temperature, allowing for a clear visualization of temperature variations over time and space. Cooler temperatures are represented by blue contours, while warmer temperatures are shown in yellow. This plot provides a more detailed view of temperature changes over time and space compared to the 3D surface plot.

Lastly, the line plots at t=0 and t=1 focus on the temperature distribution along the spatial domain x at specific time points. At t=0, the temperature distribution is sinusoidal, reflecting the initial condition f(x,0)=3sin(2πx). As time progresses to t=1, the temperature distribution decays due to the heat equation's diffusion process, resulting in a smoother curve compared to t=0. These line plots provide insight into how the temperature evolves at discrete time points along the spatial domain.

Overall, these visualizations offer a comprehensive understanding of how the temperature evolves over time and space according to the heat equation and given initial conditions. They highlight the process of heat diffusion and how it influences the temperature distribution within the domain.



Application-4.2. [WAVE EQUATION]:

Consider the wave equation [6]

$$\frac{\partial^2 f(x,t)}{\partial t^2} = \frac{\partial^2 f(x,t)}{\partial x^2} \qquad (20)$$

with the condition

$$f(x,0) = \sin x,\ f(0,t) = 0,\ f(\pi,t) = 0, \frac{\partial f(x,0)}{\partial t} \qquad (21)$$

$$0 < x < \pi \quad and \quad t > 0.$$

Relating the Iman transform to both the side of equation (20), we get

$$I\left(\frac{\partial^2 f(x,t)}{\partial t^2}\right) = I\left(\frac{\partial^2 f(x,t)}{\partial x^2}\right) \qquad (22)$$

$$v^2 B(x,v) - \frac{f(x,0)}{v^2} = I\left(\frac{\partial^2 f(x,t)}{\partial x^2}\right)$$

$$\frac{\partial^2}{\partial x^2} B(x,v) - v^2 B(x,v) = -\frac{\sin x}{v^2} \qquad (23)$$

The complementary solution of equation (23), is the solution of homogeneous equation

$\frac{\partial^2}{\partial x^2} B(x,v) - v^2 B(x,v) = 0$, which is given by

$$B(x,v) = k_1 e^{v\,x} + k_2 e^{-v\,x} \qquad (24)$$

where, k_1 and k_2 are constants

Calculating these constants by using the given initial condition as below:

$$f(0,v) = 0,\ f(\pi,v) = 0\,.$$

Applying the above condition and the condition $B(0,v) = 0$, in equation (24), we get

$$0 = k_1 e^{v\,(0)} + k_2 e^{-v\,(0)}$$

$$k_1 = -k_2$$

Again, applying the boundary condition $B(\pi,v) = 0$, in equation (24), we get

$$0 = k_2\left(e^{-v\,(\pi)} - e^{v\,(\pi)}\right)$$

But $\left(e^{-v\,(\pi)} - e^{v\,(\pi)}\right) \neq 0$, therefore, $k_2 = 0$.

Therefore, the solution of equation (23) can be written as

$$B(x,v) = M\cos x + N\sin x \qquad (25)$$

By using the technique of undetermined coefficients to order determine the constants M and N .

$$\frac{\partial}{\partial x}(B(x,v)) = -M\sin x + N\cos x \qquad (26)$$

$$\frac{\partial^2}{\partial x^2}(B(x,v)) = -M\cos x - N\sin x \qquad (27)$$

Substituting equations (25) and (27) in equation (23), we get

$$-M\cos x - N\sin x - v^2(M\cos x + Q\sin x) = -\frac{\sin x}{v^2}$$

$$(1+v^2)M\cos x + (1+v^2)N\sin x = \frac{\sin x}{v^2} \qquad (28)$$

Equating like term to both the sides, we get

$$(1+v^2)M = 0; \qquad \text{and} \qquad (1+v^2)N = \frac{1}{v^2};.$$

$$M = 0.\ \text{and} \qquad N = \frac{1}{v^2(1+v^2)}$$

Hence, the solution of equation (23) becomes

$$B(x,v) = \frac{1}{v^2(1+v^2)}\sin x \qquad (29)$$

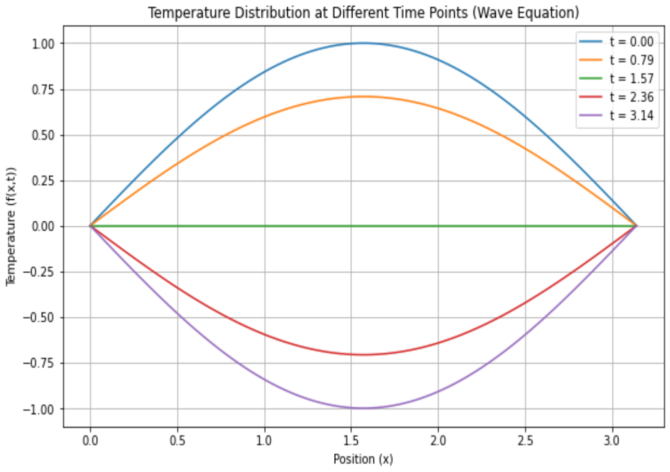
Relating inverse Iman transform of equation (29) both the sides, we get

$$I^{-1}(B(x,v)) = \sin x\, I^{-1}\left(\frac{1}{v^2(1+v^2)}\right)$$

$f(x,t) = \sin x \cos t$, which is the required solution of (20).

It can be checked that this is certainly the solution of the wave equation, as $t = 0,\ f(x,0) = \sin x$

Wave Equation:



Temperature Distribution at Different Time Points (Application 4.2)

The provided graph depicts the temperature distribution across a one-dimensional spatial domain (x) at various time points, as described by the wave equation $f(x,t) = sin(x)cos(t)$.

At $t = 0$, the temperature distribution $f(x,0) = sin(x)cos(0) = sin(x))$ is sinusoidal, exhibiting peaks and troughs along the spatial domain x. As time progresses, the cosine term $(cos(t))$ modulates the amplitude of this sine wave. Consequently, the temperature distribution undergoes sinusoidal oscillations in both space and time.

At $t = \pi/2$, the cosine term reaches its maximum value of 1, leading to maximal modulation of the sine wave's amplitude. This modulation significantly alters the temperature distribution compared to its initial state at $t = 0$, resulting in observable changes in the peaks and troughs along x.

The graph effectively illustrates the wave-like behavior of the temperature distribution over time, showcasing how it propagates and evolves according to the wave equation. It demonstrates the sinusoidal nature of the temperature variations and emphasizes their dependence on both spatial position and time.

Overall, the graph provides a clear visual representation of the dynamic nature of wave propagation and how it influences the temperature distribution within the medium.

V. DISCUSSION AND CONCLUSION

This chapter highlights the Iman transform technique as a dependable method for addressing well-known differential equations associated with heat and wave phenomena. By employing the Iman transform technique, precise solutions to these equations are obtained, showcasing the method's reliability. The Iman transform streamlines essential calculations involved in solving these differential equations, leading to accurate and efficient results.

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Data Availability Statement: Not applicable.

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