A Tau Method for Solving Second-Order Partial Integro-Differential Equations with Weakly Singular Kernels

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Abstract—*This paper introduces a numerical method to approximate the solutions of initial-boundary value problems for a specific class of partial integro-differential equations. The approach utilizes Volterra Fredholm Integro Differential with Weakly singular kernel for spatial derivatives and the backward Tau method for temporal derivatives. The study delves into detailed Converting BVP to Volterra Fredholm Integral Differential Equation of the Second Kind with tau methode and using the Legendre or Chebyshev polynomials, proving their convergence and stability. The proposed method is then applied to several test cases, with the numerical outcomes compared with The Finite Element Method . The results demonstrate the computational efficiency of the method, leading to the conclusion that it is effective for solving initial-boundary value problems.*

Keywords— Volterra-Fredholm Integro Differential Equation ,Tau Method,Weakly singular Kernel,Finite element method Method.

I. INTRODUCTION

Integro-differential equations arise naturally in the modeling of various physical, biological, and engineering phenomena where both local and non-local effects play a crucial role [11, 10]. Among these, *Volterra-Fredholm integro-differential equations (VFIDEs)* are particularly important due to their ability to describe processes that depend on both past history (via Volterra-type integral terms) and global spatial behavior (via Fredholm-type integrals) [6, 7].

These equations have wide applications in fields such as heat conduction in materials with memory, viscoelasticity, population dynamics, chemical kinetics, and control theory [12, 18]. In particular, *partial Volterra–Fredholm integro-differential equations with weakly singular kernels* appear in models where the system's response exhibits hereditary properties with a singularity in the kernel function. A typical example of such an equation is:

$$\beta\Psi_t(x,t) = \gamma\Psi_{xx}(x,t) + \int_0^t V(t-s)\Psi_x(x,s)ds + \nu(x,t), \quad x \in [a,b], \quad t \geq 0 \qquad (1.1)$$

Where β and γ constant with the initial condition

$$\Psi(x,0) = \vartheta(x)$$

Boundary condition $\Psi(a,t) = g_0(t)$, $\Psi(b,t) = g_1(t)$ and $V(t-s) = (t-s)^{-\alpha}$ is the kernel $0 < \alpha < 1$,subject to appropriate initial and boundary conditions. The presence of the weakly singular kernel $(t-s)^{-\alpha}$ introduces significant challenges in both theoretical analysis and numerical approximation [13, 15]. Specifically, the solution may exhibit limited regularity near $t = 0$, which can degrade the accuracy and convergence rates of standard numerical methods if not handled carefully [14].

Over the years, numerous analytical and numerical techniques have been proposed to solve such problems. However, due to the complexity introduced by the integral term and the singularity in the kernel, exact solutions are rarely available, and therefore, efficient and accurate numerical methods are essential [16, 17].

Among the most used numerical approaches are finite difference methods, finite element methods, spectral methods, and collocation techniques based on polynomial or collocation approximations [19, 22]. In this context, Tau collocation methods have gained popularity due to their smoothness, local support, and ease of implementation [9]. These methods provide high accuracy while maintaining computational efficiency, especially when combined with time-stepping schemes such as the backward Euler method [8].

Additionally, the Tau method, a spectral approximation technique, has proven effective in handling weakly singular kernels without requiring special transformations or mesh refinements [21, 20]. This method allows for the direct computation of the integral terms, preserving the structure of the equation while achieving high-order accuracy.

This paper presents a robust and efficient numerical scheme for solving second-order Volterra–Fredholm integro-differential equations with weakly singular kernels. The spatial derivatives are approximated using the collocation method, while the temporal derivative is discretized via the backward Tau method. The Tau method is employed to compute the integral term involving the singular kernel accurately. The stability and convergence of the proposed method are rigorously analyzed, and several test cases are considered to demonstrate its effectiveness and reliability.

Definition 1.1 (Volttera and Fredholm Integral Equations) *[34]*

VIEs and FIEs arise from parabolic boundary value problems, which are mathematical models of spatio-temporal epidemic progression, as well as diverse physical and biological models [33, 34]. In literature, they manifest in two distinct forms.

$$u(x) = f(x) + \lambda_1 \int_a^x K_1(x,t)u(t)\,dt + \lambda_2 \int_a^b K_2(x,t)u(t)\,dt,$$

And $u(x,t) = f(x,t) + \lambda \int_0^t \int_\Omega F\big(x,t,\varsigma,\tau,u(\varsigma,\tau)\big)d\varsigma\,d\tau,$

$(x,t) \in \Omega \times [0,T],$ (1.3)

where $f(x,t)$ and $F(x,t,\varsigma,\tau,u(\varsigma,\tau))$ They are defined as analytic functions on $D = \Omega \times [0,T]$; $\mathbb{R}^n$, where $n = 1,2,3$, is the closed subset of Ω.

Equation 1.2 consists of distinct VIEs and FIEs, whereas Equation 2.1 deals with a combination of VIEs and FIEs. In both scenarios, the unidentified functions $u(x)$ and $u(x,t)$ are present both within and outside the integral signs, a discriminatory of a second type IE. If the unidentified functions are present only within the integral signs, the resulting equations are of the first type; however, such cases will not be discussed in this text.

Definition 1.2 (Volttera -Fredholme Integro Differential Equations) *[34]*

VFIDEs arise similarly to VFIEs but include one or more ODs in addition to integral operators. These equations are found in the literature in two forms:

$u^{(n)}(x) =$

$f(x) + \lambda_1 \int_a^x K_1(x,t)u(t)\,dt + \lambda_2 \int_a^b K_2(x,t)u(t)\,dt,$ (1.4)

and

$$u^{(n)}(x,t) = f(x,t) + \lambda \int_0^t \int_\Omega F\big(x,t,\varsigma,\tau,u(\varsigma,\tau)\big)d\varsigma\,d\tau,$$

$(x,t) \in \Omega \times [0,T],$ (1.5)

where $f(x,t)$ and $F(x,t,\varsigma,\tau,u(\varsigma,\tau))$ are analytic funs Ω is a closed subset of $\mathbb{R}^n$, with $n = 1,2,3$, and is defined on $D = \Omega \times [0,T]$. Equation 2.2 includes separate VIEs and FIEs, whereas 2.3 involves mixed integrals. Lower-order derivatives may also appear on the left-hand side. Furthermore, The unidentified functions $u(x)$ and $u(x,t)$ present in both the inner and outer integral signs, indicating a 2^{nd}-type IE. The equations are considered first type if these unidentified funs are solely present inside the integral signs. starting points play a critical role in determining specific solutions.

Definition 1.3 (Grönwall's Inequality)

Grönwall's inequality is a fundamental result in differential and integral equations, providing bounds on functions that satisfy certain integral inequalities. It is widely used in stability analysis, control theory, and the study of differential equations [35, 36].

Definition 1.4 (Classical Grönwall Inequality (Differential Form))

Let $u(t)$ and $\alpha(t)$ be continuous and nonnegative real-valued functions on $[a,b]$. Suppose $u(t)$ satisfies the differential inequality:

$$u'(t) \le \beta(t)u(t) + \alpha(t), \quad \forall t \in [a,b],$$

where $\beta(t)$ is continuous. Then,

$$u(t) \le u(a)e^{\int_a^t \beta(s)\,ds} + \int_a^t \alpha(s)e^{\int_s^t \beta(\tau)\,d\tau}\,ds, \quad \forall t \in [a,b]. \tag{1.2}$$

Definition 1.5 (Integral Form (Grönwall-Bellman Inequality))

Let $u(t)$, $\alpha(t)$, and $\beta(t)$ be nonnegative continuous functions on $[a,b]$. If

$$u(t) \le \alpha(t) + \int_a^t \beta(s)u(s)\,ds, \quad \forall t \in [a,b],$$

then

$$u(t) \le \alpha(t) + \int_a^t \alpha(s)\beta(s)e^{\int_s^t \beta(\tau)\,d\tau}\,ds.$$

This form is particularly useful in stability analysis [38].

Definition 1.6 (Discrete Grönwall Inequality)

Let $\{u_n\}$, $\{\alpha_n\}$, and $\{\beta_n\}$ be nonnegative sequences. If

$$u_{n+1} \le \alpha_n + \beta_n u_n, \quad \forall n \ge 0,$$

then

$$u_n \le u_0 \prod_{k=0}^{n-1} \beta_k + \sum_{k=0}^{n-1} \alpha_k \prod_{j=k+1}^{n-1} \beta_j.$$

The remainder of this paper is organized as follows: Subsection a describes the numerical method in detail, Section 2 analyzes stability and convergence, Section 3 presents numerical experiments, and Section 4 concludes the work with final remarks.

II. Numerical Investigations

We rewrite equation 1.1 as

$$\Psi_{xx}(x,t) = \frac{\beta}{\gamma}\Psi_t(x,t) - \frac{1}{\gamma}\int_0^t V(t-s)\Psi_x(x,s)ds - \frac{1}{\gamma}\nu(x,t), \quad x \in [a,b], \quad t \ge 0 \tag{2.1}$$

Now, let $\Psi_{xx}(x,t) = \Lambda(x,t)$

$$\Psi_x(x,t) = \Psi_x(a,t) + \int_a^x \Lambda(\varsigma,t)d\varsigma \tag{2.2}$$

$$\Psi(x,t) = \Psi(a,t) + (x-a)\Psi_x(a,t) + \int_a^x \int_a^\varsigma \Lambda(\varsigma_1,t)d\varsigma_1 d\varsigma$$

$$\Psi(x,t) = \Psi(a,t) + (x-a)\Psi_x(a,t) + \int_a^x (x-\varsigma)\Lambda(\varsigma,t)d\varsigma \tag{2.3}$$

By substitution $x = b$ and using the boundary condition

$\Psi(b,t) = \Psi(x,t) + (b-a)\Psi_x(a,t) + \int_a^b (b-\varsigma)\Lambda(\varsigma,t)d\varsigma$

$\Psi_x(a,t) = \frac{1}{(b-a)}[g_1(t) - g_0(t)] - \frac{1}{(b-a)}\int_a^b (b-\varsigma)\Lambda(\varsigma,t)d\varsigma$ (2.4)

By substituting 2.4 in equations 2.2 and 2.3, we obtain

$\Psi_x(x,t) = \frac{1}{(b-a)}[g_1(t) - g_0(t)] - \frac{1}{(b-a)}\int_a^b (b-\varsigma)\Lambda(\varsigma,t)d\varsigma + \int_a^x \Lambda(\varsigma,t)d\varsigma$ (2.5)

$\Psi(x,t) = g_0(t) + \frac{(x-a)}{(b-a)}[g_1(t) - g_0(t)] - \frac{(x-a)}{(b-a)}\int_a^b (b-\varsigma)\Lambda(\varsigma,t)d\varsigma + \int_a^x (x-\varsigma)\Lambda(\varsigma,t)d\varsigma$ (2.6)

Now, derivative equation 2.6 we obtain

$\Psi_t(x,t) = g'_0(t) + \frac{(x-a)}{(b-a)}[g'_1(t) - g'_0(t)] - \frac{(x-a)}{(b-a)}\int_a^b (b-\varsigma)\Lambda_t(\varsigma,t)d\varsigma + \int_a^x (x-\varsigma)\Lambda_t(\varsigma,t)d\varsigma$ (2.7)

will the equation 1.6 with respect to s its to be as

$\Psi_x(x,s) = \frac{1}{(b-a)}[g_1(s) - g_0(s)] - \frac{1}{(b-a)}\int_a^b (b-\varsigma)\Lambda(\varsigma,s)d\varsigma + \int_a^x \Lambda(\varsigma,s)d\varsigma$ (2.8)

We substitute equations 2.7 and 2.8 in equation 2.1, we obtain

$$\Lambda(x,t) = \frac{\beta}{\gamma}g'_0(t) + \frac{\beta(x-a)}{\gamma(b-a)}[g'_1(t) - g'_0(t) - \frac{\beta(x-a)}{\gamma(b-a)}\int_a^b (b-\varsigma)\Lambda_t(\varsigma,t)d\varsigma + \frac{\beta}{\gamma}\int_a^x (x-\varsigma)\Lambda_t(\varsigma,t)d\varsigma - \frac{1}{\gamma}\int_0^t V(t-s)\left\{\frac{1}{(b-a)}[g_1(s) - g_0(s)] - \frac{1}{(b-a)}\int_a^b (b-\varsigma)\Lambda(\varsigma,s)d\varsigma + \int_a^x \Lambda(\varsigma,s)d\varsigma\right\}ds - \frac{1}{\gamma}v(x,t)$$

$\Lambda(x,t) = \frac{\beta}{\gamma}g'_0(t) + \frac{\beta(x-a)}{\gamma(b-a)}[g'_1(t) - g'_0(t)] - \frac{\beta(x-a)}{\gamma(b-a)}\int_a^b (b-\varsigma)\Lambda_t(\varsigma,t)d\varsigma + \frac{\beta}{\gamma}\int_a^x (x-\varsigma)\Lambda_t(\varsigma,t)d\varsigma - \frac{1}{\gamma(b-a)}\int_0^t V(t-s)[g_1(s) - g_0(s)]ds + \frac{1}{\gamma(b-a)}\int_0^t \int_a^b V(t-s)(b-\varsigma)\Lambda(\varsigma,s)d\varsigma\, ds - \frac{1}{\gamma}\int_0^t \int_a^x V(t-s)\Lambda(\varsigma,s)d\varsigma]ds - \frac{1}{\gamma}v(x,t)$ (2.9)

Then equation 2.9

$\Lambda(x,t) = \hat{v}(x,t) + \int_a^b K_1(x,\varsigma)\Lambda_t(\varsigma,t)d\varsigma + \int_a^x K_2(x,\varsigma)\Lambda_t(\varsigma,t)d\varsigma + \int_0^t \int_a^b V(t-s)K_3(x,\varsigma)\Lambda(\varsigma,s)d\varsigma\, ds + \int_0^t \int_a^x V(t-s)K_4(x,\varsigma)\Lambda(\varsigma,s)d\varsigma]ds$

were

$$\hat{v}(x,t) = -\frac{1}{\gamma}f(x,t) + \frac{\beta}{\gamma}g'_0(t) + \frac{\beta(x-a)}{\gamma(b-a)}[g'_1(t) - g'_0(t)] - \frac{1}{\gamma(b-a)}\int_0^t V(t-s)[g_1(s) - g_0(s)]ds$$

$$K_1(x,t) = -\frac{\beta(x-a)}{\gamma(b-a)}(b-\varsigma), K_2(x,t) = \frac{\beta}{\gamma}(x-\varsigma), K_3(x,t) = \frac{1}{\gamma(b-a)}(b-\varsigma), K_4(x,t) = -\frac{1}{\gamma}$$

We now examine the approximate solution. $\Lambda_N(x,t) \in \Omega_N$ of equation 2.9 as

$\Lambda_N(x,t) = \sum_{k=0}^{N} \theta_k(t)\Theta_k(x)$ (2.10)

If the orthogonal polynomial $\Theta_k(x)$ is extracted from space

$$\Omega_N = span\left\{\Theta_i(x) | \langle\Theta_i(x), \Theta_j(x)\rangle_w = \int_a^b \Theta_i(x)\Theta_j(x)w(x)dx = 0, \text{if } i \neq j\right\}$$

Now, we require the residual

$R_N(x,t) = \sum_{k=0}^{N} \theta_k(t)\Theta_k(x) - \hat{v}(x,t) - \sum_{k=0}^{N} \theta'_k(t)\left(\int_a^b K_1(x,t)\Theta_k(t)dt + \int_a^x K_2(x,t)\Theta_k(t)dt\right) - \sum_{k=0}^{N}\left(\int_0^t H(t-s)\theta_k(s)\,ds\right)\left(\int_a^b K_3(x,t)\Theta_k(t)dt + \int_a^x K_4(x,t)\Theta_k(t)dt\right)$ (2.11)

is orthogonal to Ω_N .This procedure yields

$\theta_l(t)\langle\Theta_l(x), \Theta_l(x)\rangle_w = \langle\hat{v}(x,t), \Theta_l(x)\rangle_w + \sum_{k=0}^{N} \theta'_k(t)\Phi_{kl} + \sum_{k=0}^{N} \Psi_{kl}\left(\int_0^t V(t-s)\theta_k(s)\,ds\right)$ (2.12)

were

$$\Phi_{kl} = \left\langle\int_a^b K_1(x,t)\Theta_k(t)dt + \int_a^x K_2(x,t)\Theta_k(t)dt, \Theta_l(x)\right\rangle_w$$

$$\Psi_{kl} = \left\langle\int_a^b K_3(x,t)\Theta_k(t)dt + \int_a^x K_4(x,t)\Theta_k(t)dt, \Theta_l(x)\right\rangle_w$$

Define

$\chi = diag(\langle\Theta_0, \Theta_0\rangle_w, \langle\Theta_1, \Theta_1\rangle_w, \cdots, \langle\Theta_N, \Theta_N\rangle_w)$,

$\Upsilon = \{\Phi_{kl}\}_{k,l=0}^{N}$, $\Gamma = \{\Psi_{kl}\}_{k,l=0}^{N}$,

$F = (\langle\hat{v}(x,t), \Theta_0\rangle_w, \langle\hat{v}(x,t), \Theta_1\rangle_w, \cdots, \langle\hat{v}(x,t), \Theta_N\rangle_w)$,

Let $\hat{\theta} = (\theta_0, \theta_1, \cdots, \theta_N)^T$. For this result, we get the Volterra-Fredholm integro-differential equations with weakly singular lernels as follows:

$\Psi^T\bar{\theta}'(t) = \chi\bar{\theta}(t) - F - \Gamma^T\int_0^t V(t-s)\bar{\theta}(s)\,ds$ (2.13)

with the initial conditions $\bar{\theta}(0) = \frac{\langle\vartheta''(x), \Theta_l(x)\rangle_w}{\langle\Theta_l(x), \Theta_l(x)\rangle_w}$

We have now considered the piecewise polynomial collation tau technique within the interval $[0,\hat{I}]$ for the issue 2.13's numerical solution. For a given real number $r \leq 1$ and integer $M_1 \leq 2$, we define the mesh.

$$I_h^r = \left\{t_n = \left(\frac{n}{M_1}\right)^r \hat{I}, n = 0, \cdots, M_1\right\}$$

A graded mesh on the interval $[0,\hat{T}]$ with a grading exponent $r > 1$ is denoted by I_h^r, indicating that the mesh is refined based on the exponent r. When $r = 1$, the mesh is referred to as uniform, meaning that all subintervals have equal length. Furthermore, let ϖ be computed as described below, and define the step size as $h_n = t_{n+1} - t_n$.

$$\varpi = \{t_{nj} = t_n + q_{jh_n}:\ 0 \leq q_1 < q_2 <\ldots< q_m \leq 1,\quad 0 \leq n \leq M_1 - 1\}.$$

Collocation solution: The function $\bar{\theta}_h \in S_m^0(I_h^r)$ represents the approximate solution, where S_m^0 denotes the space of piecewise polynomials of degree $m \geq 0$ defined over the graded mesh I_h^r. This solution satisfies equation (1.14) in accordance with the collocation method.

$$\Psi^T \bar{\theta}'_h(t) = \chi\bar{\theta}_h(t) - F - \Gamma^T \int_0^t V(t-s)\bar{\theta}_h(s)\, ds, t \in \varpi, \bar{\theta}_h(0) = \bar{\theta}(0) = \bar{\theta}_0 \qquad (2.14)$$

For $\varepsilon \in [0,1]$, the following equations apply since $\bar{\theta}'_h|_{t_n,t_{n+1}} \in \Pi_{m-1}$ (where $\Pi_m < m$ real-coefficient occupy)

$$\bar{\theta}'_h(t_n + \varepsilon h_n) = \sum_{j=1}^m L_j(\varepsilon)\bar{\vartheta}_{nj},\quad \bar{\vartheta}_{nj} = \bar{\theta}'_h(t_n + q_{jh_n}), \qquad (2.15)$$

where $L_j(\varepsilon) = \prod_{k\neq j}^m \frac{(\varepsilon - c_k)}{(c_j - c_k)},\quad j = 1, \ldots, m.$ (2.16)

Suppose that $\bar{\theta}_n = \bar{\theta}_{h(t_n)}$ and $\alpha_j(\varepsilon) = \int_0^\varepsilon L_j(s)ds,\ j = 1, \ldots, m$, then

$$\bar{\theta}_h(t_n + \varepsilon h_n) = \bar{\theta}_n + h_n \sum_{j=1}^m \alpha_j(\varepsilon)\bar{\vartheta}_{nj},\quad \varepsilon \in [0,1]. \qquad (2.17)$$

When (2.15) and (2.17) are substituted into the collocation equation (2.14), the result is

$$\Psi^T \bar{\vartheta}_{ni} = \chi\bar{\theta}_h(t)\left(\bar{\theta}_n + h_n \sum_{j=1}^m \alpha_j(c_i)\bar{\vartheta}_{nj}\right) - F(t_{ni})$$
$$-\Gamma^T \left\{ \begin{array}{c} \sum_{l=0}^{n-1} h_l \left(\int_0^1 V\big(t_{ni} - (t_l + sh_l)\big)ds\right)\bar{\theta}_l + \\ \sum_{l=0}^{n-1} h_l^2 \sum_{j=1}^m \left(\int_0^1 V(t_{ni} - (t_l + sh_l))\alpha_j(s)ds\right)\bar{\vartheta}_{lj} \end{array} \right\}$$
$$-\Gamma^T \left\{ \begin{array}{c} h_n\left(\int_0^{c_i} V\big(t_{ni} - (t_n + sh_n)\big)ds\right)\bar{\theta}_n \\ +h_n^2 \sum_{j=1}^m \left(\int_0^{c_i} V(t_{ni} - (t_n + sh_n))\alpha_j(s)ds\right)\bar{\vartheta}_{nj} \end{array} \right\}, \qquad (2.18)$$

$$n = 0, \ldots, M_1 - 1,\ i = 1, \ldots, m.$$

Therefore, to obtain the numerical solution of the differential equation 2.14 for a $\varepsilon \in [0,1]$, we substitute $\bar{\vartheta}_{nj}$ .The solution of 2.18 into equation 2.17.

III. Convergence Analysis

The following lemmas are required to demonstrate the error estimate: Gronvall's disparity appears to be as follows at first glance:

Theorem 3.1 *[2].*

Suppose that $t \geqslant r \geqslant x \geqslant a$ and θ_1, θ_2, and θ_3 be nonnegative constants that are not all zero, and let $\Psi(t), \Phi(t), Q(t,r)$, and $V(t,r,x)$be funs. that are not negative. If$u(t) \geqslant \theta_1 + \theta_2 \int_a^t [\Phi(s)\Upsilon(s)ds + \int_a^s Q(s,r)\Upsilon(r)dr]ds + \theta_3 \int_a^t \int_a^s \int_a^r V(s,r,x)\Upsilon(x)dxdrds,$

Then, for

$\Upsilon(t) \leq a\Upsilon(t) \leq \theta_1 exp\{\theta_2 \int_a^s [\Phi(s)ds + \int_a^s Q(s,r)dr]ds + \theta_3 \int_a^t \int_a^s \int_a^r V(s,r,x)dxdrds\}.$

By taking into consideration that $V(s,r,x) = 0$, the following result can be immediately deduced based on Theorem 3.1.

Lemma 3.2 *[3]*

Assume that the nonnegative function $\Upsilon(t)$ satisfies the inequality:

$$\Upsilon(t) \leq \theta + \int_{t_0}^t f(t,s)\Upsilon(s)ds + \int_{t_0}^t \int_{t_0}^s \hat{f}(t,\sigma)(s - \sigma)^{-\alpha}\Upsilon(\sigma)d\sigma ds, \qquad (3.1)$$

$t \geqslant s \geqslant \sigma \geq t_0 > 0$ and $c > 0$, where the funs. of $f(t,s)$ and $\hat{f}(t,\sigma)$ are nonnegative, then

$$\Upsilon(t) \leq \theta\, exp\left\{\int_{t_0}^t f(s,s)ds + \int_{t_0}^t \int_{t_0}^\tau \frac{\partial f(\tau,s)}{\partial\tau} ds d\tau + \int_{t_0}^t \int_{t_0}^\tau \hat{f}(\tau,\sigma)(\tau - \sigma)^{-\alpha} d\sigma d\tau\right\}$$
$$exp\left\{\int_{t_0}^t \int_{t_0}^\tau \int_{t_0}^s \frac{\partial \hat{f}(\tau,\sigma)}{\partial\tau}(s - \sigma)^{-\alpha} d\sigma ds d\tau\right\}.$$

Lemma 3.3 *[1]*

The collection of all functions $\Psi(x)(x = (x_1 \cdots, x_p)$ is known as the Sobolev space $W_{w(\Omega)}^m$. Let $\mathbb{P}_M(\Lambda)$ Let this be the space of all polynomials defined on $\Omega = (0,1)^p$ with degree at most N, where $p = 1,2$. The funs $\Psi(\mathbf{x})$ and their weak derivatives up to order m belong to the weighted space $L_{w(\Omega)}^2$. The orthogonal projection operator from $L_{w(\Omega)}^2$ onto $\mathbb{P}_M(\Omega)$ is denoted by P_M. The following estimate applies for all $\Psi \in W_{w(\Omega)}^m, m \geq 1$,

$$||\Psi - P_M\Psi\,||_{L_{w(\Omega)}^2} \leq \Theta\, M^{-m}|\Psi|_{W_{w(\Omega)}^{m,M}}, \qquad (3.2)$$

where the definition of the semi-norm $|.|$ is

$$|\Psi|_{W_{w(\Omega)}^{m,M}} = \left(\sum_{j=min(m,M+1)}^m \sum_{i=1}^p ||D_i^j\Psi||_{L_{w(\Omega)}^2}^2\right)^{1/2},$$

such that $\alpha = (\alpha_1, \cdots, \alpha_p)$ is a nonnegative multi-index with $D^\alpha\Psi = \frac{\partial^{\alpha_1+\cdots+\alpha_p}\Psi}{\partial x_1^{\alpha_1}\cdots\partial x_p^{\alpha_p}}$. The following estimate carries over

(3.2) to higher-order Sobolev norms when the derivatives' truncation error is substantial:

$$\| \Psi - P_M \Psi \|_{W_w^r(\Lambda)} \leq C\, M^{2r-\frac{1}{2}-m} |\Psi|_{W_w^{m,M}}(\Lambda), \quad (3.3)$$

for every r rsatisfying $1 \leq r \leq m$.

The numerical Tau technique's error estimate, which has the L_w^2 weighted norm and is described in Section 1, is investigated in the following. As you can see, The approximate solution is taken into consideration. $\Psi_N(x,t) = \sum_{k=0}^{M} \theta_k(t)\Theta_k(x)$ of eq. 1.9. So that we construct a system of Volterra integro-differential equations with weakly singular kernels is obtained to determine $\bar{\theta} = (\theta_0, \cdots, \theta_M)T$. The piecewise polynomial collocation method is used to approximate the solution of this. These two approximation techniques are entirely different, the first about x and secondly, concerning t. Then, we are unable to conduct a convergence analysis at the same time. The impact of approximation concerning t is not taken into consideration in this position, where we get $\| \Psi - \Psi_N \|_{L^2_{w(0,1)}} \to 0$ in relation to x. Stated otherwise, in this case, t It is taken to be a constant value.

Theorem 3.4

Let $\Psi(x,t)$ be a smooth function that solves the PDE (1.1) exactly, and let $\Psi_N(x,t)$ denote the numerical approximation of $\Psi(x,t)$ obtained via the Tau method, as described. Then, $\forall \epsilon > 0,\ \exists N_0 \in \mathbb{N}$ such that $N \geq N_0, \| \Psi - \Psi_N \|_{L_w^2(0,1)} < \epsilon$. Alternatively, in limit notation: $\lim_{N\to\infty} \| \Psi - \Psi_N \|_{L_w^2(0,1)} = 0$. Consequently, whenever N If it is sufficiently large, the following holds $\| \Psi - \Psi_N \|_{L^2_{w(0,1)}} \to 0$.

Proof.

We assume that $b = 1$ and $a = 0$. In the following, Eq. (2.1) can be taken into consideration.

$$\begin{cases} \Psi_{xx}(x,t) = \frac{1}{\gamma}\Psi_t(x,t) + \frac{\lambda}{\gamma}\Psi_x(x,t) - \frac{1}{\gamma}\int_0^t H(t-s)V(x,s)ds - \frac{1}{\gamma}\nu(x,t), \\ \Psi(x,0) = \vartheta(x,0), \\ V(t-s) = (t-s)^{-\alpha},\ 0 < \alpha < 1, \\ \Psi(0,t) = g_0(t),\ \Psi(1,t) = g_1(t),\ t \geq 0. \end{cases}$$
(3.4)

Using the same process as in Section 1, we then obtain

$$\Psi(x,t) = g_0(t) + x[g_1(t) - g_0(t)] - x\int_0^1 (1-\varsigma) \quad \Lambda(\varsigma,t)d\varsigma + \int_0^x (x-\varsigma)\Lambda(\varsigma,t)d\varsigma \quad (3.5)$$

and

$$\Lambda(x,t) = \hat{v}(x,t) + \int_0^1 K_1(x,\varsigma)\Lambda_t(\varsigma,t)d\varsigma + \int_0^x K_2(x,\varsigma)\Lambda_t(\varsigma,t)d\varsigma + \int_0^t \int_0^1 V(t-s)K_3(x,\varsigma)\Lambda(\varsigma,s)d\varsigma\, ds + \int_0^t \int_0^x V(t-s)K_4(x,\varsigma)\Lambda(\varsigma,s)d\varsigma ds \quad (3.6)$$

Also,

$$\Psi_M(x,t) = g_0(t) + x[g_1(t) - g_0(t)] - x\int_0^1 (1-\varsigma)\Lambda_M(\varsigma,t)d\varsigma + \int_0^x (x-\varsigma)\Lambda_M(\varsigma,t)d\varsigma \quad (3.7)$$

and

$$\Lambda_M(x,t) = \hat{v}(x,t) + \int_0^1 K_{1,M}(x,\varsigma)\Lambda_{t,M}(\varsigma,t)d\varsigma + \int_0^x K_{2,M}(x,\varsigma)\Lambda_{t,M}(\varsigma,t)d\varsigma + \int_0^t \int_0^1 V(t-s)K_{3,M}(x,\varsigma)\Lambda(\varsigma,s)d\varsigma\, ds + \int_0^t \int_0^x V(t-s)K_{4,M}(x,\varsigma)\Lambda(\varsigma,s)d\varsigma ds \quad (3.8)$$

When (2.6) is subtracted from (2.8), the result is

$$e(x,t) = \int_0^1 \int_0^t V(t-s)K_3(x,\varsigma)e(\varsigma,s)dsd\varsigma + \int_0^x \int_0^t V(t-s)K_4(x,\varsigma)e(\varsigma,s)dsd\varsigma + e_1 + e_2 + e_3 + e_4 + e_5, \quad (3.9)$$

So $e(x,t) = \Lambda(x,t) - \Lambda_M(x,t)$

$e_1 = \hat{v}(x,t) - \hat{v}_M(x,t)$,

$e_2 = \int_0^1 K_{1,M}(x,\varsigma)[\Lambda_t(\varsigma,t) - \Lambda_{t,M}(\varsigma,t)]d\varsigma + \int_0^1 [K_1(x,\varsigma) - K_{1,M}(x,\varsigma)]\Lambda_t(\varsigma,t)d\varsigma$,

$e_3 = \int_0^x K_{2,M}(x,\varsigma)[\Lambda_t(\varsigma,t) - \Lambda_{t,M}(\varsigma,t)]d\varsigma + \int_0^x [K_2(x,\varsigma) - K_{2,M}(x,\varsigma)]\Lambda_t(\varsigma,t)d\varsigma$.

$e_4 = \int_0^1 \int_0^t [K_3(x,\varsigma) - K_{3,M}(x,\varsigma)]V(t-s)\Lambda_M(\varsigma,s)dsd\varsigma$,

$e_5 = \int_0^x \int_0^t [K_4(x,\varsigma) - K_{4,M}(x,\varsigma)]V(t-s)\Lambda_M(\varsigma,s)dsd\varsigma$,

Applying the generalized Hardy's inequality from[33] and Gronvall's inequality from Theorem 2.1, as well as Lemma 3.2 by taking into consideration $V(t-s) = (t-s)^{-\alpha}$ for (3.9), yields

$$\| e \|_{L^2_{w(0,1)}} \leq C \| e_1 + e_2 + e_3 + e_4 + e_5 \|_{L^2_{w(0,1)}}. \quad (3.10)$$

Using the inequality (3.3) from Lemma 3.3,

$$\| e_1 \|_{L^2_{w(0,1)}} \leq CM^{-m} |\hat{v}|_{W^{m,M}_{w(0,1)}}. \quad (3.11)$$

The inequality (3.3) and generalised Hardy's inequality imply that

$$\| e_2 \|_{L^2_{w(0,1)}} \leq C[\| K_{1,M} \|_{L^2_{w(0,1)}} \| \Lambda_t - \Lambda_{t,M} \|_{L^2_{w(0,1)}} + \| K_1 - K_{1,M} \|_{L^2_{w(0,1)}} \| \Lambda_t \|_{L^2_{w(0,1)}}],$$

By usin the inequality 3.3 for $r = 1$,

$$\| \Lambda_t - \Lambda_{t,M} \|_{L^2_{w(0,1)}} \leq \| \Lambda - \Lambda_M \|_{W^1_{w(0,1)}} \leq CM^{\frac{3}{2}-m} |\Lambda|_{W^{m,M}_{w(0,1)}},$$

then $\| e_2 \|_{L^2_{w(0,1)}} \leq C[M^{\frac{3}{2}-m} \| K_{1,M} \|_{L^2_{w(0,1)}} |\Lambda|_{W^{m,M}_{w(0,1)}} - M^{-m} |K_1|_{W^{m,M}_{w(0,1)}} \| \Lambda_t \|_{L^2_{w(0,1)}}]$,

and in the same method, we obtain

$$\| e_3 \|_{L^2_{w(0,1)}} \leq C[M^{\frac{3}{2}-m} \| K_{2,M} \|_{L^2_{w(0,1)}} |\Lambda|_{W^{m,M}_{w(0,1)}} - M^{-m}|K_2|_{W^{m,M}_{w(0,1)}} \| \Lambda_t \|_{L^2_{w(0,1)}}],$$

$$\| e_4 \|_{L^2_{w(0,1)}} \leq C \| K_3 - K_{3,M} \|_{L^2_{w(0,1)}} \| \Lambda_N \|_{L^2_{w(0,1)}} \leq CM^{-m}|K_3|_{W^{m,M}_{w(0,1)}} [\| \Lambda \|_{L^2_{w(0,1)}} + \| e \|_{L^2_{w(0,1)}}]$$

$$\| e_5 \|_{L^2_{w(0,1)}} \leq C \| K_4 - K_{4,M} \|_{L^2_{w(0,1)}} \| \Lambda_N \|_{L^2_{w(0,1)}} \leq CM^{-m}|K_4|_{W^{m,M}_{w(0,1)}} [\| \Lambda \|_{L^2_{w(0,1)}} + \| e \|_{L^2_{w(0,1)}}]$$

Considering 3.10, we now have $\Lambda_M \to \Lambda$ for $M \to \infty$, and the intended outcome is achieved.

IV. NUMERICAL EXAMPLES

In this section, we solve three problems using the numerical process that was presented in Section 2. Additionally, the PDE problems are solve by finite element method, and it can be found that the essential integral version of the PIDE is a numerically stable problem. by comparing the inferred results. Mathematica 13.2 wrote all of the codes. The orthogonal basis functions are Legendre polynomials were shifted into the interval [0,1], and the collocation parameters are $q_1 = \frac{1}{2}$ and $q_2 = 1$ on the interval [0,1]. Additionally, for $n = 0, \cdots, N1$, we take into account $N1 = 10$ with consistent meshing, $h_n = \frac{1}{N1}$, and $t = t_n + \varepsilon h$ with $\varepsilon = 1$. The highest and L^2 errors in two scenarios - the finite element method and converted integral forms with $m = 2, N_1 = 20$ are displayed in Tables 3.1–3.3. Additionally, Figures 3.1–3.3 show the plot of the error function for various values of N in both the converted integral and finite element method versions.

Example 4.1 *Let* $\psi(x,t) = tcoshx + t^2 sinhx$ *where* $\alpha = 0.5, \beta = 0.05$ *and* $\gamma = 0.04$ *with the initial condition* $\Psi(x,0) = 0$, *boundary condition* $\Psi(a,t) = t$, $\Psi(b,t) = tcosh1 + t^3 sinh1$ *and* $V(t-s) = (t-s)^{-\alpha}$ *is the kernel* $0 < \alpha < 1$

Table 1: Errors $L^2(0,1)$ and maximal in Ex. 4.1 for the converted integral form with $m = 2, N_1 = 10$.

N	Max error	$L^2(0,1)$ error
1	1.72×10^{-3}	9.63×10^{-5}
3	5.14×10^{-6}	2.88×10^{-7}
5	4.17×10^{-6}	2.64×10^{-7}
7	9.14×10^{-9}	4.66×10^{-10}
9	4.82×10^{-9}	2.71×10^{-10}

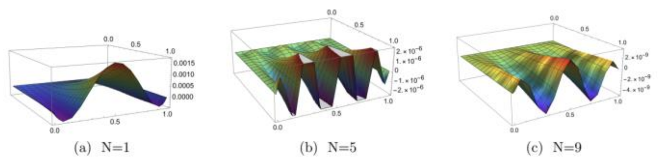

Figure 1: Plot of error function in Example 4.1 for the converted integral form.

Table 2: Max and $L^2(0,1)$ errors in Example 4.1 for the Finite Element Method form with $m = 2, N_1 = 20$.

N	Max error	$L^2(0,1)$ error
1	0.9943	0.0607
3	0.8798	0.0565
5	0.8785	0.0564
7	0.8474	0.0571
9	0.2669	1.9×10^{-2}

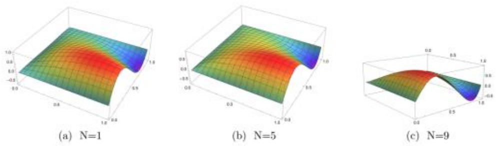

Figure 2: Plot of error function in Example 4.1 for the Finite Element Method form.

Example 4.2 *Let* $\psi(x,t) = tsin\pi xe^{-x} sinhx$ *where* $\alpha = 0.5, \beta = 0.05$ *and* $\gamma = 0.04$ *with the initial condition* $\Psi(x,0) = 0$, *boundary condition* $\Psi(a,t) = 0$, $\Psi(b,t) = 0$ *and* $V(t-s) = (t-s)^{-\alpha}$ *is the kernel* $0 < \alpha < 1$

Table 3: Errors $L^2(0,1)$ and maximal in Ex. 4.1 for the converted integral form with $m = 2, N_1 = 10$.

N	Max error	$L^2(0,1)$ error
1	3.37×10^{-2}	1.83×10^{-3}
3	4.55×10^{-4}	2.7×10^{-5}
5	1.68×10^{-5}	9.35×10^{-7}
7	1.583×10^{-7}	8.9×10^{-9}
9	4.248×10^{-9}	2.622×10^{-10}

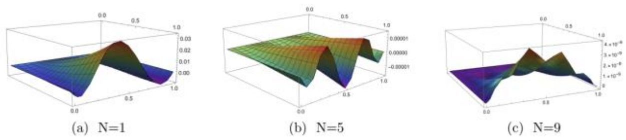

Figure 3: Plot of error function in Example 4.2 for the converted integral form.

Table 4: Max and $L^2(0,1)$ errors in Example 4.2 for the Finite Element Method form with $m = 2, N_1 = 20$.

N	Max error	$L^2(0,1)$ error
1	0.637	0.00443
3	3.3×10^{-2}	1.83×10^{-3}
5	4.55×10^{-4}	2.7×10^{-5}

7	1.68×10^{-5}	9.35×10^{-7}
9	1.58×10^{-7}	8.9×10^{-9}

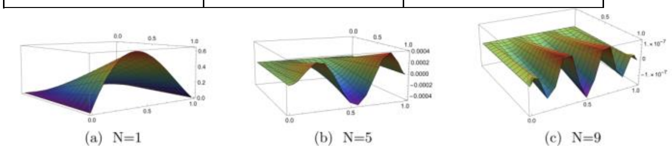

Figure 4: Plot of error function in Example 4.2 for the Finite Element Method form.

Example 4.3 *Let* $\psi(x,t) = tcos\pi x + sinh(-x)$ *where* $\alpha = 0.5, \beta = 0.05$ *and* $\gamma = 0.04$ *with the initial condition* $\Psi(x,0) = 0$, *boundary condition* $\Psi(a,t) = t,\ \Psi(b,t) = -t(1 + sinh(1))$ *and* $V(t-s) = (t-s)^{-\alpha}$ *is the kernel* $0 < \alpha < 1$

Table 5: Errors $L^2(0,1)$ and maximal in Ex. 4.3 for the converted integral form with $m = 2, N_1 = 10$.

N	Max error	$L^2(0,1)$ error
1	1.7×10^{-2}	1.12×10^{-3}
3	2.4×10^{-4}	1.46×10^{-5}
5	2.076×10^{-6}	1.18×10^{-7}
7	1.55×10^{-6}	8.5×10^{-8}
9	1.33×10^{-8}	6.98×10^{-10}

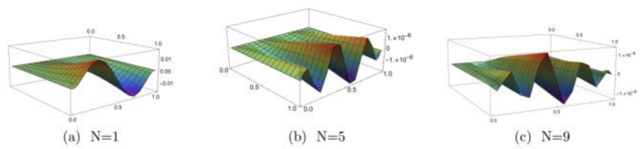

Figure 5: Plot of error function in Example 4.3 for the converted integral form.

Table 6: Max and $L^2(0,1)$ errors in Example 4.3 for the Finite Element Method form with $m = 2, N_1 = 20$.

N	Max error	$L^2(0,1)$ error
1	0.248	0.0152
3	1.7×10^{-2}	1.12×10^{-3}
5	2.43×10^{-4}	1.46×10^{-5}
7	2.08×10^{-6}	1.18×10^{-7}
9	1.66×10^{-7}	6.76×10^{-9}

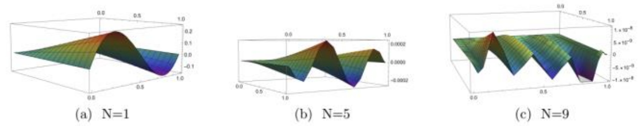

Figure 6: Plot of error function in Example 4.3 for the Finite Element Method form.

V. Remark and Conclusions

Remark 5.1 *The derivatives of the approximation are required when attempting to directly discover an approximate solution of the PIDE. For the first or second derivatives, we can use the relation 2.3 (truncation error of the derivatives) to let* $r = 1$ *or* $r = 2$ *and write*

$$\| \Psi' - (P_M\Psi)' \|_{L^2_w(\Omega)} \leq CN^{\frac{3}{2}-m}|\Psi|_{W^{m,M}_w(\Lambda)},$$

$$\| \Psi'' - (P_M\Psi)'' \|_{L^2_w(\Omega)} \leq CM^{\frac{7}{2}-m}|\Psi|_{W^{m,M}_w(\Lambda)},$$

whereas by the the relation 2.2, we have

$$\| \Psi - P_M\Psi \|_{L^2_w(\Omega)} \leq CN^{-m}|\Psi|_{W^{m,N}_w(\Omega)}.$$

Comparing these obtained relations, we observe that for truncation error of the derivatives, the order of convergence has worsened. Due to the elimination of these derivatives in VFIE form, the reported errors in this form are better than PIDE form and show a good numerical stability compared to that.

This paper investigates the numerical solution of a partial integro-differential equation (PIDE) with a weakly singular kernel. First, the equation is transformed into a Volterra-Fredholm integral equation (VFIE). The Tau method, based on orthogonal polynomials, is then applied in two different scenarios: first, to solve the VFIE numerically, and second, to address the problem directly in its original PIDE form. The convergence of the numerical solution is thoroughly analyzed. The results demonstrate that the VFIE formulation exhibits significantly higher numerical stability compared to the original PIDE problem.

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